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Interpolation in L^p with boundary conditions

by

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Dedicated to Professor Antoni Zygmund

Abstract. This paper characterizes the complex interpolation spaces between L^p and $L^p_{B,\omega}$, where $L^p_{B,\omega}$ denotes all those k-tuples of functions on a compact manifold G with boundary ∂G which have derivatives of order $\leqslant \omega$ in L^p, and which satisfy homogeneous boundary conditions $Bu = 0$ on ∂G. The system $Bu = 0$ must be "normal" in a certain natural sense; the space $[L^p, L^p_{B,\omega}]_\theta$ then consists of those functions in $L^p_{\theta\omega}$ which satisfy the boundary conditions involving normal derivatives of order $\leqslant \theta\omega - 1/p$. When $\theta\omega - 1/p$ is an integer, the boundary conditions may be satisfied in a weak sense. This generalizes results of Grisvard [Arch. Rat. Mech. Anal. 25 (1967), pp. 40–63], and gives the domains of fractional powers of elliptic operators whose domains are determined by the condition $Bu = 0$ on the boundary of G. It tells which boundary conditions must be satisfied by a function u in order to guarantee convergence of eigenfunctions in $L^p_{\theta\omega}$. It shows that regularity theorems for fractional powers are necessarily more complicated than for integer powers.

A theorem of Grisvard [9] characterizes the interpolation spaces between $L^2(G)$ and $H^\omega_B(G)$, where G is an open compact subset in R^n with C^∞ boundary ∂G, and $H^\omega_B(G)$ is the space of functions u in $H^\omega(G)$ satisfying $Bu = 0$ on ∂G. This paper obtains corresponding results for systems of functions in L^p, $1 < p < \infty$. Whereas Grisvard interpolates by the trace method, we use the complex method, outlined brifly in § 1. This is natural in view of the applications to domains of fractional powers; if A_B is an elliptic operator with domain defined by $Bu = 0$, and the fractional powers $(A_B)^\theta$ can be defined as in [14], then the domain of $(A_B)^\theta$ is precisely the complex interpolation space we are considering here. Thus, we obtain the general version of some results of Fujiwara [6], [7]. § 2 reviews the basic properties of Bessel potential spaces L^p_k and their restriction spaces $B^p_{k-1/p}$, due to Calderón, Stein, Taibleson, and Strichartz. § 3 gives a reasonable generalization of the concept of "normal boundary system" from the case $q = 1$ to general q. Our definition of normal system is more

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natural and slightly more general than Geymonat's [8], which in turn is more general than the one in [5]. Moreover, it is automatically satisfied when fractional powers can be defined as in [14]. § 4 proves the main result, which says roughly that the space $[L^p, L^p_{B,\omega}]_\theta$ consists of those functions in $L^p_{\theta\omega}$ satisfying that part of the boundary conditions B which makes sense in $L^p_{\theta\omega}$. When $\theta = 1$, all conditions apply, and as θ decreases to 0 they drop out one by one until none are left for $\omega\theta < 1/p$. § 5 gives a corollary on the convergence of eigenfunction expansions in various norms. These results seem to be new even for ordinary differential equations.

It is of course a great pleasure and honor to join in honoring Professor Antoni Zygmund in this special volume. The outline above suggests how much the results in this article depend on the large body of mathematics generated by Professor Zygmund and his mathematical descendants.

The author is also grateful to Professor E. Magenes for bringing Grisvard's paper to his attention.

§ 1. Complex interpolation. Let $X_1 \subset X_0$ be a continuous dense injection of Banach spaces. Let $S = \{z\colon 0 \leqslant \operatorname{Re}(z) \leqslant 1\}$, and S^0 denote the interior of S. Let $H(X_0, X_1)$ denote all continuous bounded functions $F\colon S \to X_0$ which are holomorphic in S^0, and such that the following norm is finite:

$$\|F\| = \max_j \sup_y \|F(j+iy)\|_j$$

where $j = 0$ or 1; $-\infty < y < \infty$; and $\|\ \|_j$ is the norm in X_j. For $0 \leqslant \theta \leqslant 1$, the interpolation space X_θ is defined by

$$X_\theta = [X_0, X_1]_\theta = \{F(\theta)\colon F \text{ in } H(X_0, X_1)\},$$

with norm $\|f\|_\theta = \inf\{\|F\|\colon F(\theta) = f\}$ (In [4], it is not required that $X_0 \supset X_1$; however, this simplifying assumption is satisfied in the case we are investigating.) The following facts are proved by Calderón in [4]:

(1.1) X_1 is densely and continuously injected in X_θ, and X_θ in X_0.

(1.2) If $\theta_1 \leqslant \theta \leqslant \theta_2$, then $[X_0, X_1]_\theta = [X_{\theta_1}, X_{\theta_2}]_s$, where $\theta = \theta_1 + s(\theta_2 - \theta_1)$.

(1.3) Let $Y_1 \subset Y_0$ be a continuous dense injection of Banach spaces, and let $A\colon X_0 \to Y_0$ be a linear map such that $\|Ax\|_0 \leqslant C_0\|x\|_0$, and $\|Ax\|_1 \leqslant C_1\|x\|_1$. Then $\|Ax\|_\theta \leqslant C_\theta\|x\|_\theta$, $C_\theta = C_0^{1-\theta}C_1^\theta$.

Property (1.2) is the *iteration property*, and (1.3) is the *interpolation property*.

It is easy to check that for $F \in H(X_0, X_1)$ and y real,

$$\|F(x+iy)\|_x \leqslant \|F\|; \tag{1.4}$$

simply replace $F(z)$ by $F(z+iy)$.

§ 2. The basic spaces. Two examples of interpolation spaces are based on the operators Λ^z and Λ_0^z, defined by

$$\widehat{\Lambda^z f}(\xi) = (1+|\xi|^2)^{z/2}\hat{f}(\xi), \tag{2.1}$$

$$\widehat{\Lambda_0^z f}(\xi) = (i\tau+\sqrt{1+|\xi'|^2})^z\hat{f}(\xi), \tag{2.2}$$

where $\xi = (\tau, \xi_2, \ldots, \xi_n)$ and $\xi' = (\xi_2, \ldots, \xi_n)$. Let

$$L^p(R^n_+) = \{f \text{ in } L^p(R^n)\colon f(x) = 0 \text{ if } x_1 < 0\}.$$

We define for α real and $1 < p < \infty$ (see [3]),

$$L^p_\alpha(R^n) = \Lambda^{-\alpha}(L^p(E^n))$$
$$L^p_{0\alpha}(R^n_+) = \Lambda_0^{-\alpha}(L^p(R^n_+))$$
$$L^p_\alpha(R^n_+) = \text{restrictions to } R^n_+ \text{ of members of } L^p_\alpha(R^n).$$

By a strong version of Mihlin's theorem ([11] Theorem 6, or [12] Theorem 4.5), $\Lambda_0^z\Lambda^{-z}$ is bounded on L^p, $1 < p < \infty$; since $L^p(R^n_+)$ is a closed complemented subspace of $L^p(R^n)$, it follows that $L^p_{0\alpha}(R^n_+)$ is a closed complemented subspace of $L^p_\alpha(R^n)$. Further, since $(i\tau+\sqrt{1+|\xi'|^2})^z$ extends analytically to $\operatorname{Im}(\tau) \leqslant 0$, it follows easily that

$$L^p_{0\alpha}(R^n_+) = \{f \in L^p_\alpha\colon f(t, x_2, \ldots, x_n) = 0 \text{ for } t < 0\}. \tag{2.3}$$

When $\alpha < 0$, the condition "$f(t, \ldots) = 0$ for $t < 0$" is taken in the sense of distributions.

A further appeal to Mihlin's theorem shows easily that if $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, then

$$L^p_\alpha(R^n) = [L^p_{\alpha_0}(R^n), L^p_{\alpha_1}(R^n)]_\theta, \tag{2.4}$$

$$L^p_{0\alpha}(R^n_+) = [L^p_{0\alpha_0}(R^n_+), L^p_{0\alpha_1}(R^n_+)]_\theta, \tag{2.5}$$

$$L^p_\alpha(R^n_+) = [L^p_{\alpha_0}(R^n_+), L^p_{\alpha_1}(R^n_+)]_\theta. \tag{2.6}$$

In taking boundary values we encounter the Besov spaces $B^p_\alpha(R^{n-1})$, also denoted W^p_α in [15], and $\Lambda(\alpha, p, p)$ in [17]. Let R_t denote the restriction map

$$R_t f(x_2, \ldots, x_n) = f(t, x_2, \ldots, x_n).$$

Stein [15] has proved that the map

$$(t, f) \to R_t f \text{ is continuous: } R^1 \times L^p_{\alpha+1/p}(R^n) \to B^p_\alpha(R^{n-1}) \tag{2.7}$$

for $\alpha > 0$. Further, there is a continuous linear extension map $\mathscr{E}\colon B^p_\alpha(R^{n-1}) \to L^p_{\alpha+1/p}(R^n)$, independent of α, such that for any real α, $\mathscr{E}g$ is C^∞ for $t \neq 0$ when g is in $B^p_\alpha(R^{n-1})$, and

$$g = \lim_{t\to 0} R_t\mathscr{E}g \text{ in the } B^p_\alpha \text{ norm.} \tag{2.8}$$

Thus, for $\alpha > 0$, $B^p_\alpha(R^{n-1})$ could be defined as $R_0(L^p_{\alpha+1/p}(R^n))$.

The result (2.8) extends easily to Cauchy data of any order (see [13]). Let

$$\gamma_k f(x_2, \ldots, x_n) = (R_0 f, R_0 D_t f, \ldots R_0 D_t^k f).$$

Then there is a continuous map

(2.8a) $$\mathscr{E}^k\colon B_a^p \oplus \ldots \oplus B_{a-k}^p \to L_{a+1/p}^p(R^n),$$

defined for all real a, such that for $a > k$, $\gamma_k \mathscr{E}^k$ is the identity operator. For $a \leqslant k$ this relation continues to hold in the limit sense of (2.8). (In [13], equation (3), p. 786, should read

$$\int A^{k-1}\psi(y')\,dy' = \delta_{1k}, \quad k = 1, \ldots, \omega.$$

To satisfy this equation it suffices that ψ have compact support and $\int \psi(y')\,dy' = 1$, since $\int A^{k-1}\psi = \int \big(\sum_{j=2}^{n} D_j(y_j\psi)\big)^{k-1} dy'$ is automatically zero for $k = 2, \ldots, \omega$, by the fundamental theorem of calculus.)

The map

(2.9) $$\Lambda^z\colon B_a^p(R^{n-1}) \to B_{a-\mathrm{Re}(z)}^p(R^{n-1})$$

is an isomorphism ([18]), where Λ^z now denotes the operator (2.1) with ξ in R^{n-1} instead of R^n. Using this, (2.4), (2.7), and (2.8), it is easy to show that for $a = (1-\theta)a_0 + \theta a_1$,

(2.10) $$B_a^p = [B_{a_0}^p, B_{a_1}^p]_\theta$$

(see also [18] and [4]).

The space $\mathscr{S}(R^n)$ of smooth functions of rapid decay is dense in $L_a^p(R^n)$, and the pairing

$$(f, g) = \int f\bar{g}.$$

extends from $\mathscr{S} \times \mathscr{S}$ to $L_a^p \times L_{-a}^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$; and the pairing realizes each space as the (anti-) dual of the other ([3]). This induces a pairing between $L_a^p(R_+^n)$ and $L_{0,-a}^{p'}(R_+^n)$.

From the density of $\mathscr{S}$ in $L_a^p(R^n)$, it follows that the space $\mathscr{S}(R_+^n)$ of smooth functions in the closed half space R_+^n is dense in $L_a^p(R_+^n)$. Further, since translation is continuous in L_a^p, the subspace $\mathscr{S}_0(R_+^n)$ of smooth functions vanishing in a neighborhood of R^{n-1} is dense in $L_{0a}^p(R_+^n)$.

Many operations on $\mathscr{S}(R^n)$ induce bounded operators on the various spaces considered here. Let X_a denote any of $L_a^p(R^n)$, $L_a^p(R_+^n)$, $L_{0a}^p(R_+^n)$, $B_a^p(R^n)$. Then the following are bounded:

(2.11) $$D^\beta\colon X_a \to X_{a-|\beta|}, \quad \text{where } D^\beta = (-i)^{|\beta|}\left(\frac{\partial}{\partial x_1}\right)^{\beta_n} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n};$$

(2.12) $$M_\varphi\colon X_a \to X_a, \quad \text{where } \varphi \in \mathscr{S}(R^n) \text{ and } M_\varphi f = \varphi f;$$

(2.13) $$F^{-1}M_\varphi F\colon X_a \to X_a,$$

where F is the Fourier transform and φ is a multiplier on L^p;

(2.14) $$\chi^* M_\varphi\colon X_a \to X_a,$$

where χ is a C^∞ map of a neighborhood of $\operatorname{supp}(\varphi)$ into R^n, and $\chi^* f(x) = f(\chi(x))$. In the case of R_+^n, χ is a map of the closed half space R_+^n into R_+^n, and φ is in $\mathscr{S}(R_+^n)$.

For $L_a^p(R^n)$ this is well known, and it follows easily for $L_a^p(R_+^n)$ and $L_{0a}^p(R_+^n)$. For B_a^p with $a > 0$, the results can be deduced by using the extension $\mathscr{E}$ into L_a^p. For $a < 0$, they follow by duality, as in [18]; and for $a = 0$, by interpolation.

There is an obvious injection of $L_{0a}^p(R_+^n)$ into $L_a^p(R_+^n)$; however, when $a - \frac{1}{p}$ is an integer, the image is *not* closed, as [10], shows for $p = 2$. This explains the "exceptional values" $a = k + 1/p$ in the following characterization of L_{0a}^p.

For f in $\mathscr{S}(R_+^n)$, let

$$E_0 f(t, x_2, \ldots, x_n) = \begin{cases} f(t, x_2, \ldots, x_n) & \text{if } t \geqslant 0, \\ 0 & \text{if } t < 0. \end{cases}$$

LEMMA 2.1. *Let $k = 0, 1, 2, \ldots$, and $f \in L_a^p(R_+^n)$.*

i) *For $\frac{1}{p} - 1 < a < \frac{1}{p}$, the map E_0 extends by continuity to a map from $L_a^p(R_+^n)$ to $L_{0a}^p(R_+^n)$.*

ii) *If $k + \frac{1}{p} - 1 < a < k + \frac{1}{p}$, then*

(2.15) $$E_0 f \in L_{0a}^p(R_+^n) \Leftrightarrow \gamma_{k-1} f = 0.$$

iii) *If $a = k + \frac{1}{p}$, then*

(2.16) $$E_0 f \in L_{0a}^p \Leftrightarrow \gamma_{k-1} f \equiv 0 \quad \text{and} \quad E_0 D_t^k f \in L_{0,1/p}^p(R_+^n).$$

Proof. Strichartz [16] proves the following: If χ_+ denotes the characteristic function of R_+^n, then the map $f \to \chi_+ f$ is continuous on $L_a^p(R^n)$, $0 \leqslant a < \frac{1}{p}$. This same map is thus continuous for $\frac{1}{p} - 1 < a < 0$, in view of the duality between L_a^p and $L_{-a}^{p'}$, and the fact that $(\chi_+ f, g) = (f, \chi_+ g)$. Part (i) follows easily: if $f \in \mathscr{S}(R_+^n)$, let $\tilde{f} \in L_a^p(R^n)$ with $\tilde{f} = f$ in R_+^n and $\|\tilde{f}\|_a \leqslant 2\|f\|_a$, the latter norm being in $L_a^p(R_+^n)$. Then

$$\|E_0 f\|_a = \|\chi_+ \tilde{f}\|_a \leqslant C\|f\|_a,$$

which proves (i).

In parts (ii) and (iii) the direct implication ($\Rightarrow$) follows from (2.7) by letting $t \to 0-$. The reverse implication can be reduced to (i) by induction, using

LEMMA 2.2. *If $f \in L^p_a(R^n_+)$, $a > 1/p$, and $\gamma_0 f \equiv 0$, then*

$$DE_0 f = E_0 Df \tag{2.17}$$

for any first derivative D.

Proof. Let $f_m \in \mathcal{S}(R^n_+)$ and $f_m \to f$ in $L^p_a(R^n_+)$. Then for φ in $\mathcal{S}(R^n)$,

$$-\int_{R^n} (E_0 f_m) D_t \varphi = \int_{R^{n-1}} (\gamma_0 f_m)\varphi + \int_{R^n} E_0 D f_m.$$

As $m \to \infty$, $\gamma f_m \to \gamma_0 f = 0$ in $B^{a-1/p}(R^{n-1})$, and (1.16) follows.

Now when $k = 0$, (ii) and (iii) reduce to (i), since the condition $\gamma_{k-1} f = 0$ drops out. Suppose then that $k > 0$, and the right hand side of (2.15) or (2.16) holds. Then by the induction assumption, $E_0 f \in L^p_{a-1}(R^n)$, and by the induction assumption and (2.17), $DE_0 f \in L^p_{a-1}(R^n)$; hence (see [3]) $E_0 f \in L^p_a(R^n)$. Since $E_0 f$ vanishes in R^n_-, Lemma 2.1 is proved.

Lemma 2.3 gives another source of functions in the "exceptional" space $L^p_{0,1/p}$.

LEMMA 2.3. *Suppose that $f \in L^p_{m+1/p}(R^n_+)$, $m = 1, 2, \ldots$, and that*

$$\gamma_{m-1} f = 0. \tag{2.18}$$

Let D^a_x be any tangential derivative (i.e. involving derivatives only along R^{n-1}) with $0 < |a| \leqslant m$. Then $D^a_x f \in L^p_{0,1/p}(R^n_+)$.

Proof. Let $F_1 \in H(L^p_m, L^p_{m+1})$ with $F_1(1/p) = f$. Let $F = F_1 - \mathcal{E}^{m-1}\gamma_{m-1}F_1$. Then $F \in H(L^p_m, L^p_{m+1})$, and by (2.18), $F(1/p) = F_1(1/p) = f$; and

$$\gamma_{m-1} F(z) = 0 \quad \text{for all } z. \tag{2.19}$$

Since $m \geqslant 1$, (2.19) implies that $\lim_{t\to 0+} R_t F(1+iy) = 0$ in $B^p_{m+1-1/p}(R^{n-1})$. Hence

$$\lim_{t\to 0+} R_t D^a_x F(1+iy) = 0 \quad \text{in } B^p_{1-1/p}.$$

Hence by Lemma 2.1, $D^a_x F(1+iy) \in L^p_{0,1}$, so $D^a_x F \in H(L^p, L^p_{0,1})$, and $D^a_x F(1/p) = D^a_x f \in L^p_{0,1/p}$. ∎

In view of (2.12) and (2.14), all the results in this section can be carried over (by a partition of unity) to a compact C^∞ manifold with boundary.

§ 3. Normal boundary systems. Let G be a bounded domain in R^n with smooth boundary ∂G. (More generally, let G be a compact C^∞ manifold with boundary ∂G.) Let t denote a "normal variable" such that $t > 0$ inside G, and $t = 0$ on ∂G; thus, a neighborhood U of ∂G in G has the form $\partial G \times \{0 \leqslant t < 1\}$. Consider operators B_j from q-tuples of functions on U to r_j-tuples of functions on U, having the form

$$B_j = \sum_{\nu=0}^{\omega_j} b^\nu_j D_t^{\omega_j-\nu}, \quad \text{where } D_t = -i\partial/\partial t, \tag{3.1}$$

and b^ν_j is a differential operator on ∂G of order $\leqslant \nu$. In particular, b^0_j is an $r_j \times q$ matrix of functions on U. (More generally, B_j can map sections of a q-dimensional vector bundle over G into sections of an r_j-dimensional vector bundle over U.)

DEFINITION 3.1. A system of boundary operators $B_1, \ldots, B_k$ is called *normal* iff

(i) the orders ω_j of the B_j satisfy $\omega_1 < \omega_2 < \ldots < \omega_k$, and

(ii) the coefficient b^0_j of $D_t^{\omega_j}$ in B_j is an $r_j \times q$ matrix of rank r_j at every point of ∂G.

Remark. Generally, one considers a system of boundary operators with arbitrary indexing. However, for convenience in stating our results, we have lumped together all operators of a given order ω_j into one system B_j, and indexed the resulting systems monotonically according to their orders. This can be done with any system; the crucial part of Definition 1 is the rank condition (ii). This implies that $r_j \leqslant q$. In particular, when $q = 1$ each B_j maps into *functions* on U (since $r_j = 1$), all the B_j's have different orders, and each B_j is noncharacteristic on ∂G. This is the usual definition of a normal system for $q = 1$.

Definition 3.1 arises naturally from Agmon's condition [1] for minimum growth of the resolvent. Let A be a $q \times q$ system of differential operators defined in G. In analogy with (3.1), write

$$A = \sum_{\nu=0}^{\omega} a_\nu D_t^{\omega-\nu},$$

where a_ν is a differential operator on ∂G or order $\leqslant \nu$, with coefficients depending on $t \geqslant 0$ and $x \in \partial G$. Let $\hat{a}_\nu(x, t, \xi)$ be the characteristic polynomial of a_ν, homogeneous in ξ of degree ν; similarly for the b^ν_j in (3.1). Let $-\pi \leqslant \theta < \pi$. Then the ray $R = \{\lambda; \arg\lambda = \theta\}$ in the complex plane is called a *ray of minimal growth* for $(A; B_1, B_2, \ldots, B_k)$ iff

(3.2) the characteristic polynomial of A has no eigenvalues on R, and

(3.3) the problem

$$\sum \hat{a}_\nu(x, 0, \xi) D_t^{\omega-\nu} u(t) = \lambda u(t), \quad t > 0,$$
$$\sum \widehat{b^\nu_j}(x, \xi)\,(D_t^{\omega_j-\nu} u)(0) = g_j, \quad j = 1, 2, \ldots, k,$$
$$u(+\infty) = 0$$

has a unique solution for each g_j in C^{r_j}, x in ∂G, and $(\xi, \lambda) \neq 0$ with λ on the ray R.

If the boundary problem for A defined by $B_j u = 0$, for $t = 0$, and all $j = 1, \ldots, k$, is self-adjoint, then every ray, except those with $\theta = 0$ and $\theta = \pi$, is a ray of minimal growth.

Suppose that $A, B_1, \ldots, B_k$ has a ray of minimal growth, and $B_1, \ldots, B_k$ has been arranged as in (ii); *then this is a normal system.* To prove this, take $\xi = 0$, $\lambda \neq 0$ in (3.3). Since $b_j^\nu(x, \xi)$ is homogeneous of degree ν, it follows that $b_j^\nu(x, 0) = 0$ for $\nu > 0$, and $b_j^0(x, \xi) = b_j^0(x)$. Hence, from (3.3), b_j^0 is surjective, and this is the condition (ii) in Definition 3.1. (This argument was suggested by T. Burak, in the case of functions, i.e. $q = 1$.)

We are interested only in the null spaces of the B_j; hence they can be replaced by any other operators with the same null spaces. Since the coefficient b_j^0 in Definition 3.1 is a surjective matrix, the adjoint b_j^{0*} is $1-1$; hence multiplying B_j by b_j^{0*} does not affect the null space. Multiplying further by a positive definite $q \times q$ matrix, *we can replace B_j by a $q \times q$ matrix operator such that the coefficient b_j^0 of $D_t^{\omega_j}$ is a projection-valued matrix function, and*

(3.4) $$b_j^0 b_j^\nu = b_j^0 \quad \text{for all } \nu.$$

§ 4. Interpolation between $L^p_{B,\omega}$ and L^p. Our main result concerns interpolation between the following spaces.

Definition 4.1. Let $B = (B_1, \ldots, B_k)$ be a normal system on ∂G. We define the space $L^p_{Bs}(G)$ as follows, for $s \geqslant 0$:

(i) If $s - 1/p < \omega_1$, then $L^p_{Bs} = L^p_s(G)$.

(ii) If for some l, $\omega_l < s - 1/p < \omega_{l+1}$, then
$$L^p_{Bs} = \{u \in L^p_s(G) \colon \gamma_0 B_j u = 0 \quad \text{for } j \leqslant l\}.$$

(iii) If $s - 1/p = \omega_l$, then
$$L^p_{Bs} = \{u \in L^p_s(G) \colon \gamma_0 B_j u = 0 \text{ for } j < l, \text{ and } E_0 B_l u \in L^p_{0,1/p}(G)\}.$$

Here γ_0 denotes restriction to ∂G, and $E_0 B_l u \equiv 0$ outside G.

Remark. In parts (i) and (ii), we apply precisely those boundary conditions which are well-defined, in view of Stein's result (2.7) on restrictions. Since the map $u \to \gamma_0 B_j u$ is continuous for $j \leqslant l$, the spaces defined in (i) and (ii) are closed subspaces of L^p_s. Part (iii) covers the situation at the special values of s where some boundary operator B_l is dropping out; the condition in (iii) requires that $B_l u$ vanish on the boundary in a weak sense. For these values of s, $L^p_{Bs}(G)$ is *not* a closed subspace of $L^p_s(G)$. (See [10], p. 67).

Theorem 4.1. *Let $B = (B_1, \ldots, B_k)$ be a normal system* (§ 3), *and let $\omega_k + 1 \leqslant \omega$, $0 < \theta < 1$. Then*
$$[L^p(G), L^p_{B,\omega}(G)]_\theta = L^p_{B,\theta\omega}(G),$$
where $[\,,\,]_\theta$ denotes the complex interpolation method of § 1.

Proof. Let $X_0 = L^p(G)$, $X_1 = L^p_{B,\omega}(G)$, and $X_\theta = [X_0, X_1]_\theta$; we have to prove that $X_\theta = L^p_{B,\theta\omega}(G)$. Let $f \in X_\theta$, and $F \in H(X_0, X_1)$ with $F(\theta) = f$. Since $X_1 \subset L^p_\omega(G)$, it follows from (1.4) and (2.5) that the restriction of F to $\theta \leqslant \mathrm{Re}(z) \leqslant 1$ is a continuous map into $L^p_{\omega\theta}(G)$, and is holomorphic for $\theta < \mathrm{Re}(z) < 1$. Thus if $\omega_j < \omega\theta - 1/p$, the map $z \to \gamma_0 B_j F(z)$ is continuous into $B^p_{\omega\theta - 1/p - \omega_j}(\partial G)$ for $\theta \leqslant \mathrm{Re}(z) \leqslant 1$, holomorphic for $0 \leqslant \mathrm{Re}(z) < 1$, and vanishes identically for $\mathrm{Re}(z) = 1$. Hence $F(z) \in L^p_{\omega\theta}(G)$ and $\gamma_0 B_j F(z) = 0$ for $\omega_j < \omega\theta - 1/p$ and $\mathrm{Re}(z) \geqslant \theta$, which proves that $f = F(\theta) \in L^p_{B,\omega\theta}(G)$, *except* in case (iii) of Definition 4.1, where

(4.0) $$\omega\theta - 1/p = \omega_l \quad \text{for some } l.$$

In particular, when $\theta\omega$ is an integer, $X_\theta \subset L^p_{B,\theta\omega}(G)$, with continuous inclusion. Now suppose that (4.0) holds, and $f \in X_\theta$. By the iteration property (1.2), and the cases already proved, we have
$$f \in [X_{\theta_1}, X_{\theta_2}]_s = [L^p_{B,\omega_l}, L^p_{B,\omega_l+1}]_s,$$
where $\theta_1\omega = \omega_l$ and $\theta_2\omega = \omega_l + 1$, and $\theta = \theta_1 + s(\theta_2 - \theta_1)$. Let $F \in H(L^p_{B,\omega_l}, L^p_{B,\omega_l+1})$ with $F(s) = f$. Then $\gamma_0 B_l F(z)$ vanishes identically for $\mathrm{Re}(z) = 1$, so by Lemma 2.1, the map $z \to B_l F(z)$ lies in $H(L^p, L^p_{0,1})$. Hence $B_l f = B_l F(s) \in [L^p, L^p_{0,1}]_\theta = L^p_{0,1/p}$, by (2.5), since $L^p = L^p_{0,0}$. This concludes the proof that $X_\theta \subset L^p_{B,\omega\theta}$.

The converse uses the simple idea of "extending the Cauchy data". Suppose that

(4.1) $$(g_0, \ldots, g_k) \in B^p_s(\partial G) \oplus \ldots \oplus B^p_{s-k}(\partial G),$$

is given. Referring to (3.1), define for $\omega_j \leqslant k$

(4.2) $$B_j^\sim(g_0, \ldots, g_k) = \sum_{\nu=0}^{\omega_j} b_j^\nu g_{\omega_j - \nu}.$$

Thus $\gamma_0 B_j f = B_j^\sim \gamma_k f$.

Lemma 4.1. *Given* (4.1) *with $s = (1-\theta)s_1 + \theta s_2$, suppose that $B_j^\sim(g_0, \ldots, g_k) = 0$ for $\omega_j \leqslant k$. Then one can define functions*

(4.3) $$G_m \in H\big(B^p_{s_1 - m}(\partial G), B^p_{s_2 - m}(\partial G)\big), \quad 0 \leqslant m < \omega$$

such that

(4.4) $$B_j^\sim(G_0, \ldots, G_{\omega-1}) = 0 \quad \textit{for all } j,$$

(4.5) $$G_m(\theta) = g_m \quad \textit{for } 0 \leqslant m \leqslant k,$$

(4.6) $$\|(G_0, \ldots, G_{\omega-1})\| \leqslant C \|(g_0, \ldots, g_k)\|.$$

Proof. We can (and do) assume that the boundary operators $B_1, \ldots$ have the special form deduced at the end of § 3. If $k < 0$ we can take $G_m = 0$, so assume $k \geqslant 0$. If $\omega_1 > 0$, take G_0 in $H(B^p_{s_1}, B^p_{s_2})$ with $G_0(\theta) = g_0$; such a G_0 exists, by (2.10). If $\omega_1 = 0$, replace G_0 by $G_0' = (I - b_1^0)G_0$;

since b_1^0 is a projection, and $b_1^0 g_0 = \tilde{B}_1 g_0 = 0$, we have $G_0'(\theta) = g_0$ and $b_1^0 G_0 \equiv 0$, as desired. Proceeding by induction, suppose that $G_0, \ldots, G_{m-1}$ are defined satisfying (4.4) for $\omega_j < m$, (4.5), and (4.6). The definition of G_m now depends on m.

(i) If $m = \omega_j$ for no j and $m > k$, take $G_m = 0$.

(ii) If $m = \omega_j$ for no j and $m \leqslant k$, take any G_m in $H(B^p_{s_1-m}, B^p_{s_2-m})$ such that $G_m(\theta) = g_m$.

(iii) If $m = \omega_l$ and $m > k$, take $G_m = -\sum_{\nu=1}^{\omega} b_l^\nu G_{\omega_l-\nu}$.

Since $b_l^0 b_l^\nu = b_l^\nu$, (4.4) will now hold for $\omega_j \leqslant m$, while (4.5) and (4.6) are trivial.

(iv) If $m = \omega_l$ and $m \leqslant k$, take any G_m' in $H(B^p_{s_1-m}, B^p_{s_2-m})$ such that

$$(I - b_l^0) G_m'(\theta) = (I - b_l^0) g_m$$

and set

$$G_m = (I - b_l^0) G_m' - \sum_{\nu=1}^{\omega_l} b_l^\nu G_{\omega_l-\nu}.$$

Then (4.4) holds for $\omega_j \leqslant m$ as in (iii) above, and (4.5) holds since

$$b_l^0 G_m(\theta) = -\sum_{\nu=1}^{\omega_l} b_l^\nu G_{\omega_l-\nu}(\theta) = -\sum_{\nu=1}^{\omega_l} b_l^\nu g_{\omega_l-\nu} = b_l^0 g_m,$$

and

$$(I - b_l^0) G_m(\theta) = (I - b_l^0) G_m'(\theta) = (I - b_l^0) g_m.$$

Again, (4.6) is trivial, so Lemma 4.1 is proved.

Now, given $f \in L^p_{B,\theta\omega}(G)$, we must find $F \in H(L^p, L^p_{B,\omega})$ with $F(\theta) = f$. Suppose at first that $\theta\omega - 1/p$ is not an integer. This excludes Case (iii) of Definition 4.1; suppose we are in Case (ii). (Case (i) is similar, and will be left to the reader.) Thus

$$f \in L_{0\omega}(G) \tag{4.7}$$

$$\omega_l < \theta\,\omega - 1/p < \omega_{l+1} \tag{4.8}$$

$$\gamma_0 B_j f = 0 \quad \text{for } j \leqslant l. \tag{4.9}$$

From (4.7) and the restriction theorem (2.7),

$$(g_0, \ldots, g_k) = \gamma_k f \in B_{\theta\omega-1/p} \otimes \ldots \otimes B^p_{\theta\omega-1/p-k}, \tag{4.10}$$

where $\theta\,\omega - 1/p - 1 < k < \theta\,\omega - 1/p$. Further, from (4.8) and (4.9), $B_j^\sim(g_0, \ldots, g_k) = 0$ for $\omega_j \leqslant k$. Hence by Lemma 4.1 we have G_m in $H(B^p_{-1/p-m}, B^p_{\omega-1/p-m})$, $0 \leqslant m < \omega$, satisfying (4.4)–(4.6). Using the extension $\mathscr{E}^{\omega-1}$ in (2.8a), let $F_1 = \mathscr{E}^{\omega-1}(G_0, \ldots, G_{\omega-1})$. Then $F_1 \in H(L^p, L^p_\omega)$, and since $\gamma_{\omega-1}\mathscr{E}^{\omega-1} = $ identity, we get from (4.10) and (4.5) that $\gamma_k F_1(\theta) = \gamma_k f$.

Since $\omega\theta - 1/p$ is not an integer, Lemma 2.1 shows that $f - F_1(\theta) \in L^p_{0,\omega\theta}(G)$; hence by (2.4) there is an F_2 in $H(L^p, L^p_{0,\omega})$ such that $F_2(\theta) = f - F_1(\theta)$. Let $F = F_1 + F_2$. Then $\gamma_0 B_j F_1(1+iy) \equiv 0$ for all j, and $\gamma_{\omega-1} F_2(1+iy) \equiv 0$, so $\gamma_0 B_j F(1+iy) \equiv 0$ for all j, and $F \in H\big(L^p(G), L^p_{B,\omega}(G)\big)$, as was to be proved.

The case $\omega\theta - 1/p = m =$ integer remains. In view of the case already proved ($\omega\theta - 1/p \neq$ integer) and the iteration property (1.2), it suffices to prove that

$$f \in L^p_{B,\omega\theta} \Rightarrow f \in [L^p_{B,m} L^p_{B,m+1}]_s \tag{4.11}$$

where $s = 1/p$. Let $F_1 \in H(L^p_m, L^p_{m+1})$ with $F_1(s) = f$, and let $\gamma_{m-1} F_1 = (G_0, \ldots, G_{m-1})$. Proceeding inductively as in Lemma 4.1, we can then define G_l', $0 \leqslant l < m$, such that

$$B_j^\sim(G_0 + G_0', \ldots, G_{m-1} + G_{m-1}') \equiv 0, \qquad \omega_j < m \tag{4.12}$$

and

$$G_l'(s) = 0, \qquad 0 \leqslant l < m. \tag{4.13}$$

Set $F_2 = F_1 + \mathscr{E}^m(G_0', \ldots, G_{m-1}', 0)$. By (4.13), $F_2(s) = f$. Further, *if $m = \omega_j$ for no j*, then $\gamma_0 B_j F_2 = 0$ for $\omega_j \leqslant m$, so $F_2 \in H(L^p_{B,m}, L^p_{B,m+1})$, and (4.11) is proved.

In the remaining case where $\omega\theta - \dfrac{1}{p} = m = \omega_l$ for some l, the construction is more complicated. First, as in Lemma 4.1, we construct G_j in $H(B^p_{m-1/p-j}, B^p_{m+1-1/p-j})$, $0 \leqslant j \leqslant m$, such that $B_j^\sim(G_0, \ldots, G_m) = 0$ for $0 \leqslant j \leqslant l$ and $\gamma_{m-1} f = \big(G_0(s), \ldots, G_{m-1}(s)\big)$. Let $F_1 = \mathscr{E}^m(G_0, \ldots, G_m)$. Then

$$\gamma_{m-1} f = \gamma_{m-1} F_1(s) \tag{4.14}$$

$$\gamma_0 B_j F_1(iy) = B_j^\sim\big(G_0(iy), \ldots, G_{m-1}(iy)\big) \equiv 0 \quad \text{for } 0 \leqslant j < l, \tag{4.15}$$

$$\gamma_0 B_j F_1(1+iy) \equiv B_j^\sim\big(G_0(1+iy), \ldots, G_m(1+iy)\big) \equiv 0 \quad \text{for } 0 \leqslant j \leqslant l, \tag{4.16}$$

so $F_1 \in H(L^p_{B,m}, L^p_{B,m+1})$. We now have to modify F_1 into an F such that $F(s) = f$. From (4.16) and Lemma 2.1, $B_l F_1(1+iy) \in L^p_{0,1}$, so $B_l F_1 \in H(L^p, L^p_{0,1})$ and by (2.4), $B_l F_1(s) \in L^p_{0,1/p}$, hence $B_l\big(f - F_1(s)\big) \in L^p_{0,1/p}$. Since b_l^ν is a tangential differential operator of order $\leqslant \nu$, it follows from (4.14) and Lemma 2.3 that $\sum_{\nu=1}^{m} b_l^\nu D_t^{m-\nu}\big(f - F_1(s)\big)$ is in $L^p_{0,1/p}$; hence $b_l^0 D_t^m\big(f - F_1(s)\big) \in L^p_{0,1/p}$. By (4.14) and Lemma 2.1, then, $b_l^0\big(f - F_1(s)\big) \in L^p_{0,\omega\theta}$. Hence there is

$$F_2 \text{ in } H(L^p_{0,m} L^p_{0,m+1}) \text{ with } F_2(s) = b_l^0\big(f - F_1(s)\big) \tag{4.17}$$

and

$$F_3 \text{ in } H(L^p_m, L^p_{m+1}) \text{ with } F_3(s) = (I - b_l^0)\big(f - F_1(s)\big). \tag{4.18}$$

Let

$$F_4 = b_l^0 F_2 + (I - b_l^0)\big(F_3 - \mathscr{E}^m(\gamma_{m-1}(F_3, 0))\big)$$

Then

(4.19) $$\gamma_{m-1} F_4 \equiv 0,$$

so

(4.20) $$\gamma_0 B_l F_4(1+iy) = \gamma_0 b_l^0 D_t^m F_4(1+iy) = \gamma_0 b_l^0 D_t^m F_2(1+iy) = 0,$$

in view of (4.17). Further, from (4.14) and (4.18), $\mathscr{E}^m\big(\gamma_{m-1}(F_3(s), 0)\big) = 0$, so $F_4(s) = f - F_1(s)$. Hence, setting $F = F_1 + F_4$, we have

$$F(s) = f,$$

$$\gamma_0 B_j F = \gamma_0 B_j F_1 + \gamma_0 B_j F_4 = 0 + 0, \quad 0 \leqslant j < l,$$

$$\gamma_0 B_l F(1+iy) = 0,$$

by (4.16) and (4.20). Thus $F \in H(L^p_{B,m}, L^p_{B,m+1})$, so $f = F(s) \in [L^p_{B,m}, L^p_{B,m+1}]_s = [L^p, L^p_{B,\omega}]_\theta$, and Theorem 4.1 is proved.

§ 5. Eigenfunction expansions. Agmon [1] has given conditions for completeness of eigenfunctions in L^p. Here, assuming completeness in L^p, we characterize the functions f which have an eigenfunction expansion converging in $L^p_{\theta\omega}$, when θ is not one of the exceptional values in Definition 4.1.

Let A be an elliptic system of order ω on G, and let A and B satisfy the conditions in [14], so that the realization A_B of A under the boundary condition $Bu = 0$ is a Fredholm operator, and the complex powers $(A_B)^z$ can be defined for $\mathrm{Re}(z) < 0$. Since the eigenfunctions of A_B are the same as those of $A_B + cI$, we can assume that A_B is invertible; hence A_B^z is defined for all z, and $A_B^0 = I$ (see [13]). Let $\lambda_1, \ldots,$ be the eigenvalues of A_B, in any order, and φ_k be the eigenfunctions, $A_B\varphi_k = \lambda_k\varphi_k$; then $A_B^z\varphi_k = \lambda_k^z\varphi_k$. From [12], A_B^θ is an isomorphism of $D(A_B^\theta) = [L^p, L^p_{B,\omega}]_\theta$ onto L^p. Hence from Theorem 4.1, we have

(5.1) $$\sum a_k\varphi_k \to f \text{ in } L^p_{B,\theta\omega} \Leftrightarrow \sum a_k\lambda_k^\theta\varphi_k \to A_B^\theta f \text{ in } L^p.$$

Thus the eigenfunctions are complete in $L^p_{B,\theta\omega}$ if and only if they are complete in L^p. Moreover, if linear combinations of eigenfunctions are to converge to a function f in $L^p_{\theta\omega}$ (where $\theta\omega$ is not one of the exceptional exponents in Definition 4.1), then f must satisfy those boundary conditions of order $< \theta\omega - 1/p$.

§ 6. Regularity problems. The question of domains answered above grew out of a regularity question posed by Agmon. He proposed to reduce the eigenvalue problem $A_B u = \lambda C u$ to $v = \lambda(A_B^{-1/2} C A_B^{-1/2})v$, where $v = A_B^{1/2} u$. To carry this through easily would require regularity results for $A_B^{1/2}$ analogous to those for A_B. Unfortunately, the fractional powers are not so well-behaved in this respect.

Suppose, for example, that $A_B u = f$ is a second order Dirichlet problem, that is A is second order, and $Bu = u$. Thus $\mathrm{Dom}(A_B^\theta)$ requires $Bu = 0$ for $\theta > 1/2p$, but not for $\theta < 1/2p$. To answer the regularity problem, we consider u in L^p_s, and ask where $A_B^{-1/2}u$ lies. We find:

(i) If $s < 1/p$, then $A_B^{-1/2}u$ does indeed lie in $L^p_{B,s+1}$. For in this case, u is in the domain of $(A_B)^{s/2}$, so $u = A_B^{-s/2}w$ for some w in L^p, and $A_B^{-1/2}u = A_B^{-1/2-s/2}w \in L^p_{B,s+1}$.

(ii) If $\dfrac{1}{p} < s$, then $A_B^{-1/2}u$ is in $L^p_{B,1+1/p-\varepsilon}$ for all $\varepsilon > 0$; this follows from (i).

(iii) If $\dfrac{1}{p} < s < 1 + \dfrac{1}{p}$, then the full regularity conclusion $A_B^{-1/2}u \in L^p_{s+1}$ will hold *if and only if u satisfies the boundary condition* $Bu = 0$. In fact, $A_B^{-1/2}u \in L^p_{1+1/p+\varepsilon}$ for *any* $\varepsilon > 0$ implies that $Bu = 0$ on the boundary. For, any function v in the range of $A_B^{-1/2}$ automatically satisfies $Bu = 0$, so if $A_B^{-1/2}u$ is in $L^p_{1+1/p+\varepsilon}$, then actually it is in $L^p_{B,1+1/p+\varepsilon}$, hence $A_B^{-1/2}u = A_B^{-1/2-1/2p-\varepsilon/2}w$ for a w in L^p, and it follows that $u = A_B^{-1/2p-\varepsilon/2}w$ is in $L^p_{B,1/p+\varepsilon}$, so $Bu = 0$ on the boundary.

By contrast, the integer power A_B^{-1} maps L^p_s into L^p_{s+2} for *every* $s \geqslant 0$. Let $u \in L^p_s$, and set $v = A_B^{-1}u$. Then $Bv = 0$ on the boundary, and $Av = u \in L^p_s$, so the standard regularity theorem for elliptic equations says v lies in L^p_{s+2}.

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On the localization property of square partial sums for multiple Fourier series

by

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To A. Zygmund on the 50th anniversary of his first mathematical publication

Abstract. It seems that localization and convergence of multiple Fourier series are related to the Sobolev spaces W_p^1. This paper establishes the existence of such a relation regarding the square partial sums. It is shown that for $f \in W_p^1$, $p \geqslant n-1$, this sort of localization holds for the n-torus. For each $p < n-1$ there is an $f \in W_p^1$ for which localization fails. Examples are given of an everywhere differentiable periodic function of 2 variables for which localization by square partial sums fails and of a function in W_2^1 for which localization by rectangular partial sums fails.

1. In the study of Fourier series, a primary feature is the localization property, which has been known to hold in the case of functions of one variable since Riemann. That localization does not generally hold for functions of several variables has also been known for a long time. Our purpose is to obtain precise information regarding the functions of n variables which have this property.

Tonelli, [2], observed that, for $n = 2$, localization holds for those functions now known as the functions whose partial derivatives (in the distribution sense) are measures; this includes the Sobolev space W_1^1. An example by Torrigiani, [3], shows that a condition given by Tonelli, which guarantees convergence at a point, and holds almost everywhere for $n = 2$ for functions in W_1^1, may hold nowhere for $n = 3$.

In a recent paper, Igari [1], settles the localization problem for the square $(C, 1)$ partial sums of a multiple Fourier series. He shows that this sort of localization holds for $f \in L^p$, $p \geqslant n-1$, and fails to hold for $p < n-1$. For the square partial sums themselves — not the averages — he points out that there are continuous functions for which localization fails.

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