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On spaces of measurable functions

by

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Dedicated to Antoni Zygmund

Abstract. The main result is Theorem 4.1 stating that, for every separable metric space X having more than one point, the space M_X, of measurable functions from the interval $[0;1]$ into X, is homeomorphic to the Hilbert space l_2. One of the corollaries is that every separable complete metric group is algebraically and topologically isomorphic to a closed subgroup of a group homeomorphic to l_2.

The present paper deals with the spaces M_X of all Lebesgue measurable functions from the interval $[0;1]$ into a metric space X. The topology of M_X is that of convergence in measure. The main result is Theorem 4.1 which asserts that M_X is homeomorphic to the separable infinite-dimensional Hilbert space l_2 if and only if X is a separable complete-metrizable space with more than one point. Particular cases of this theorem related to the following specifications of X:

(a) $X = R$, the real line,

(b) $X = [-1;1]$, the closed unit interval, and

(c) $X = \mathbf{2}$, the two-point space,

have been obtained in [9]. Let us mention that M_R is the linear metric space denoted in Banach's book [6] by S; the space $M_{[-1;1]}$ is isometric to the unit ball of the space L_∞ (of bounded measurable functions) regarded in the L_2 metric, and $M_\mathbf{2}$ can be identified with the measure algebra of all Lebesgue measurable subsets of $[0;1]$. Our theorem is also a strengthening of a result of [13] stating that $M_\mathbf{2}$ is universal for separable metric spaces.

If G is a metric group, then M_G is also a group under point-wise multiplication; the elements of G can be identified with constant functions in M_G. Hence if G is non-trivial and separable, then it can be isomorphically embedded into the group $M_{\overline{G}}$ ($\overline{G}$ = the completion of G), which is homeomorphic to l_2. In particular every separable metric group admits a free transformation group action in l_2. These results answer a question of E. Michael [19].

The proof of Theorem 4.1 combines the technique of R. D. Anderson's Z-sets (cf. [1]–[4] and [7]–[9]) with convexity arguments. The latter appear in § 2 — a generalization of Keller's [15] theorem on homeomorphism of compact convex sets, and related criteria of recognizing spaces hemeomorphic to l_2, and in § 4 — proof of Lemma 4.3 which concerns the weak approximation of measurable functions from $[0;1]$ to a simplex by functions with values in the vertices of the simplex (cf. a similar result of Dvoretzky, Wald and Wolfowitz [12]). We also employ a theorem on the existence of closed linearly independent homeomorphic embeddings of metric spaces into the spheres of Hilbert spaces (cf. Arens and Eells [5]) which is established in § 3.

§ 1. Notation and preliminaries. By N we shall denote the set of positive integers; by R, the real line, $I=[-1;1]$, the closed interval; $R^+=\{t\in R\colon t\geqslant 0\}$; $\emptyset$ denotes the empty set, while 0 stands for the number zero and the zero vector of any linear space. If $a_\lambda\in R^+$ for $\lambda\in\Lambda$, then $\sum\limits_{\lambda\in\Lambda}a=\sup\sum\limits_{\lambda\in S}a$, the supremum taken over all finite subsets $S\subset\Lambda$.

For any topological space X, we denote by X^N the Cartesian product of $\aleph_0$ copies of X labelled by positive integers, i.e. the elements of X^N are sequences $x=\big(x(n)\big)$ and the topology of X^N is the product one.

The symbols $\cup,\cap,\setminus$ denote the set-theoretical operations, and $+,-,\cdot$ are symbols of algebraic operations on numbers, vectors of a linear space E and on sets of numbers and vectors. For instance: $A-x=\{a-x\colon a\in A\}$, $R^+\cdot A=\{t\cdot a\colon t\in R^+,\ a\in A\}$, etc.

The symbol $\simeq$ denotes the relation of being homeomorphic for topological spaces and also for pairs consisting of topological spaces and their subsets, i.e. $(X,A)\simeq(Y,B)$ if and only if there is a homeomorphism f of X onto Y which caries A onto B.

The closure, the interior and the boundary of a set A in a topological space are denoted by $\mathrm{cl}\,A$, $\mathrm{Int}A$ and ∂A.

Let X be a metric space. A function $f\colon[0;1]\to X$ is said to be measurable if $f^{-1}(U)$ is a Borel subset of $[0;1]$ for every open set $U\subset X$. Measurable functions $f,\ g\colon[0;1]\to X$ are said to be equivalent if $|\{t\in[0;1]\colon f(t)\neq g(t)\}|=0$. Here $|A|$ denotes the Lebesgue measure of the set $A\subset[0;1]$. By M_X we denote the topological space of equivalence classes of measurable functions $f\colon[0;1]\to X$. The topology of M_X is that of convergence in measure, otherwords it is defined by the metric

$$\varrho(f,g)=\Big(\int_0^1\big(d\big(f(t),g(t)\big)\big)^2dt\Big)^{1/2},\tag{1}$$

where d is any bounded metric for X. If we replace d by an equivalent metric, say d', then the corresponding metric ϱ' for M_X is equivalent to the metric ϱ. For this observe that if (f_n) is a sequence in M_X, then $\lim\limits_n\varrho(f_n,f_1)=0$ if and only if the sequence of real-valued functions $\varphi_n(\cdot)=d\big(f_n(\cdot),f_1(\cdot)\big)$ tends to zero in measure or, equivalently, every subsequence of the sequence (φ_n) contains a subsequence which converges to zero almost everywhere (cf. Halmos [14], § 22). The last property is obviously independent on the particular choice of the metric for X.

1.1. PROPOSITION. *The space M_X is complete-metrizable and separable if and only if X is so.*

Proof. The "only if" statement follows from the fact that the set of constant functions is closed in M_X and isometric to X. Conversely, if d_0 is any complete metric fo X, then $d=d_0/(1+d_0)$ is a bounded complete metric for X and the metric (1) for M_X is complete. Finally, if $\{x_n\colon n\in N\}$ is a dense set in X, then the set of all linear combinations $\sum\limits_i c_i\chi_{A_i}$, where χ_{A_i} are characteristic functions of Borel subsets of $[0;1]$, are dense in M_X.

All linear spaces appearing in this paper are over the field R. If A is a subset of a linear space E, then span A denotes the linear subspace of E generated by A and $\mathrm{conv}A$ denotes the convex hull of A.

For every normed linear space $E=(E,\|\cdot\|)$, we denote $B_E=\{x\in E\colon\|x\|\leqslant 1\}$ and $S_E=\{x\in E\colon\|x\|=1\}$ the closed unit ball and the unit sphere of E.

By a pre-Hilbert space we mean a normed linear space E whose norm is induced by a scalar product: $\|x\|=\sqrt{\langle x,x\rangle}$. The completion of a pre-Hilbert space is a Hilbert space.

We shall use the following special Hilbert spaces.

The Euclidean n-space R^n; $\langle x,y\rangle=\sum\limits_{i=1}^n x(i)\cdot y(i)$ for $x,y\in R^n$.

The space $l_2(\Lambda)$ consisting of all real functions $x=\big(x(\lambda)\big)$ defined on Λ such that $\|x\|=\big(\sum\limits_{\lambda\in\Lambda}|x(\lambda)|^2\big)^{1/2}<\infty$, $l_2=l_2(N)$.

$L_2[l_2]$ the Hilbert space of equivalence classes of measurable functions $\boldsymbol{f}=f(\cdot)\colon[0;1]\to l_2$ such that

$$|\!|\!|\boldsymbol{f}|\!|\!|=\Big(\int_0^1\|f(t)\|^2dt\Big)^{1/2}<\infty;\tag{2}$$

the scalar product in $L_2[l_2]$ is defined by $\langle\boldsymbol{f},\boldsymbol{g}\rangle=\int_0^1\langle f(t),g(t)\rangle\,dt$.

The space $L_2[R^n]$ of measurable functions $\boldsymbol{f}\colon[0;1]\to R^n$ satisfying the condition (2).

The weak topology of a Hilbert space H is determined by the basis of open sets: $U(y;f_1,\ldots,f_k)=\{x\in H\colon|\langle x-y,f_i\rangle|<1$ for $i\leqslant k\}$, for all possible finite systems of vectors $y,f_1,\ldots,f_k\in H$.

For any subset A of a Hilbert space H, we shall use the same symbol A when regarding A as topological space equipped with the norm-topology; and we shall write $A^{\sim}$ for denoting the set A equipped with the topology induced by the weak topology of H.

In the next two propositions we recall some well-known properties of weak topologies in Hilbert spaces. These properties will be used in §§ 2, 3, 4.

1.2. PROPOSITION. *If A is a bounded subset of a separable Hilbert space H, then $A^{\sim}$ is metrizable and pre-compact. The metric for $A^{\sim}$ can be given by the formula*

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n}\langle x-y, f_n\rangle,$$

where $\{f_n: n \in N\}$ is an arbitrary countable dense subset of S_H.

For the proof see Dunford–Schwartz [11], Chapt. V, sect. 4 and 5.

1.3. PROPOSITION. *Let H be an arbitrary Hilbert space and $(f_\lambda)_{\lambda\in\Lambda}$ an orthonormal basis for H. Then the norm topology of the unit sphere S_H coincides with the topology determined by the basis of neighbourhoods:*

$$V(y, n, \lambda_1, \ldots, \lambda_k) = \{x \in S_H: |\langle x-y, f_{\lambda_i}\rangle| < 1/n \text{ for } i \leqslant k\},$$

$y \in H$, $\lambda_1, \ldots, \lambda_k \in \Lambda$, $k, n \in N$, and coincides with the weak topology of S_H.

Proof. Given $y \in S_H$ and $\varepsilon > 0$. We shall show that there are $\lambda_1, \ldots, \lambda_k \in \Lambda$ and $n \in N$ such that

$$V(y, n, \lambda_1, \ldots, \lambda_k) \subset \{x \in S_H: \|x-y\| < \varepsilon\}. \tag{3}$$

To this end let $A = \{\lambda_1, \ldots, \lambda_k\}$ be a finite subset of Λ such that $\sum_{\lambda\notin A} |\langle y, f_\lambda\rangle|^2 = 1 - \sum_{\lambda\in A} |\langle y, f_\lambda\rangle|^2 < \varepsilon^2/4$. Pick an $n \in N$ with $n > \varepsilon^2/8$. For any $x \in V(n, y, \lambda_1, \ldots, \lambda_k)$, we obviously have

$$|\langle x-y, f_\lambda\rangle| < 1/n \text{ and } \big||\langle x, f_\lambda\rangle|^2 - |\langle y, f_\lambda\rangle|^2\big| < 2/n \quad \text{for } \lambda \in A,$$

whence

$$\sum_{\lambda\in A} |\langle x-y, f_\lambda\rangle|^2 < \varepsilon^2/8; \qquad \sum_{\lambda\in A} |\langle y, f_\lambda\rangle|^2 - \sum_{\lambda\in A} |\langle x, f_\lambda\rangle|^2 < \varepsilon^2/4.$$

Therefore

$$\begin{aligned}\|x-y\|^2 &= \sum_{\lambda\in A} |\langle x-y, f_\lambda\rangle|^2 < \varepsilon^2/4 + \sum_{\lambda\notin A} |\langle x-y, f_\lambda\rangle|^2 \\ &\leqslant \varepsilon^2/4 + 2\sum_{\lambda\notin A} |\langle y, f_\lambda\rangle|^2 + \sum_{\lambda\notin A} |\langle x, f_\lambda\rangle|^2 - \sum_{\lambda\in A} |\langle y, f_\lambda\rangle|^2 \\ &< 3\varepsilon^2/4 + 1 - \sum_{\lambda\in A} |\langle x, f_\lambda\rangle|^2 - \Big(1 - \sum_{\lambda\in A} |\langle y, f_\lambda\rangle|^2\Big) \\ &\leqslant 3\varepsilon^2/4 + \Big|\sum_{\lambda\in A} |\langle y, f_\lambda\rangle|^2 - \sum_{\lambda\in A} |\langle x, f_\lambda\rangle|^2\Big| \leqslant \varepsilon^2.\end{aligned}$$

This establishes (3). The remaining part of the proof is trivial.

Let $E = (E, \|\cdot\|)$ be a normed linear space, $A \subset E$, $x \in E$. We denote $d(x, A) = \inf\{\|x-y\|: y \in A\}$.

We shall need the following version of Dugundji's [10] result.

1.4. THEOREM OF DUGUDJI. *If A is a convex subset of a normed linear space E, then there is a retraction r: $E \to A$ such that $\|r(x)-x\| \leqslant 2\cdot d(x, A)$ for all $x \in X$.*

Finally we shall prsent some facts concerning Z-sets in the sense of R. D. Anderson [2].

Let $Q = I^N$ be the Hilbert cube. Suppose that $X = (X, d)$ is a metric space homeomorphic to Q. By a Z-set in X we shall mean any closed subset A of X satisfying the following condition:

(*) *For every $\varepsilon > 0$, every integer $m \geqslant 0$ and every continuous map f: $I^m \to Q$, there exists a continuous map g: $I^m \to Q\setminus A$ such that $d\big(f(x), g(x)\big) < \varepsilon$ for $x \in X$.*

The class of all Z-sets in X will be denoted by $\mathscr{Z}(X)$. It is easy to see that the class of Z-sets is invariant under homemorphisms, i.e. if h is a homeomorphism of X onto Y, then $\{h(A): A \in \mathscr{Z}(X)\} = \mathscr{Z}(Y)$. In particular the property of being a Z-set does not depend on the choice of the metric on X, provided the metric induces the same topology.

The results, wee need, can be summarized in the following proposition.

1.5. PROPOSITION. *Let $X = (X, d)$ be a metric space homeomorphic to Q. Then the following statements hold:*

(i) *If $A \in \mathscr{Z}(X)$ and $A \simeq X$, then $(X, A) \simeq (Q, Q_{\text{odd}})$, where $Q_{\text{odd}} = \{x \in Q: x(2n) = 0$ for $n = 1, 2, \ldots\}$.*

(ii) *If $A_1 \subset A_2 \subset A_3 \subset \ldots \subset X$ are such that:* (a) *$(A_{n+1}, A_n) \simeq (Q, Q_{\text{odd}})$ and* (b) *for every $\varepsilon > 0$, $m \in N$ and $A \in \mathscr{Z}(X)$, there is a continuous* map *f: $A \to A_n$ such that $f(x) = x$ for $x \in A \cap A_m$ and $d\big(f(x), x\big) < \varepsilon$ for all x in A, then $(X, \bigcup_{n\in N} A_n) \simeq (Q, P)$, where $P = (-1; 1)^N$, the pseudo-interior of the Hilbert cube.*

(iii) *If $(X, B) \simeq (Q, Q\setminus P)$, $B \subset A \subset X$ and A is a countable union of Z-sets, then $(X, A) \simeq (Q, Q\setminus P)$ and $X\setminus A \simeq l_2$.*

Proof. The assertion (i) follows from Anderson's extension theorem ([2], Corollary 10.3) which states that homeomorphisms between Z-sets in Q can be extended to autohomeomorphisms of Q, whence any homeomorphism from A onto Q_{odd} admits an extension which is a homeomorphism from X onto Q.

The conditions (a) and (b) are evidently equivalent to the conditions (4.11) and (4.12) of [8]. (To derive (4.12) from (b) we use [8], Proposition 1.1). Therefore the sequence (A_n) appearing in (ii) is a $\mathscr{Z}(X)$-skeleton. Hence, by [8], Propositions 4.3, 6.2 and 6.5, $(X, \bigcup_{n\in N} A_n) \simeq (Q, Q\setminus P)$.

The general statement (iii) immediately reduces to the case where $X = Q$. In this case the homeomorphism $(X, A) \simeq (Q, Q\setminus P)$ have been asserted by Anderson [3], whence, by [1], $l_2 \simeq P \simeq X\setminus A$.

§ 2. Keller spaces and their central points. The main lemma. A convex subset K of a linear topological space E is said to be a Keller space if there exists a homeomorphic embedding $f\colon K \to l_2$ which is affine (i.e. $f(tx+(1-t)y) = tf(x)+(1-t)f(y)$ for $x, y \in K$, $0 \leqslant t \leqslant 1$) and such that $f(K)$ is infinite-dimensional and compact. Keller spaces will be usually regarded with the affine structure (= the operation of forming convex linear combinations) and the topological one.

2.1. EXAMPLE. The Hilbert cube $Q = [-1;1]^N$ is a Keller space; an embedding $f\colon Q \to l_2$ can be defined by $f(x) = \sum_{n=1}^{\infty} n^{-1}x(n)\cdot v_n$, where $v_i = (0, \ldots, 1, 0, \ldots)$ is the ith unit vector in l_2.

2.2. EXAMPLE. Let K be an infinite-dimensional, weakly closed and bounded convex subset of a separable Hilbert space H, and let $K^{\sim}$ be the set K regarded as topological space under the topology induced by the weak topology of H. Then $K^{\sim}$ is a Keller space.

In fact, since H is separable, there is a countable set $\{f_n\colon n \in N\}$ of linear functionals separating points of K and such that $\|f_n\| \leqslant 1/n$ for $n = 1, 2, \ldots$ Thus the required homeomorphic embedding can be given by $K^{\sim} \ni x \xrightarrow{f} (f_n(x)) \in l_2$. Notice that the continuity of $f^{-1}\colon f(K^{\sim}) \to K^{\sim}$ follows from the fact that K, being a bounded and weakly closed subset of a Hilbert space, is compact in the weak topology, see Proposition 1.2.

2.3. THEOREM OF KELLER. *Every Keller space is homeomorphic to the Hilbert cube Q.*

For the proof see [15].

Let K be a Keller space. A subset A of K is said to be a T-set, provided that there exists a sequence of continuous maps $g_n\colon K \to K$ such that $g_n(K) \subset A$ for all n, and $\lim_n g_n(x) = x$ uniformly in K. The class of all T-sets in K will be denoted by $\mathscr{T}(K)$. Evidently:

(*) *If L is a closed subset of K and $K\setminus L \in \mathscr{T}(K)$, then $L \in \mathscr{Z}(K)$.* (In fact also the converse implication is true).

Let $y \in K$. Denote

$$\operatorname{aur}_y K = \bigcup_{x\in K} \{y+t\cdot(y-x)\colon 0 \leqslant t < 1\}.$$

The point y is said to be central if $K\setminus \operatorname{aur}_y K \in \mathscr{T}(K)$. The set of all central points will be denoted by $\operatorname{cent} K$.

2.4. GENERALIZED KELLER THEOREM. *If X and Y are Keller spaces, $x_0 \in \operatorname{cent} X$ and $y_0 \in \operatorname{cent} Y$, then $(X, \operatorname{aur}_{x_0} X) \simeq (Y, \operatorname{aur}_{y_0} Y) \simeq (Q, Q\setminus P)$.*

Proof. Consider the sequence of the homotheths

$$X_n = x_0+(1-2^{-n})\cdot(X-x_0).$$

Obviously, $X_n \simeq X$ and, by Theorem 2.3,

(1) $$X_n \simeq Q \quad \text{for } n = 1, 2, \ldots$$

Given $n \in N$. Since $x_0 \in \operatorname{cent} X$, we conclude that $x_0 \in \operatorname{cent} X_{n+1}$, i.e. $X_{n+1}\setminus \operatorname{aur}_{x_0} X_{n+1} \in \mathscr{T}(X_{n+1})$. But $X_n \subset \operatorname{aur}_{x_0} X_{n+1}$ and therefore $X_{n+1}\setminus X_n \in \mathscr{T}(X_{n+1})$. Hence, by (*), $X_n \in \mathscr{Z}(X_{n+1})$. Now, by (1) and by Proposition 1.5, (i), we obtain

(2) $$(X_{n+1}, X_n) \simeq (Q, Q_{\text{odd}}).$$

Assume that $A \in \mathscr{Z}(X)$, $m \in N$ and $\varepsilon > 0$. Regard X as a convex subset of l_2 and let $d(x, z) = \|x-z\|$, the metric on X induced by the norm of l_2. By Theorem 1.2, there is an $n > m$ and a retraction r of X onto X_n such that $d(x, r(x)) < \epsilon$ for all $x \in X$. Let $f = r_{|A}\colon A \to X$. The map f has the properties

(3) $f(A) \subset X_n$, $\quad f(x) = x \quad$ for $x \in A \cap X_m$, $d(f(x), x) < \varepsilon$ for $x \in A$.

By (2), (3) and Proposition 1.5, (ii), we have $(X, \bigcup_{n\in N} X_n) \simeq (Q, Q\setminus P)$. Since obviously $\bigcup_{n\in N} X_n = \operatorname{aur}_{x_0} X$, we obtain the assertion of Theorem 2.4.

2.5. THEOREM. *Every Keller space K admits a point y such that $y \in \operatorname{cent} K\setminus \operatorname{Ext} K$.*

In the statement above $\operatorname{Ext} K$ denotes the set of all extreme points of K. Recall that $z \in K$ is extreme if $u = 0$ is the only vector u such that $z+u \in K$ and $z-u \in K$.

Proof. Assume that K is an infinite dimensional compact convex subset of l_2; $\|\cdot\|$ denotes the standard norm in l_2 and $\langle\cdot,\cdot\rangle$ the scalar product. For every $z \in l_2$ define

$$\operatorname{diam}_z K = \sup\{\|x-u\|\colon x, u \in K, \ z \in R\cdot(x-u)\}.$$

Let $\operatorname{reg} K = \{x \in K\colon \inf_{z\in K} \operatorname{diam}_{z-x} K = 0\}$.

The following proposition is an obvious strengthening of the theorem.

2.6. PROPOSITION. $\operatorname{reg} K\setminus \operatorname{Ext} K \neq \emptyset$ *and* $\operatorname{reg} K \subset \operatorname{cent} K$.

Proof. Pick $x_n \in K$, $n = 1, 2, \ldots$ so that

(4) $$\operatorname{cl}\{x_n\colon n \in N\} = K.$$

Let

$$y = \sum_{n=1}^{\infty} 2^{-n} x_n, \quad z_n = (x_1-x_n)/2^n \quad \text{for } n = 1, 2, \ldots$$

Since the set K is closed and convex, we have $\sum_{n=1}^{\infty} t_n x_n \in K$, whenever $t_i \geqslant 0$, $\sum_{i=1}^{\infty} t_i = 1$. Hence in particular $y \in K$, $y + z_n \in K$ and $y - z_n \in K$ for $n = 1, 2, \ldots$ Therefore

$$(6) \qquad y \in K \setminus \operatorname{Ext} K, \quad \{z_1, -z_1, z_2, -z_2, \ldots\} \subset (K-y) \cap (K+y).$$

The set K, being a Keller space, is infinite-dimensional. Hence, by (4), among the vectors z_n there are infinitely many linearly independent. Let (z_n') be a linearly independent subsequence of the sequence (z_n), and let

$$y_n = \sum_{i=1}^{\infty} c_{ni} z_i', \qquad n = 1, 2, \ldots$$

be the orthonormal vectors obtained from the sequence (z_i') by the Schmidt orthogonalization procedure. Finally let $c_n = \sum_{i=1}^{n} c_{ni}$. Obviously, by (6),

$$c_n y_n \in \operatorname{conv}\{z_1, -z_1, z_2, -z_2, \ldots\} \subset \operatorname{conv}[(K-y) \cap (K+y)] = (K-y) \cap (K+y).$$

Hence

$$(7) \qquad y + c_n y_n \in K \quad \text{for} \quad n = 1, 2, \ldots$$

By the Bessel inequality: $\sum_{n=1}^{\infty} |\langle x, y_n\rangle|^2 \leqslant \|x\|$, we have

$$(8) \qquad \lim_n \langle x, y_n\rangle = 0 \quad \text{for every } x \in l_2.$$

Let $A_n \subset K$ be a finite set which is an $\frac{1}{n}$-net for K. Then

$$\operatorname{diam}_{c_n y_n} K \leqslant \sup_{x \in K} \langle x, y_n\rangle - \inf_{x\in K} \langle x, y_n\rangle \leqslant 2 \cdot \sup_{x \in K} |\langle x, y_n\rangle| \leqslant 2 \cdot \sup_{x \in A_n} |\langle x, y_n\rangle| + 2 \cdot \frac{1}{n}.$$

Hence, by (8) $\lim_n \operatorname{diam}_{c_n y_n} K = 0$. This, together with (6) and (7), gives $y \in \operatorname{reg} K \setminus \operatorname{Ext} K$, the first statement of Proposition 2.6.

The proof of the second statement of the proposition is based on the following lemma.

2.7. Lemma. *Suppose that B is a compact convex body in a finite-dimensional Euclidean space E, $0 \in B$, L is a subspace of E such that $L \cap \operatorname{Int} B \neq \emptyset$, z is a point of $\operatorname{Int} B \setminus L$, and finally A is a compact subset of $L \cap \operatorname{Int} B$. For each $x \in A$, let $f(x)$ denote the (unique) point of intersection of the ray $x + R^+ \cdot z$ with the boundary ∂B. Then the map $f\colon A \to B$ is continuous and $f(A) \subset B \setminus \operatorname{aur}_0 B$.*

Proof. Let $h\colon E \to L$ be the parallel projection in the direction of the vector z. Obviously h is continuous and $f^{-1}(y) = h(y)$ for $y \in f(A)$. Hence $f^{-1}\colon f(A) \to A$ is continuous. We have $f(A) = \partial B \cap h^{-1}(A)$, and therefore $f(A)$ is compact. Thus we infer that f is continuous.

Finally, since $z \in \operatorname{Int} B \setminus L$, we conclude that, for every $x \in A \setminus \{0\}$, the interior of the tetragon with the vertices $0, x, f(x), z$ lies entirely in $\operatorname{Int} B$. Hence, for every $x \in A$, we have $\partial B \cap R^+ \cdot f(x) = \{f(x)\}$, i.e. $f(x) \in \operatorname{aur}_0 B$.

Let us return to the proof of Proposition 2.6. Given a point $y \in \operatorname{reg} K$. We have to show that $K \setminus \operatorname{aur}_y K \in \mathscr{T}(K)$. Since the notions involved are invariant under translations, we may without loss of generality assume that $y = 0$. Let $\{x_1, \ldots, x_n\} \subset K$ be an $\varepsilon/8$-net for K. Since $0 \in K$, we have $\inf_{x \in K} \operatorname{diam}_x K = 0$. Using the fact that $K_1 = K \cap \operatorname{span}\{x_1, \ldots, x_n\}$ is a finite-dimensional compact convex set, we conclude that

$$\inf\{\operatorname{diam}_x K\colon x \in K_1\} \geqslant \inf\{\operatorname{diam}_x K_1\colon x \in K_1\} > 0.$$

Therefore there is a point $x_0 \in K \setminus \operatorname{span}\{x_1, \ldots, x_n\}$ such that $\operatorname{diam}_{x_0} K < \varepsilon/8$. We let

$$E = \operatorname{span}\{x_0, \ldots, x_n\}, \quad L = \operatorname{span}\{x_1, \ldots, x_n\}, \quad B = K \cap E.$$

Obviously $\operatorname{diam}_{x_0} B \leqslant \operatorname{diam}_{x_0} K < \varepsilon/8$. Therefore there exists a point $z \in \operatorname{Int} B \cap L$ (the interior taken with respect to E) so close to x_0 that

$$(9) \qquad \operatorname{diam}_z B < \varepsilon/4.$$

Let $A \subset L \cap \operatorname{Int} B$ be a compact convex set such that

$$\sup\{d(x, A)\colon x \in L \cap \operatorname{Int} B\} < \varepsilon/4.$$

Since $L \cap \operatorname{Int} B$ contains the $\varepsilon/8$-net $\{x_1, \ldots, x_n\}$, we have

$$\sup_{x \in K} d(x, A) < \varepsilon/2.$$

Therefore, by the Theorem of Dugundji 1.4, there is a retraction $r\colon K \to A$ such that $\|r(x) - x\| < \varepsilon$. Define $g = f \circ r\colon K \to K$, where g is the map of Lemma 2.6. Obviously $g(K) \subset K \setminus \operatorname{aur}_0 K$ and, by (9), $d(g(x), x) < \varepsilon + \varepsilon/4$ for all $x \in K$.

Since the map g with the above properties can be constructed for every $\varepsilon > 0$, we conclude that $K \setminus \operatorname{aur}_0 K \in \mathscr{T}(K)$. This completes the proof of Proposition 2.6 and the proof of Theorem 2.5.

2.8. Corollary. *If K is a Keller space, A is a subset of K of type G_δ such that $A \in \mathscr{T}(K)$ and $A \subset \operatorname{Ext} K$, then $A \simeq l_2$.*

Proof. Pick a point $x_0 \in \operatorname{cent} K \setminus \operatorname{Ext} K$. It is clear that the points of $\operatorname{aur}_{x_0} K$ cannot be extreme for K. Therefore $K \setminus A \supset \operatorname{aur}_{x_0} K$. Since $A \in T(K)$ and A is of type G_δ, we conlude that $K \setminus A$ is a countable union of Z-sets (cf. property (*)). Applying Theorem 4.2 and Proposition 1.5, (iii), we obtain $A \simeq l_2$.

2.9. THE MAIN LEMMA. *Suppose that H is a separable Hilbert space, D is an infinite-dimensional convex and weakly closed subset of the unit ball B_H, M is a closed subset of $D \cap S_H$ such that $M^{\sim} \in \mathcal{T}(D^{\sim})$. Then $M \simeq l_2$.*

Proof. Since $M \subset S_H$, we obtain, by Proposition 1.3,

(10) $$M^{\sim} \simeq M.$$

Thus M is complete-metrizable, and by the Lavrentiev–Sierpiński theorem ([16], § 31, III), M is a G_δ-subset of D. Since $M \subset S_H$, we also conclude that $M^{\sim} \subset \operatorname{Ext} D^{\sim}$. The D is a Keller space (Example 2.2). Hence, by Corollary 2.8, $M^{\sim} \simeq l_2$ and, by (10), $M \simeq l_2$.

We shall conclude this section with some examples concerning central points in Keller spaces.

2.10. EXAMPLE. We have $\operatorname{cent} Q = \operatorname{reg} Q = Q$.

2.11. EXAMPLE. If $K = S_{l_2}^{\sim}$, then $\operatorname{cent} K = \operatorname{reg} K = K$.

2.12. EXAMPLE. Let $K = \{(x, t) \in Q \times [0; 1]: \sup_n |x(n)| \leqslant t\}$. Then $\operatorname{cent} K = \operatorname{reg} K = K \setminus \{(0, 0)\}$.

In [8] and [9] we have used the notion of a radial interior of a Keller space, which can be defined as follows

$$\operatorname{rint} K = \{y \in K: R^+ \cdot [(K-y) \cap (y-K)] \supset K-y\}.$$

It is easy to see that $\operatorname{rint} K \subset \operatorname{cent} K$ and $\operatorname{aur}_y K = \operatorname{rint} K$ for every $y \in \operatorname{rint} K$. Hence, by Theorem 2.4, $(K, \operatorname{rint} K) \simeq (Q, Q \setminus P)$, provided that $\operatorname{rint} K$ is non-empty. Let us mention here that there exist Keller spaces whose radial interior is empty.

2.13. EXAMPLE. Let $C(I)$ be the Banach space of continuous functions defined on $I = [-1; 1]$, and let K be the subset of the conjugate $(C(I))^*$ consisting of functionals represented as probabilistic measures on I, and let the topology of K be that induced by the weak-star topology of $(C(I))^*$. Then $\operatorname{rint} K = \emptyset$.

The proofs of the statements in the above examples are of a routine type; therefore we leave them to the reader.

§ 3. An embedding theorem.

3.1. THEOREM. *Every [complete] metric space X is homeomorphic to a closed linearly independent subset of the unit sphere of a [complete] pre-Hilbert space H such that*

(1) $$\dim H = \operatorname{wght} X.$$

By $\dim H$ we mean the cardinality of an orthogonal basis for H; $\operatorname{wght} X = \inf\{\operatorname{card} A: \operatorname{cl} A = X\}$, the topological weight of X.

Proof. First assume that

(2) X is a complete metric space.

Using the fact that metric spaces are paracompact, we construct, for each $n \in N$, a partion of unity $\{\varphi_\lambda\}_{\lambda \in \Lambda_n}$ such that

(3) $$\varphi_\lambda(x) \cdot \varphi_\lambda(y) = 0 \quad \text{if} \quad d(x, y) \geqslant 2^{-n}.$$

Let $g_\lambda(x) = 2^{-n}\varphi_\lambda(x)$ for $\lambda \in \Lambda_n$. Assume that the indexing sets Λ_n are pair-wise disjoint and let $\Lambda = \bigcup_{n \in N} \Lambda_n$.

We define $h: X \to l_2(\Lambda)$ by the formula

$$h(x) = (\sqrt{g_\lambda(x)})_{\lambda \in \Lambda}.$$

We have

$$\|h(x)\|^2 = \sum_{\lambda \in \Lambda} g_\lambda(x) = \sum_{n=1}^{\infty} 2^{-n} \sum_{\lambda \in \Lambda_n} \varphi_\lambda(x) = \sum_{n=1}^{\infty} 2^{-n} = 1,$$

i.e.

(4) $$h(x) \in S_{l_2(\Lambda)} \quad \text{for every } x \in X.$$

Furthermore the coordinates $\sqrt{g_\lambda}$ of the map h are continuous, and since on the unit sphere $S_{l_2(\Lambda)}$ the coordinate-wise convergence topology and the norm topology coincide (Proposition 1.3), we conclude that h is continuous.

Assume that $x, y \in X$, $x \neq y$ and, say, $2^{-n} \leqslant d(x, y) < 2^{-n+1}$. Then, by (3),

$$\sum_{\lambda \in \Lambda_n} |\sqrt{g_\lambda(x)} - \sqrt{g_\lambda(y)}|^2 = \sum_{\lambda \in \Lambda_n} (g_\lambda(x) + g_\lambda(y)) = 2^{-n} \sum_{\lambda \in \Lambda_n} (\varphi_\lambda(x) + \varphi_\lambda(y)) = 2^{-n+1},$$

whence $\|h(x) - h(y)\| \geqslant (2^{-n+1})^{1/2} \geqslant 2^{-n+1} \geqslant d(x, y)$. Thus

(5) h is a homeomorphism.

Moreover $h^{-1}: h(X) \to X$ satisfies the Lipschitz condition; therefore h^{-1} takes Cauchy sequences in $h(X)$ into Cauchy sequences in X. Hence, under the assumption (2), we have

(6) $h(X)$ is closed in $l_2(\Lambda)$.

Finally assume that $x_1, \ldots, x_n$ are distinct points of X. Pick $k \in N$ with $2^{-k} < \inf\{d(x_i, x_j): i \neq j\}$. By (3), for every $i \leqslant n$, there is a $\lambda \in \Lambda_k$ such that $g_\lambda(x_i) \neq 0$ and $g_\lambda(x_j) = 0$ for $j \neq i$, $j \leqslant n$. Thus

(7) $h(x_1), \ldots, h(x_n)$ are linearly independent.

Letting $H = \operatorname{cl}\operatorname{span} h(X)$, we obtain the statement (1). This completes the proof of the statement concerning complete metric spaces of Theorem 3.1.

Now suppose that X is not complete. Let $X^{\wedge}$ be the completion of X. By the statement already proved, there exists a homeomorphic embedding f of the space $X^{\wedge}$ into a Hilbert space H_0 such that $f(X^{\wedge})$ is a closed linearly independent subset of the unit sphere of H_0. We let $H = \operatorname{span} f(X)$. Obviously $\dim H = \operatorname{wght} X$. Since $f(X^{\wedge})$ is a linearly independent set, we have $f(X) = H \cap f(X^{\wedge})$. Therefore $f(X)$ is closed relative H.

§ 4. Spaces M_X homeomorphic to the Hilbert space l_2. This section is devoted to the proof of the following theorem.

4.1. THEOREM. *Let X be a metric space. Then $M_X \simeq l_2$ if and only if X is complete-metrizable, separable, and X has more than one point.*

Proof. The "only if" statement is a direct consequence of Proposition 1.1.

We shall establish the "if" statement. Denote by S, $\boldsymbol{S}$ and $\boldsymbol{B}$ the unit sphere of the Hilbert space l_2, the unit sphere of the Hilbert space $L_2[l_2]$ and the closed unit ball of the space $L_2[l_2]$. Suppose that X is a separable complete metric space with $\operatorname{card} X \geqslant 2$. By Theorem 2.1, we may assume that

(1) X is a closed linearly independent subset of S.

Hence, for each $f = f(\cdot) \in M_X$, we have

$$\Big(\int_0^1 \|f(t)\|^2 dt\Big)^{1/2} = \Big(\int_0^1 1\, dt\Big)^{1/2} = 1,$$

i.e. f may be regarded as an element of the sphere $\boldsymbol{S}$. Moreover, by the definition of the metric ϱ on M_X, the distance $\varrho(f, g)$ is the same as the distance between f and g measured by the norm of the space $L_2[l_2]$. Hence, we may and shall assume that

$$(2) \qquad M_X \subset \boldsymbol{S}.$$

For any set $A \subset L_2[l_2]$, we shall denote by $A^{\sim}$ the same set equipped with the topology induced by the weak topology of $L_2(l_2]$. Let

$$D_X^{\sim} = \operatorname{cl}\operatorname{conv} M_X^{\sim},$$

the closure in the weak topology of the space $L_2[l_2]$.

According to Proposition 1.2, the proof of Theorem 4.1 is reduced to the following result.

4.2. PROPOSITION. *With the notation above we have $M_X^{\sim} \in \mathscr{T}(D_X^{\sim})$.*

We shall begin with the particular case, where X is a finite set.

4.3. LEMMA. *Let $X_n = \{x_1, \ldots, x_n\}$ be a linearly independent finite subset of S. Then $M_{X_n}^{\sim} \in T(D_{X_n}^{\sim})$.*

Proof. Observe that $D_{X_n} = \{f \in L_2[l_2]\colon f([0;1]) \subset \operatorname{conv} X_n\}$. We shall identify the Euclidean n-space R^n with the subspace $\{y \in l_2\colon y(i) = 0$ for $i > n\}$ of l_2 and we shall regard $L_2[R^n]$ as the corresponding subspace of the Hilbert space $L_2[l_2]$. By taking a suitable orthogonal basis in l_2 and regarding it as the unit vector basis, we may assume that

$$X_n \subset R^n \quad \text{and} \quad M_{X_n}, D_{X_n} \subset L_2[R^n].$$

Since obviously, for any subset A of $L_2[R^n]$, the weak topology of A induced from the space $L_2[R^n]$ coincides with the weak topology induced from $L_2[l_2]$, we may assume that $M_{X_n}^{\sim}$ and $D_{X_n}^{\sim}$ are equipped with the weak topology of the space $L_2[R^n]$.

For any $k \in N$, denote by Y_k the $n\cdot 2^k$ — dimensional linear subspace of $L_2[R^n]$ consisting of the functions which are constant on each interval

$$\Delta(k, i) = \big[i\cdot 2^{-k};\ (i+1)\cdot 2^{-k}\big), \qquad i = 0, \ldots, 2^k - 1.$$

Since the set of all linear combinations of the characteristic functions $\chi_{\Delta(k,i)}$, $k \in N$, $i = 0, \ldots, 2^{k-1}$, is dense in L_2, we conclude that $\bigcup_{k\in N} Y_k$ is dense in $L_2[R^n]$, and therefore there exists a countable set $\{g_k\colon k \in N\}$ such that

$$(4) \qquad \operatorname{cl}\{g_k\colon k \in N\} = S_{L_2[R^n]}; \qquad g_k \in Y_k \supset S_{L_2[R^n]} \quad \text{for } h = 1, 2, \ldots$$

Since the set $\{g_k\colon k \in N\}$ is dense in the unit sphere of the space $L_2[R^n]$, the metric

$$(5) \qquad \bar d(f, g) = \sum_{k=1}^{\infty} 2^{-k} |\langle f - g, g_n\rangle|, \quad f, g \in D_{X_n}^{\sim}.$$

Let $P_k\colon L_2[R^n] \xrightarrow[\text{onto}]{} Y_k$ be the orthogonal projection, i.e.

$$(6) \qquad P_k(f) = \sum_{i=0}^{2^k - 1} 2^k \int_{\Delta(k,i)} f(t)\, dt \cdot \chi_{\Delta(k,i)} \quad \text{for } f \in L_2[R^n]$$

(the integral of R^n-valued functions is understood coordinatewise). Observe that

$$(7) \qquad P_k(D_{X_n}^{\sim}) \subset D_{X_n}^{\sim} \quad \text{and} \quad \langle P_k(f), g\rangle = \langle f, g\rangle \quad \text{for} \quad g \in Y_k.$$

Suppose that, for each $k \in N$, we have constructed a continuous map $F_k\colon D_{X_n}^{\sim} \cap Y_k^{\sim} \to M_{X_n}^{\sim}$ such that

$$(8) \qquad \langle F_k(f), g\rangle = \langle f, g\rangle \quad \text{for any} \quad f \in D_{X_n}^{\sim} \cap Y_k^{\sim},\ g \in Y.$$

Let $\varphi_k\colon D_{X_n}^{\sim} \to M_{X_n}^{\sim}$ be the map $\varphi_k(f) = F_k P_k(f)$ for $f \in D_{X_n}^{\sim}$. Since the projection operators with finite dimensional range are weakly continuous, we conclude that φ_k are continuous. By (4), (5), (7) and (8), we have

$$\bar d(\varphi_k(f), f) = \sum_{i=1}^{\infty} 2^{-i} |\langle \varphi_k(f) - f, g_i\rangle| = \sum_{i=k+1}^{\infty} 2^{-i} |\langle \varphi_k(f) - f, g_i\rangle| \leqslant 2^{1-k}$$

for $f \in D_{X_n}$, i.e. $\lim_k \varphi_k(f) = f$ uniformly on D_{X_n}. This gives $M_{X_n}^{\sim} \in \mathscr{T}(D_{X_n}^{\sim})$.

To complete the proof of the lemma we have to construct the maps F_k with the property (8). For every $y \in \operatorname{conv} X_n$, let $b_1(y), \ldots, b_n(y)$ denote the barycentric coordinates of the point y, i.e. $b_s(y) \geqslant 0$ for $s = 1, \ldots, n$, $\sum_{s=1}^{n} b_s(y) = 1$, $y = \sum_{s=1}^{n} b_s(y) \cdot y_s$. Since $x_1, \ldots, x_n$ are linearly independent, the barycentric coordinates are continuous affine functions on $\operatorname{conv} X_n$. Hence there is a $c > 0$ such that

$$\sum_{s=1}^{n} |b_s(y) - b_s(y')| \leqslant c \cdot \|y - y'\| \quad \text{for } y, y' \in \operatorname{conv} X_n. \tag{9}$$

Let us put $a_0(y) = 0$, $a_s(y) = \sum_{i \leqslant s} b_i(y)$ for $s = 1, \ldots, n$. Now for a given $k \in N$ and for

$$f = \sum_{i=0}^{2^k-1} y_i \cdot \chi_{\Delta(k,i)} \in D_{X_n}^{\sim} \cap Y_k^{\sim}, \tag{10}$$

we define

$$A_s(f) = \bigcup_{i=0}^{2^k-1} \left[(i + a_{s-1}(y)) \cdot 2^{-k};\ (i + a_s(y_i)) \cdot 2^{-k} \right] \tag{11}$$

and

$$F_k(f) = \sum_{s=1}^{n} x_s \cdot \chi_{A_s(f)}. \tag{12}$$

Clearly $F_k(f) \in M_{X_n}^{\sim}$. Furthermore, if $z \in R^n$,

$$g = z \cdot \chi_{\Delta(k,i)} \in Y_k, \tag{13}$$

and $a_s = (i + a_s(y_i)) \cdot 2^{-k}$ for $s = 1, \ldots, n$, then, by (10), (11) and (12), we have

$$\langle F_k(f), g \rangle = \int_{\Delta(k,1)} \langle F_k(f)(t), z \rangle\, dt = \sum_{s=1}^{n} \int_{a_{s-1}}^{a_s} \langle x_s, z \rangle\, dt$$
$$= \sum_{s=1}^{n} b_s(y_i) \cdot 2^{-k} \cdot \langle x_s, z \rangle = 2^{-k} \cdot \langle y_i, z \rangle = \int_{\Delta(k,i)} \langle f(t), z \rangle\, dt = \langle f, g \rangle.$$

Since the space Y_k is linearly generated by functions of the form (13), we obtain (8).

It remains to check the continuity of F_k. Suppose that

$$f, f' \in D_{X_n}^{\sim} \cap Y_k^{\sim},\ f = \sum_{i=1}^{2^k-1} y_i \cdot \chi_{\Delta(k,i)}, \qquad f' = \sum_{i=1}^{2^k-1} y_i' \cdot \chi_{\Delta(k,i)}.$$

Then

$$\|f - f'\| = \Big(\int_0^1 \|f(t) - f'(t)\|\, dt \Big)^{1/2} = 2^{-k/2} \Big(\sum_{i=0}^{2^k-1} \|y_i - y_i'\|^2 \Big)^{1/2}.$$

Remembering that $\|x_s\| = 1$ for $s = 1, \ldots, n$ and employing the Schwartz inequality, estimation (9) and again the Schwartz inequality, we obtain

$$\|F_k(f) - F_k(f')\|^2 = \sum_{s=1}^{n} \|x_s \cdot (\chi_{A_s(f)} - \chi_{A_s(f')})\|^2$$
$$= \Big[\sum_{s=1}^{n} \Big(\int_0^1 |(\chi_{A_s(f)} - \chi_{A_s(f')})(t)|^2 dt \Big)^{1/2} \Big]^2$$
$$= \Big[\sum_{s=1}^{n} \Big(\int_0^1 |(\chi_{A_s(f)} - \chi_{A_s(f')})(t)|\, dt \Big)^{1/2} \Big]^2$$
$$\leqslant n \cdot \sum_{s=1}^{n} \int_0^1 |\chi_{A_s(f)}(t) - \chi_{A_s(f')}(t)|\, dt$$
$$= n \cdot \sum_{i=0}^{2^k-1} 2^{-k} \sum_{s=1}^{n} |b_s(y_i) - b_s(y_i')| \leqslant c \cdot n \sum_{i=0}^{2^k-1} 2^{-k} \cdot \|y_i - y_i'\|$$
$$\leqslant c \cdot n \cdot 2^{-k/2} \cdot \Big(\sum_{i=0}^{2^k-1} \|y_i - y_i'\|^2 \Big)^{1/2}.$$

Thus

$$\|F_k(f) - F_k(f')\| \leqslant (c \cdot n \cdot \|f - f'\|)^{1/2} \quad \text{for} \quad f, f' \in D_{X_n} \cap Y_k.$$

Therefore F_k is continuous in the norm topology. Using the facts that Y_k is a finite dimensional space, M_{X_n} is a subset of the unit sphere of the Hilbert space $L_2[R^n]$, we conclude that $D_{X_n} \cap Y_k \simeq D_{X_n}^{\sim} \cap Y_k^{\sim}$, $M_{X_n} \simeq M_{X_n}^{\sim}$. Hence $F_k\colon D_{X_n}^{\sim} \cap Y_k^{\sim} \to M_{X_n}^{\sim}$ is continuous. This completes the proof of Lemma 4.3.

Let us return to Proposition 4.2. Recall that X is a linearly independent subset of the unit sphere S of the space l_2, M_X is the subset of $L_2[l_2]$ consisting of X-valued functions, D_X is the weak closure of $\operatorname{conv} M_X$.

Let $\{x_i\colon i \in N\}$ be a countable dense subset of X, $X_n = \{x_1, \ldots, x_n\}$. Obviously

$$M_{X_n} \subset M_X \text{ for each } n \in N, \quad \text{and} \quad D_\infty = \bigcup_{n \in N} D_{X_n} \subset D_X.$$

We shall first prove the following lemma.

4.4. Lemma. *D_∞ is dense in D_X and therefore $D_\infty^{\sim}$ is dense in $D_X^{\sim}$.*

Proof. By the definition, the set $M_X^{\sim}$ is dense in $D_X^{\sim}$. Hence, by the theorem of Mazur ([11], V. 3. Theorem 10), $\operatorname{conv} M_X$ is norm-dense in D_X. Thus it is enough to show that D_∞ is dense in $\operatorname{conv} M_X$. To this end take a countable dense subset $\{y_j\colon j \in N\}$ of the set $\bigcup_{n \in N} \operatorname{conv}\{x_1, \ldots, x_n\}$. For a given $f \in D_X$ and $\varepsilon > 0$, define

$$A_1(f, \varepsilon) = f^{-1}\big(K(y_1, \varepsilon/2)\big)$$

and, for $j > 1$,

$$A_j(f, \varepsilon) = f^{-1}\big(K(y_j, \varepsilon/2)\big) \setminus \bigcup_{i=1}^{j} f^{-1}\big(K(y_i, \varepsilon/2)\big),$$

$$B_j(f, \varepsilon) = [0;1] \setminus \bigcup_{i=1}^{j} A_i(f, \varepsilon),$$

where $K(y, a)$ is the open ball in l_2 of radius a centred at y. Clearly $A_i(f, \varepsilon) \cap A_j(f, \varepsilon) = \emptyset$ for $i \neq j$, and $\lim_j |B_j(f, \varepsilon)| = 0$. Thus there is an $n = n(f, \varepsilon)$ such that $|B_n(f, \varepsilon)| < \varepsilon^2/8$. Let us put

$$g = \sum_{j=1}^{n-1} \chi_{A_j(f,\varepsilon)} \cdot y_j + \chi_{B_n(f,\varepsilon)} \cdot y_n.$$

Clearly $g \in D_\infty$ and

$$\|f-g\| = \Big(\sum_{j=1}^{n-1} \int_{A_j(f,\varepsilon)} \|f(t)-y_j\|^2 dt + \int_{B_n(f,\varepsilon)} \|f(t)-y_n\|^2 dt\Big)^{1/2}$$

$$\leqslant \Big(\sum_{j=1}^{n-1} \frac{\varepsilon^2}{4} |A_j(f, \varepsilon)| + 4|B_n(f, \varepsilon)|\Big)^{1/2} \leqslant (\varepsilon^2/4 + \varepsilon^2/2)^{1/2} < \varepsilon.$$

This completes the proof of the lemma.

Proof of Proposition 4.2. Let $d^{\sim}$ be a metric for $D_X^{\sim}$ induced by an affine homeomorphic embedding of $D_X^{\sim}$ into l_2. Let $k \in N$. Since $D_X^{\sim}$ is compact, it follows from Lemma 4.4 that there is an $n = n(\varepsilon)$ such that $D_{X_n}^{\sim}$ is an $1/k$-net for $D_X^{\sim}$ with respect to the metric $d^{\sim}$. Therefore, by the Theorem of Dugundji 1.4, there is a retraction r_k of $D_X^{\sim}$ onto the (convex) set $D_{X_n}^{\sim}$ such that

(14) $$d^{\sim}\big(f, r_k(f)\big) \leqslant 2/k \quad \text{for all } f \in D_X^{\sim}.$$

By Lemma 4.3, there is a map $\psi_k\colon D_{X_n}^{\sim} \to M_{X_n}^{\sim}$ which is continuous and such that

(15) $$d^{\sim}\big(f, \psi_k(f)\big) \leqslant 1/k \quad \text{for all } f \in D_{X_n}^{\sim}.$$

We have $\psi_k r_k\colon D_X^{\sim} \to M_{X_n}^{\sim} \subset M_X^{\sim}$, and by (14) and (15), $\lim_k \psi_k r_k(x) = x$ uniformly on $D_X^{\sim}$. Hence $M_X^{\sim} \in T(D_X^{\sim})$. This completes the proof of Proposition 4.2 and the proof of Theorem 4.4.

§ 5. Applications.

Transformation group actions. Suppose that G is a topological group, X is a topological space, and to each $g \in G$ a homeomorphism F_g of the space X is assigned in such a way that:

(j) the map $G \times X \ni (g, x) \to F_g(x) \in X$ is continuous,

(jj) $F_{gh} = F_g \circ F_h$ for every $g, h \in X$.

The assignement $g \to F_g$ with the above properties is called a transformation group action (briefly: an action) of the group G in the space X. The action $g \to F_g$ is said to be free, if $g \neq e$, the unity of the group G, implies that $F_g(x) \neq x$ for all $x \in X$.

The following is a consequence of Theorem 4.1.

5.1. Theorem. *Every separable (complete) metric group G is algebraically and topologicaly isomorphic to a (closed) subgroup of a group $\Gamma = \Gamma_G$ homeomorphic to the Hilbert space l_2.*

Proof. Since, for the trivial group $G = \{e\}$ the asertion is evident, we may assume that card $G \geqslant 2$. Let $\bar{G}$ be the completion of G and let $\Gamma = M_{\bar{G}}$ with the point-wise group operation. By Theorem 4.1, $\Gamma \simeq l_2$. Obviously G is isomorphic to the subgroup of Γ consisting of constant functions.

5.2. Corollary. *Every separable metric group admits a free action in the Hilbert space l_2.*

Proof. Regard G as a subgroup of $\Gamma \simeq l_2$ and define $F_g(x) = x \cdot g$ for $x \in \Gamma$.

It has been known before that every compact metric group admits a free action in l_2, see West [19].

Topological classification of separable normalized measure algebras. Assume that $(K, \mathscr{K}, m)$ is a normalized measure space, i.e. K is a set, $\mathscr{K}$ is a sigma field of subsets of K and m is a non-negative, σ-additive measure on $\mathscr{K}$ with the property $m(K) = 1$. The normalized measure algebra $\mathfrak{R} = (\mathfrak{R}, m)$ associated with the measure space $(K, \mathscr{K}, m)$ is the Bollean algebra of equivalence classes

$$[A] = \{B \in \mathscr{K}\colon m(B \setminus A) + m(A \setminus B) = 0\}, \quad \text{where } A \in \mathscr{K},$$

equipped with the measure induced by m and with the topology induced by the metric

(1) $$d([A], [B]) = m(A \setminus B) + m(B \setminus A).$$

A class $[A] \in \mathfrak{R}$ (and also a set $A \in \mathscr{K}$) is said to be an m-atom if $m(A) > 0$ and, for any $B \in \mathscr{K}$ with $B \subset A$, either $m(B) = m(A)$ or $m(B) = 0$ holds. Let $a(m)$ denote the cardinality of the set of atoms in the algebra $\mathfrak{R}$. Since the measure m is normalized, we have $a(m) \leqslant \aleph_0$. The measure m is said to be atomless if $a(m) = 0$ and m is said to be purely atomic if every $A \in \mathscr{K}$ is a countable union of atoms.

For more detailed informations on measure algebras the reader is referred to Halmos [14], Sections 40 and 41.

Let $\mathfrak{L}$ denote the measure algebra of equivalence classes (with respect to the Lebesgue measure) of Lebesgue measurable subsets of the interval $[0;1]$. We have

5.3. COROLLARY. *The topological space of the measure algebra $\mathfrak{L}$ is homeomorphic to the Hilbert space l_2.*

Proof. Let $\mathbf{2} = \{0,1\}$ be the two-point discrete space. By Theorem 4.1, $M_{\mathbf{2}} \simeq l_2$. Define $F\colon M_{\mathbf{2}} \to \mathfrak{L}$ by $F(f) = f^{-1}(0)$ for $f \in M_{\mathbf{2}}$. Assuming $d(0,1) = 1$, we get $\varrho(f,g) = \big(|f^{-1}(0)\setminus g^{-1}(0)| + |g^{-1}(0)\setminus f^{-1}(0)|\big)^{1/2}$ for $f, g \in M_{\mathbf{2}}$. Thus F is one-to-one and both F and F^{-1} are continuous. Since every Lebesgue measurable set is equivalent to a Borel set (e.g. to a set of type F_σ), we infer that $F(M_{\mathbf{2}}) = \mathfrak{L}$.

For every non-negative integer n, let $\mathbf{2}^n = \{0, \ldots, 2^n-1\}$, the 2^n-point discrete space; let $\mathbf{2}^{\aleph_0}$ = the Cantor discontinuum.

The next theorem gives the complete description of all topological types of separable normalized measure algebras.

5.4. THEOREM. *Let $\mathfrak{R} = (\mathfrak{R}, m)$ be a separable normalized measure algebra. Then $\mathfrak{R} \simeq \mathbf{2}^{a(m)}$ if m is purely atomic, and $\mathfrak{R} \simeq l_2 \times \mathbf{2}^{a(m)}$ if m is not purely atomic.*

Proof. The result is an immediate consequence of Corollary 5.2 and the following facts (which hold under the assumption of the theorem).

(a) If the measure m is purely atomic and $\{A_\lambda\colon \lambda \in \Lambda\}$ is the set of all atoms in $\mathfrak{R}$ (recall that card $\Lambda \leqslant \aleph_0$), then the map

$$\{0,1\}^\Lambda \ni x = \big(x(\lambda)\big) \to \Big[\bigcup_{x^{-1}(1)} A_\lambda\Big]$$

is a homeomorphism from the product space $\{0,1\}^\Lambda$, which is evidently homeomorphic to $\mathbf{2}^{a(m)}$, onto $\mathfrak{R}$.

(b) If the measure m is atomless then the measure algebra $(\mathfrak{R}, m)$ is isomorphic (as topological Boolean σ-algebra) to the algebra $\mathfrak{L}$, cf. Halmos [14], § 41.

(c) Let $\mathfrak{R}_d$ and $\mathfrak{R}_c$ denote the σ-subalgebras of $\mathfrak{R}$ generated by the atoms of $\mathfrak{R}$, and consisting of those elements of $\mathfrak{R}$ which do not contain atoms, respectively. Then $\mathfrak{R}$ is isomorphic (as a topological Boolean σ-algebra) to the product $\mathfrak{R}_d \times \mathfrak{R}_c$.

Spaces of measurable transformations. Let $(K, \mathscr{K}, m)$ be a normalized measure space and let X be a metric space. By $M_X(K, \mathscr{K}, m)$ we shall denote the space of equivalent classes of measurable transformations $f\colon K \to X$ (measurability of f means that $f^{-1}(A) \in \mathscr{K}$ for every Borel subset $A \subset X$). The metric for $M_X(K, \mathscr{K}, m)$ is defined by

$$\varrho(f,g) = \int_K d\big(f(k), g(k)\big)\, m(dk) \quad \text{for } f, g \in M_X(K, \mathscr{K}, m),$$

where d is a fixed bounded metric for X. Using Theorem 4.1 and the facts (a) – (c) stated above one easily gets

5.5. THEOREM. *If X is a separable complete metric space which has more than one point and the algebra $\mathfrak{R}$ associated with $(K, \mathscr{K}, m)$ is separable, then the space $M_X(K, \mathscr{K}, m)$ is homeomorphic to the Hilbert space l_2.*

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