where ω denotes the volume of the unit sphere. Furthermore, if $\psi(\varrho) = \sup_{|y|\geqslant\varrho} |k(y)|$, $\psi(\varrho)$ is non increasing and

$$|g_a(x)| = |\overline{K}f_a(x)| \leqslant \int N(F_2, x-y)|k(y)|\,dy \leqslant \int N(F_2, x-y)\psi(|y|)\,dy$$
$$= \int_0^\infty \psi(\varrho)\,dv(\varrho, x) = -\int_0^\infty v(\varrho, x)\,d\psi(\varrho) \leqslant -h(x)\int_0^\infty \omega\varrho^n\,d\psi(\varrho)$$
$$= nh(x)\,\omega\int_0^\infty \varrho^{n-1}\psi(\varrho)\,d\varrho \leqslant nh(x),$$

and, according to Lemma 7,

$$N(G, x) = M(g, x) = M(\overline{K}f_2, x) \leqslant ch^*(x)$$

where c is independent of $\overline{K}$. Thus

$$\sup_K M(\overline{K}f_2, x) \leqslant ch^*(x)$$

where $\|h^*\|_p \leqslant c\|h\|_p \leqslant c\|N(F_2, x)\|_p$. From here on the proof proceeds as that of Theorem 1.

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Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients

by

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Abstract. A Banach space X is isomorphic to a Hilbert space if and only if one of the following conditions holds for all sequences (x_n) in X

(a) if $\sum_{n=0}^{\infty} \|x_n\|^2 < +\infty$, then

$$\int_0^{2\pi} \Big\| x_0 + \sum_{k=1}^{\infty} (x_{2k-1}\sin kt + x_{2k}\cos kt) \Big\|^2 dt < +\infty,$$

(b) if $\int_0^{2\pi} \|x_0 + \sum_{k=1}^{\infty} (x_{2k-1}\sin kt + x_{2k}\cos kt)\|^2 dt < +\infty$, then $\sum_{n=0}^{\infty} \|x_n\|^2 dt < +\infty$,

(c) $\sum_{n=1}^{\infty} \|x_n\|^2 < +\infty$ if and only if $\int_0^1 \|\sum_{n=1}^{\infty} x_n r_n(t)\|^2 dt < +\infty$.

Here (r_n) denotes the Rademacher system of functions.

1. Introduction. In the present paper we prove the following

THEOREM 1.1. *A real or complex Banach space X is isomorphic to a Hilbert space if and only if one of the following conditions holds for all sequences (x_n) in X*

(a) *if* $\sum_{n=0}^{\infty} \|x_n\|^2 < +\infty$, *then*

$$\int_0^{2\pi} \Big\| x_0 + \sum_{k=1}^{\infty} (x_{2k-1}\sin kt + x_{2k}\cos kt) \Big\|^2 dt < +\infty,$$

(b) *if* $\int_0^{2\pi} \Big\| x_0 + \sum_{k=1}^{\infty} (x_{2k-1}\sin kt + x_{2k}\cos kt) \Big\|^2 dt < +\infty$,

then $\sum_{n=0}^{\infty} \|x_n\|^2 < +\infty$,

(c) $\sum_{n=1}^{\infty} \|x_n\|^2 < +\infty$ *if and only if* $\int_0^1 \Big\| \sum_{n=1}^{\infty} x_n r_n(t) \Big\|^2 dt < +\infty$.

Here (r_n) denotes the Rademacher system defined by

$$r^n(t) = \operatorname{sign}\sin 2^n\pi t \quad \text{for} \quad t\in[0,1] \ (n = 1, 2, \ldots).$$

Let us observe that the validity of (c) for all sequences (x_n) in a Banach space X is equivalent to the following "two sided Khinchin inequality".

(*) There exists $C \geqslant 1$ such that for any sequence (x_j) in X,

$$(1)\qquad C^{-1}\sum_{j=1}^{\infty}\|x_j\|^2 \leqslant \int_0^1 \Big\|\sum_{j=1}^{\infty} x_j r_j(t)\Big\|^2 dt \leqslant C\sum_{j=1}^{\infty}\|x_j\|^2 \qquad (n = 1, 2, \ldots).$$

Furthermore note that

$$(2)\qquad \int_0^1 \Big\|\sum_{j=1}^{n} x_j r_j(t)\Big\|^2 dt = 2^{-n}\sum_{\varepsilon(n)}\Big\|\sum_{j=1}^{n}\varepsilon_j x_j\Big\|^2,$$

where $\sum_{\varepsilon(n)}$ is extended over all sequences $(\varepsilon_j)_{j=1}^n$ with $\varepsilon_j = \pm 1$ $(j = 1, 2, \ldots, n)$.

Thus for $C = 1$ the condition (*) becomes the equality

$$(3)\qquad 2^{-n}\sum_{\varepsilon(n)}\Big\|\sum_{i=1}^{n}\varepsilon_i x_i\Big\|^2 = \sum_{j=1}^{n}\|x_j\|^2 \qquad (n = 1, 2, \ldots)$$

which for $n = 2$ is the parallelogram identity and for $n > 2$ can be easily deduced by induction from the parallelogram identity.

Next observe that the validity of (b) for all square summable sequences (x_n) in a Banach space X is equivalent to the following "Bessel type inequality"

(**) there exists $C > 0$ such that for any sequence (x_j) in X

$$C\int_0^{2\pi}\Big\|x_0 + \sum_{j=1}^{n} x_{2j-1}\sin jt + x_{2j}\cos jt\Big\|^2 dt \geqslant \sum_{j=0}^{n}\|x_j\|^2 \qquad (n = 1, 2, \ldots).$$

Similarly the condition (a) yields "the reverse Bessel type inequality". For complex Banach spaces the validity of (a) for all sequences (x_j) in X is equivalent to the boundedness of the Fourier transform in the space $L^2(X)$ of all strongly measurable square integrable (in the sense of Bochner) X-valued functions on the real line (cf. Proposition 4.2). This enables us to substantiate a conjecture of Peetre [5], p. 20.

2. A corollary to the central limit theorem. Let $(\Omega, \mathfrak{M}, P)$ denote an atomless probability space. If ξ is a random variable on Ω, then

$$E\xi = \int_\Omega \xi(\omega)\,dP(\omega).$$

By (γ_i) we shall denote a sequence of independent Gaussian random variables on Ω each distributed by the rule

$$P(\gamma_i < t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{s^2}{2}}\,ds \qquad (-\infty < t < +\infty).$$

By (δ_i) we shall denote a sequence of independent random variables on Ω each distributed by the rule

$$(4)\qquad P(\delta_i = 1) = P(\delta_i = -1) = \tfrac{1}{2}.$$

Let us fix a positive integer n and put

$$\delta_i^m = \frac{1}{\sqrt{m}}\Big(\sum_{k=0}^{m-1}\delta_{im+k}\Big) \quad \text{for} \quad i = 1, 2, \ldots, n;\ m = 1, 2, \ldots$$

By the Moivre–Laplace Theorem ([6], Chapt. VIII), the common distribution of $(\delta_1^m, \delta_2^m, \ldots, \delta_n^m)$ converges to the common distribution of $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ as $m \to \infty$ equivalently (by the Lebesgue convergence theorem)

$$\lim_{m\to\infty} Eh(\delta_1^m, \delta_2^m, \ldots, \delta_n^m) = Eh(\gamma_1, \gamma_2, \ldots, \gamma_n)$$

for any bounded continuous function $h\colon R^n \to R$. (Here R denotes the real line).

We shall need the following well known strenthening of this fact

Lemma 2.1. *If $h\colon R^n \to R$ is a continuous function such that*

$$(5)\qquad h(s_1, s_2, \ldots, s_n)e^{-\sum_{i=1}^{n}|s_i|} \to 0 \quad \text{as} \sum_{i=1}^{n}|s_i| \to \infty,$$

then

$$\lim_{m\to\infty} Eh(\delta_1^m, \delta_2^m, \ldots, \delta_n^m) = Eh(\gamma_1, \gamma_2, \ldots, \gamma_n).$$

Proof. Let us consider the Banach space B of all continuous function $h\colon R^n \to R$ satisfying (5) under the norm

$$\|h\| = \sup_{(s_1, s_2, \ldots, s_n)\in R^n}|h(s_1, s_2, \ldots, s_n)|e^{-\sum_{i=1}^{n}|s_i|}.$$

Let us set

$$F_m(h) = Eh(\delta_1^m, \delta_2^m, \ldots, \delta_n^m) \quad \text{for} \quad h \in B\ (m = 1, 2, \ldots),$$

$$F(h) = Eh(\gamma_1, \ldots, \gamma_n)$$
$$= \frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} h(s_1, \ldots, s_n)e^{-\sum_{i=1}^{n}\frac{s_i^2}{2}}\,ds_1\ldots ds_n \quad \text{for} \quad h \in B.$$

It follows from (5) that the above formulas well define linear functionals on B. Moreover by the Moivre–Laplace Theorem quoted above $\lim_{m=\infty} F_m(h) = F(h)$ for any bounded continuous h. Since the set of such functions h is dense in B to complete the proof it is enough to show that $\sup_m \|F_m\| < +\infty$. In fact using the stochastic independence of the sequence (δ_j) we have

$$\|F_m\| \leqslant E\exp\Big(\sum_{i=1}^{n}|\delta_i^m|\Big) = (E\exp|\delta_1^m|)^n$$

$$\leqslant (E\exp\delta_1^m + E\exp(-\delta_1^m))^n$$

$$= \left(\left(E\exp\frac{\delta_1}{\sqrt{m}}\right)^m + \left(E\exp\left(-\frac{\delta_1}{\sqrt{m}}\right)\right)^m\right)^n$$

$$= 2^n\left(\frac{1}{2}\left(\exp\frac{1}{\sqrt{m}} + \exp\left(-\frac{1}{\sqrt{m}}\right)\right)\right)^{mn} \leqslant 2(2\sqrt{e})^n.$$

3. Two sided estimations characterizing Hilbert spaces. We begin with the following

PROPOSITION 3.1. *For any real or complex Banach space X the following condition are equivalent*

(i) *X is isomorphic to a Hilbert space,*

(ii) *there exists a constant $C \geqslant 1$ such that for any positive integer n and any $x_1, x_2, \ldots, x_n$ in X*

$$C^{-1}\sum_{i=1}^{n}\|x_i\|^2 \leqslant E\Big(\Big\|\sum_{i=1}^{n}\delta_i x_i\Big\|^2\Big) \leqslant C\sum_{i=1}^{n}\|x_i\|^2,$$

(iii) *there exists a constant $C \geqslant 1$ such that for any positive integer n and any $x_1, x_2, \ldots, x_n$ in X*

$$C^{-1}\sum_{i=1}^{n}\|x_i\|^2 \leqslant E\Big(\Big\|\sum_{i=1}^{n}\gamma_i x_i\Big\|^2\Big) \leqslant C\sum_{i=1}^{n}\|x_i\|^2,$$

(iv) *there exists a constant $C \geqslant 1$ such that for any positive integer n and for any $n\times n$ scalar valued matrix (a_{ij}) if*

$$\sum_{j=1}^{n}\Big|\sum_{i=1}^{n}a_{ij}s_j\Big|^2 \leqslant \sum_{j=1}^{n}|s_j|^2 \quad \textit{for all } n\textit{-tuples} \tag{6}$$

$(s_1, s_2, \ldots, s_n)$ of scalars then for any sequence $x_1, x_2, \ldots, x_n$ in X,

$$\sum_{j=1}^{n}\Big\|\sum_{i=1}^{n}a_{ij}x_j\Big\|^2 \leqslant C\sum_{j=1}^{n}\|x_j\|^2. \tag{7}$$

Proof. (i) $\Rightarrow$ (ii) The Rademacher functions are an example of a sequence of independent random variables distributed by the rule (4). Hence, by (2), (3) and the remark after (3), it follows that in any Hilbert space the inequality appearing in (ii) holds with $C = 1$. Thus if X is isomorphic to a Hilbert space, then (ii) is satisfied with $C = \inf\|T\|^2$ where inf is extended on all isomorphisms T from X into a Hilbert space such that $\|T\| = \|T\|^{-1}$.

(ii) $\Rightarrow$ (iii). Fix $x_1, x_2, \ldots, x_n$ in X and put $\tilde{x}_{im+k} = m^{-1/2}x_i$ ($i = 1, 2, \ldots, n$; $k = 0, 1, \ldots, m-1$; $m = 1, 2, \ldots$). Clearly we have

$$\sum_{j=1}^{mn}\|\tilde{x}_j\|^2 = \sum_{i=1}^{n}\|x_i\|^2 \quad\text{and}\quad E\Big(\Big\|\sum_{i=1}^{n}\delta_i^m x_i\Big\|^2\Big) = E\Big(\Big\|\sum_{j=1}^{mn}\delta_j\tilde{x}_j\Big\|^2\Big)$$

for $m = 1, 2, \ldots$

Thus (ii) applied to the sequence $(x_j)_{1\leqslant j\leqslant mn}$ yields

$$C^{-1}\sum_{i=1}^{n}\|x_i\|^2 \leqslant E\Big(\Big\|\sum_{i=1}^{n}\delta_i^m x_i\Big\|^2\Big) \leqslant C\sum_{i=1}^{n}\|x_i\|^2.$$

Now (iii) follows from Lemma 2.1 applied to the function $h(s_1, s_2, \ldots, s_n) = \|\sum_{i=1}^{n}s_i x_i\|^2$ which satisfies (5) because $\|\sum_{i=1}^{n}s_i x_i\|^2 \leqslant (\sum_{i=1}^{n}|s_i|\,\|x_i\|)^2 \leqslant \max_{1\leqslant i\leqslant n}\|x_i\|^2(\sum_{i=1}^{n}|s_i|)^2$.

(iii) $\Rightarrow$ (iv). Assume first that (a_{ij}) is a real $n\times n$ matrix satisfying (6). Then (a_{ij}) represents in the n-dimensional (real!) Euclidean space l_2^n a linear operator of norm $\leqslant 1$. It is well known that the extreme points of the unit ball of the n^2 dimensional space of linear operators on l_2^n are exactly linear isometries. Hence, by the Krein–Milman and the Caratheodory theorems, any $n\times n$ real matrix satisfying (6) is a convex combination of at most n^2+1 matrices of isometries. Hence to establish that (iii) $\Rightarrow$ (iv) it is enough to show that (7) holds for any real $n\times n$ matrix (a_{ij}) which represents an isometry. Then we have $\sum_{j=1}^{n}(\sum_{i=1}^{n}s_i a_{ij})^2 = \sum_{i=1}^{n}s_i^2$ for any reals $s_1, s_2, \ldots, s_n$ and $|\mathrm{Det}(a_{ij})| = 1$. Thus using (iii) and the change of variables $t_j = \sum_{i=1}^{n}s_i a_{ij}$ for $j = 1, 2, \ldots, n$ we get

$$\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}a_{ij}x_j\Big\|^2 \leqslant CE\Big(\Big\|\sum_{i=1}^{n}\gamma_i\sum_{j=1}^{n}a_{ij}x_j\Big\|^2\Big)$$

$$= \frac{C}{(\sqrt{2\pi})^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\Big\|\sum_{i=1}^{n}s_i\sum_{j=1}^{n}a_{ij}x_j\Big\|^2 e^{-\frac{1}{2}\sum_{i=1}^{n}s_i^2}ds_1\ldots ds_n$$

$$= \frac{C}{(\sqrt{2\pi})^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\Big\|\sum_{j=1}^{n}t_j x_j\Big\|^2 e^{-\frac{1}{2}\sum_{j=1}^{n}t_j^2}|\mathrm{Det}(a_{ij})|^{-1}dt_1\ldots dt_n$$

$$= CE\Big(\Big\|\sum_{j=1}^{n}\gamma_j x_j\Big\|^2\Big) \leqslant C^2\sum_{j=1}^{n}\|x_j\|^2.$$

Now if (a_{ij}) is a complex $n\times n$ matrix satisfying (6), then $(\operatorname{Re} a_{ij})$ and $(\operatorname{Im} a_{ij})$ are real $n\times n$ matrices satisfying (6) for real $s_1, s_2, \ldots, s_n$. Thus from what was just established we get

$$\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}a_{ij}x_j\Big\|^2\leqslant 2\Big(\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}(\operatorname{Re} a_{ij})x_j\Big\|^2+\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}(\operatorname{Im} a_{ij})x_j\Big\|^2\Big)\leqslant 2C^2\sum_{i=1}^{n}\|x_i\|^2.$$

(iv) $\Rightarrow$ (i). Let (a_{ij}) be a scalar valued $n\times n$ matrix satisfying (6). Fix $x_1, x_2, \ldots, x_n$ in X and put $y_i = \sum_{j=1}^{n} a_{ij}x_j$ for $i = 1, 2, \ldots, n$.

Then, by (6), we get

$$\sum_{i=1}^{n}|x^*(y_i)|^2\leqslant\sum_{j=1}^{n}|x^*(x_j)|^2 \quad \text{for any } x^*\in X^*. \tag{8}$$

Conversely assume that sequences $(x_j)_{j=1}^n$ and $(y_i)_{i=1}^n$ in X satisfy (8). Let F be a subspace of l_2^n consisting of all sequences $(t_j)_{j=1}^n$ such that $t_j = x^*(x_j)$ for $i = 1, 2, \ldots, n$ and for some $x^*\in X^*$. Let us set

$$A\big((t_j)\big) = \big(x^*(y_i)\big) \quad \text{for} \quad (t_j) = \big(x^*(x_j)\big).$$

It follows from (8) that the above formula define a linear operator from F into l_2^n with $\|A\|\leqslant 1$. Let $A^{\sim} = AP\colon l_2^n\to l_2^n$ where P is the orthogonal projection from l_2^n onto F. Let (a_{ij}) be the $n\times n$ matrix representing A in the unit vector basis i.e. $A\big((t_j)\big) = \big(\sum_{j=1}^{n} a_{ij}t_j\big)$ for $(t_j)_{j=1}^n\in l_2^n$ then the matrix (a_{ij}) satisfies (6) because $\|A\| = \|A^{\sim}\|\leqslant 1$, and for any $x^*\in X^*$ we have

$$\big(x^*(y_i)\big)_{i=1}^n = A\Big(\big(x^*(x_j)\big)\Big) = \Big(\sum_{j=1}^{n}a_{ij}x^*(x_j)\Big)_{i=1}^n.$$

Hence $x^*(y_i) = x^*\big(\sum_{j=1}^{n}a_{ij}x_j\big)$ for $i = 1, 2, \ldots, n$ and $x^*\in X^*$ equivalently $y_i = \sum_{j=1}^{n}a_{ij}x_j$ for $i = 1, 2, \ldots, n$.

Thus (iv) is equivalent to the following condition

(v) There exists $C > 0$ such that for any positive integer n and any sequences $(x_j)_{j=1}^n$ and $(y_i)_{i=1}^n$ in X the condition (8) implies (7).

By [2], Theorem 7.3, the conditions (v) and (i) are equivalent. This completes the proof.

COROLLARY 3.2. *A Banach space X is isomorphic to a Hilbert space if and only if X satisfies the two sided Khinchine inequality* (∗).

Proof. Use the fact that Rademacher functions are independent random variables distributed by (4).

Next we shall show that in the statement of Corollary 3.2 one can replace the (non complete!) Rademacher system by an arbitrary complete

orthonormal system in L^2; here by L^2 we denote the space of all square Lebesgue integrable scalar valued functions on $[0;1]$. We shall write

$$(g, h) = \int_0^1 g(t)\cdot\overline{h(t)}\,dt \quad \text{for } g, h\in L^2.$$

LEMMA 3.3. *Let (f_i) be an orthonormal complete system in L^2 and let X be a real or complex Banach space. If for some $C > 0$ and for any $x_1, x_2, \ldots, x_n\in X$ and for $n = 1, 2, \ldots$ we have*

$$\int_0^1\Big\|\sum_{i=1}^{n}f_i(t)x_i\Big\|^2dt\leqslant C\sum_{i=1}^{n}\|x_i\|^2 \quad (\textit{resp.}\geqslant), \tag{9}$$

then for the same $C > 0$ and for any $x_1, \ldots, x_n$ and for $n = 1, 2, \ldots$ we also have

$$\int_0^1\Big\|\sum_{i=1}^{n}r_i(t)x_i\Big\|^2dt\leqslant C\sum_{i=1}^{n}\|x_i\|^2 \quad (\textit{resp.}\geqslant). \tag{10}$$

Proof. By the standard gliding hump procedure one can find for a given $\varepsilon > 0$ increasing sequences of indices (k_j) and (m_j) and an orthonormal sequence (h_j) such that

$$h_j = \sum_{k=k_j}^{k_{j+1}-1}(h_j, f_k)f_k,$$

$$\int_0^1|h_j(t)-r_{m_j}(t)|^2dt<\frac{\varepsilon}{2^{+j}} \quad \text{for} \quad j = 1, 2, \ldots$$

Now for a fixed positive integer n and fixed $x_1, x_2, \ldots, x_n$ in X we have

$$\int_0^1\Big\|\sum_{j=1}^{n}r_j(t)x_j\Big\|^2dt = \int_0^1\Big\|\sum_{j=1}^{n}r_{m_j}(t)x_j\Big\|^2dt = E(\|\delta_1x_1+\ldots+\delta_nx_n\|^2).$$

Furthemore, by the triangle inequality

$$\sqrt{\int_0^1\Big\|\sum_{j=1}^{n}r_{m_j}(t)x_j\Big\|^2dt}\leqslant\sqrt{\int_0^1\Big\|\sum_{j=1}^{n}\big(r_{m_j}(t)-h_j(t)\big)x_j\Big\|^2dt}+\sqrt{\int_0^1\Big\|\sum_{j=1}^{n}h_j(t)x_j\Big\|^2dt}$$

$$\leqslant\sqrt{\varepsilon}\sqrt{\sum_{j=1}^{n}\|x_j\|^2}+\sqrt{\int_0^1\Big\|\sum_{j=1}^{n}h_j(t)x_j\Big\|^2dt}$$

because, by the Schwartz inequality,

$$\int_0^1\Big\|\sum_{j=1}^{n}\big(r_{m_j}(t)-h_j(t)\big)x_j\Big\|^2dt\leqslant\int_0^1\sum_{j=1}^{n}|r_{m_j}(t)-h_j(t)|^2\sum_{\nu=1}^{n}\|x_\nu\|^2dt$$

$$=\sum_{j=1}^{n}\int_0^1|r_{m_j}(t)-h_j(t)|^2dt\cdot\sum_{\nu=1}^{n}\|x_\nu\|^2\leqslant\varepsilon\sum_{\nu=1}^{n}\|x_\nu\|^2.$$

On the other hand, by (9), remembering that $1 = \|h_j\|^2 = \sum_{k=k_j}^{k_{j+1}-1} |(h_j, f_k)|^2$ we get

$$\int_0^1 \Big\| \sum_{j=1}^n h_j(t) x_j \Big\|^2 dt = \int_0^1 \Big\| \sum_{j=1}^n \Big(\sum_{k=k_j}^{k_{j+1}-1} (h_j, f_k) f_k \Big) x_j \Big\|^2 dt$$

$$\leqslant C \sum_{j=1}^n \sum_{k=k_j}^{k_{j+1}-1} |(h_j, f_k)|^2 \|x_j\|^2 = C \sum_{j=1}^n \|x_j\|^2.$$

Thus

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|^2 dt \leqslant (\sqrt{\varepsilon} + \sqrt{C})^2 \sum_{j=1}^n \|x_j\|^2.$$

Let ε tend to zero, and we have (10). The proof for the reverse inequality is analogous.

Corollary 3.4. *Let (f_i) be an orthonormal complete system in L^2 and let X be a real or complex Banach space. Then X is isomorphic to a Hilbert space if and only if there exists $C > 0$ such that for any positive integer n and any $x_1, x_2, \ldots, x_n$,*

$$C^{-1} \sum_{i=1}^n \|x_i\|^2 \leqslant \int_0^1 \Big\| \sum_{i=1}^n f_i(t) x_i \Big\|^2 dt \leqslant C \sum_{i=1}^n \|x_i\|^2.$$

Remark. Corollary 3.4 is clearly not true for an arbitrary infinite orthonormal sequence of functions in L^2. Indeed let $f_i = 2^{-i}\chi_i$ where χ_i denotes the characteristic function of the interval $(2^{-i}, 2^{-i+1})$ for $i = 1, 2, \ldots$ Then for any Banach space X and any $x_1, x_2, \ldots, x_n \in X$ $(n = 1, 2, \ldots)$ we have

$$\int_1^1 \Big\| \sum_{i=1}^n f_i(t) x_i \Big\|^2 dt = \sum_{i=1}^n \|x_i\|^2.$$

4. Characterizations of a Hilbert space by the existence of the Fourier transform and Bessel type inequalities. Let X be a complex Banach space. Denote by $L_0^2(X)$ the normed linear space of all simple functions $f\colon R \to X$ under the norm $\|f\| = \big(\int_{-\infty}^{+\infty} \|f(t)\|^2 dt\big)^{1/2}$. Here by a simple function we mean any function of the form $\sum_{j=1}^n \chi_{A_j} x_j$ where $x_j \in X$; A_j are mutually disjoint measurable subsets of R of finite Lebesgue measure and χ_{A_j} denotes the characteristic function of $A_j (j = 1, 2, \ldots, n$; n — any positive integer). The completion of $L_0^2(X)$ in the norm $\|\cdot\|$ will be denoted by $L^2(X)$. The Fourier Transform $\mathscr{F}\colon L_0^2(X) \to L^2(X)$ is defined by

$$\mathscr{F}(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ist} f(s)\, ds \quad \text{for} \quad t \in R;\ f \in L_0^2(X).$$

Similarly we define the inverse Fourier Transform: $\mathscr{F}^{\sim}\colon L_0^2(X) \to L^2(X)$ by

$$\mathscr{F}^{\sim}(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ist} f(s)\, ds \quad \text{for} \quad t \in R;\ f \in L_0^2(X).$$

Clearly $\mathscr{F}$ and $\mathscr{F}^{\sim}$ are linear operators in general unbounded. Our next lemma seems to be known. The proof repeats the classical argument used in the Poisson summation formula

Lemma 4.1. *Let*

$$h = \sum_{k=-M}^{M} \frac{x_k}{\sqrt{a}} \chi_{[ka;(k+1)a)}, \tag{11}$$

where $a > 0$, $x_k \in X$ $(k = 0, \pm 1, \pm 2, \ldots, \pm M)$, M—any positive integer. Then

$$\|h\|^2 = \sum_{k=-M}^{M} \|x_k\|^2;\quad \|\mathscr{F}(h)\|^2 = \int_0^1 \Big\| \sum_{k=-M}^{M} e^{-2\pi k l i} x_k \Big\|^2 dt.$$

Proof. The computation of the norm $\|h\|$ is trivial. To establish the second formula we compute first directly $\mathscr{F}(h)$. We have

$$\mathscr{F}(h)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sum_{k=-M}^{M} \frac{x_k}{\sqrt{a}} \chi_{[ka,(k+1)a)}(s) \cdot e^{-ist} ds = \frac{1}{\sqrt{2\pi a}} \sum_{k=-M}^{M} x_k \int_{ka}^{(k+1)a} e^{-ist} ds$$

$$= \sqrt{\frac{a}{2\pi}} \frac{\sin \frac{at}{2}}{\frac{at}{2}} \cdot \big(-e^{-i\frac{at}{2}}\big) \sum_{k=-M}^{M} x_k e^{-kati}.$$

Hence, changing the variable, $u = at/2$, we get

$$\|\mathscr{F}(h)\|^2 = \int_{-\infty}^{+\infty} \frac{\sin^2 u\pi}{(u\pi)^2} \Big\| \sum_{k=-M}^{M} x_k e^{-2\pi iku} \Big\|^2 du$$

$$= \sum_{\nu=-\infty}^{+\infty} \int_{\nu}^{\nu+1} \frac{\sin^2 u\pi}{(u\pi)^2} \Big\| \sum_{k=-M}^{+M} x_k e^{-2\pi iku} \Big\|^2 du$$

$$= \int_0^1 \sum_{\nu=-\infty}^{+\infty} \frac{\sin^2 u\pi}{[\pi(u+\nu)]^2} \Big\| \sum_{k=-M}^{M} x_k e^{-2\pi iku} \Big\|^2 du.$$

Since $\sum_{\nu=-\infty}^{+\infty} \frac{\sin^2 \pi u}{[\pi(u+\nu)]^2} = 1$ for all real u, we get

$$\|\mathscr{F}(h)\|^2 = \int_0^1 \Big\| \sum_{k=-M}^{M} x_k e^{-2\pi kui} \Big\|^2 du.$$

PROPOSITION 4.1. *For any complex Banach space X the following conditions are equivalent*

(v) *X is isomorphic to a Hilbert space,*

(vi) *there exists $C>0$ such that for any positive integer n and any $x_0, x_1, x_{-1}, \ldots, x_n, x_{-n}$ in X*

$$\int_0^1 \Big\| \sum_{k=-n}^{n} e^{2\pi ikt} x_k \Big\|^2 dt \leqslant C \sum_{k=-n}^{n} \|x_k\|^2,$$

(vii) *there exists $C>0$ such that for any positive integer n and any $x_0, x_1, x_{-1}, \ldots, x_n, x_{-n}$*

$$\int_0^1 \Big\| \sum_{k=-n}^{n} e^{2\pi ikt} x_k \Big\|^2 dt \geqslant \frac{1}{C} \sum_{k=-n}^{n} \|x_k\|^2,$$

(viii) *The Fourier Transform $\mathcal{F}\colon L_0^2(X) \to L^2(X)$ is bounded.*

Proof. The direct computation shows that if X is isometrically isomorphic to a Hilbert space, then for any orthonormal functions $f_1, f_2, \ldots, f_n$ in L^2 and any $x_1, x_2, \ldots, x_n$ in X

$$\int_0^1 \Big\| \sum_{j=1}^{n} f_j(t) x_j \Big\|^2 = \sum_{j=1}^{n} \|x_j\|^2 \qquad (n = 1, 2, \ldots).$$

Hence (v) $\Rightarrow$ (vi) and (v) $\Rightarrow$ (vii).

Next we shall show the equivalence (vi) $\Leftrightarrow$ (viii). Assume first (vi). It follows from Lemma 4.1 that for any function h of form (11) we have

$$\|\mathcal{F}(h)\|^2 = \int_0^1 \Big\| \sum_{k=-M}^{M} e^{-2\pi ikt} x_k \Big\|^2 dt = \int_0^1 \Big\| \sum_{k=-M}^{M} e^{2\pi kti} x_{-k} \Big\|^2 dt \leqslant C \sum_{k=-M}^{M} \|x_k\|^2 = C\|h\|^2.$$

This proves the boundedness of $\mathcal{F}$ because the functions of form (11) are dense in $L_0^2(X)$. Hence (vi) $\Rightarrow$ (viii). Conversely assuming (viii) we get (vi).

Next we show that (viii) $\Rightarrow$ (vii). Obviously (viii) implies that the inverse Fourier Transform $\mathcal{F}^{\sim}$ is also bounded. Hence both $\mathcal{F}$ and $\mathcal{F}^{\sim}$ have the unique extensions to the bounded operators from $L^2(X)$ into $L^2(X)$ which we shall denote by the same symbols $\mathcal{F}$ and $\mathcal{F}^{\sim}$ respectively. We have

(12) $$\mathcal{F}^{\sim}\big(\mathcal{F}(f)\big) = \mathcal{F}\big(\mathcal{F}^{\sim}(f)\big) = f \quad \text{for} \quad f \in L^2(X).$$

Indeed we check (12) directly for $f = x \cdot \chi_{[a,b)}$, for $x \in X$ and $-\infty < a < b < +\infty$. Since the functions $x \cdot \chi_{[a,b)}$ are linearly dense in $L^2(X)$, the linearity and boundedness of $\mathcal{F}$ and $\mathcal{F}^{\sim}$ imply (12) in full generality.

It follows from (12) that there exists a constant $C_1 > 0$ such that $\|\mathcal{F}(f)\| \geqslant C_1 \|f\|$ for any $f \in L^2(X)$. Hence again using Lemma 4.1 we get (vii).

From what we just showed it follows that (vi) $\Rightarrow$ (viii) $\Rightarrow$ (vii). By Corollary 3.4, the conjuction (vi) and (vii) implies (v). Thus the conditions (v), (vi) and (viii) are equiqalent for any Banach space X.

Finally to prove that (vii) implies (v) we apply the following:

LEMMA 4.3. *If for some Banach space X and for some complete orthonormal system (f_n) in L^2 there exists $C>0$ such that*

$$\int_0^1 \Big\| \sum_{k=1}^{n} x_k f_k(t) \Big\|^2 dt \geqslant C \sum_{k=1}^{n} \|x_k\|^2 \quad \text{for any } x_1, \ldots, x_n \text{ in } X \quad (n = 1, 2, \ldots)$$

then

$$\int_0^1 \Big\| \sum_{k=1}^{n} x_k^* f_k(t) \Big\|^2 dt \leqslant C^{-1} \sum_{k=1}^{n} \|x_k^*\|^2$$

for any $x_1, \ldots, x_n$ in X^* $(n = 1, 2, \ldots)$.

Proof. The completness of the system (f_n) implies that the linear combinations of the f_n's are dense in L^2. Hence the set E of all vector valued functions of the form

$$\varphi = \sum_{k=1}^{n} f_k(\cdot) x_k \qquad (x_k \in X;\ k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$$

is dense in the space $L^2\big([0,1), X\big)$ of all strongly measurable Bochner square integrable functions from $[0,1)$ into X. Now it follows from a standard duality argument that for any fixed linear combination $\varphi^* = \sum_{k=1}^{n} x_k^* f_k$ and any $\varepsilon > 0$ there exists $\varphi = \sum_{k=1}^{m} x_k f_k \in E$ with $|||\varphi||| = \big(\int_0^1 \|\varphi(t)\|^2 dt\big)^{1/2} = 1$ such that

$$\Big(\int_0^1 \Big\| \sum_{k=1}^{n} x_k^* f_k(t) \Big\|^2 dt\Big)^{\frac12} \leqslant \int_0^1 \big|\big(\varphi^*(t)\varphi(t)\big)\big| dt + \varepsilon = \sum_{k=1}^{\min(m,n)} |x_k^*(x_k)| + \varepsilon$$
$$\leqslant \Big(\sum_{k=1}^{n} \|x_k^*\|^2\Big)^{\frac12} \Big(\sum_{k=1}^{m} \|x_k\|^2\Big)^{\frac12} + \varepsilon \leqslant \frac{1}{\sqrt{C}} \Big(\sum_{k=1}^{m} \|x_k\|^2\Big)^{\frac12} + \varepsilon$$

which yields the desired conclusion.

It follows from (vii) and Lemma 4.3 that the dual space X^* of X satisfies (vi). Hence, by the equivalence of (vi) and (v) for any Banach space, we conclude that X^* is isomorphic to a Hilbert space. Hence X has the same property. Thus (vii) $\Rightarrow$ (v) and this completes the proof of the Proposition.

Proof of Theorem 1.1. The equivalence of (c) and the condition (i) of Proposition 3.1 follows from Corollary 3.2. Clearly (i) implies (a) and (b). To prove that (a) implies (i) consider first the case where X is a complex Banach space. Clearly (a) implies that there is a constant $C > 0$ such that

$$\int_0^{2\pi} \Big\| x_0 + \sum_{k=1}^{n} (x_{2k-1}\sin kt + x_{2k}\cos kt)\Big\|^2 dt \leqslant C \sum_{k=0}^{2n} \|x_k\|^2$$

$$\text{for } x_k \in X \ (k = 0, 1, \ldots, 2n;\ n = 1, 2, \ldots)$$

thus using the Euler formula $e^{it} = \cos t + i\sin t$ for any $y_j \in X$ ($j = 0, \pm 1, \ldots, \pm n$; $n = 1, 2, \ldots$) we obtain

$$\int_0^1 \Big\| \sum_{k=-n}^{n} y_k e^{2\pi kti}\Big\|^2 dt = \frac{1}{(2\pi)^2}\int_0^{2\pi} \Big\| y_0 + \sum_{k=-n}^{n} (y_k - y_{-k})\sin kt + (y_k + y_{-k})\cos kt\Big\|^2 dt$$

$$\leqslant \frac{C}{4\pi^2}\Big[\|y_0\|^2 + \sum_{k=-n}^{n} (\|y_k - y_{-k}\|^2 + \|y_k + y_{-k}\|^2)\Big]$$

$$\leqslant \frac{C}{\pi^2} \sum_{k=-n}^{n} \|y_k\|^2.$$

Hence (a) implies the condition (vi) of Proposition 4.2 and therefore (a) implies (i). Now, if X is a real Banach space satisfying (a), then the complexification $\tilde{X}$ of X also satisfies (a). Thus, from what was just proved, it follows that $\tilde{X}$ is isomorphic to a complex Hilbert space equivalently X is isomorphic to a real Hilbert space.

Finally using Lemma 4.3 we deduce the implication (b) $\Rightarrow$ (i) from the implication (a) $\Rightarrow$ (i).

We end this section by the following conjecture

CONJECTURE. *Let (f_n) be a complete orthonormal system in L^2. Then a Banach space X is isomorphic to a Hilbert space if and only if*

(12)
$$\int_0^1 \Big\| \sum_{k=1}^{n} f_k(t) x_k\Big\|^2 dt \leqslant C \sum_{k=1}^{n} \|x_k\|^2 \qquad (x_k \in X,\ k = 1, 2, \ldots, n;\ n = 1, 2, \ldots),$$

where C is an universal constant.

Remark 1. (due to A. Pełczyński) The above Conjecture reduces to the case of the Haar orthonormal system because any complete orthonormal system contains a block sequence which is distributed by the same rule as the Haar functions (cf. [4] and [3], proof of Theorem 4.1).

Remark 2. The Conjecture is true for the Walsh orthonormal system (w_k) (cf. [1], Chapt. IV, § 6 for definition). This follows from the formula

$$\int_0^1 \Big\| \sum_{k=0}^{2^n-1} w_k(t) x_k\Big\|^2 dt = \sum_{j=0}^{2^n-1} \Big\| \sum_{k=0}^{2^n-1} a_{jk}^{(n)} x_k\Big\|^2$$

for any $x_1, x_2, \ldots, x_{2^n-1}$ in X ($n = 1, 2, \ldots$) where $(a_{jk}^{(n)})_{j,k=0,1,\ldots,2^n-1}$ is an orthogonal matrix of an involution.

This fact enables to deduce from the inequality (12) the reverse inequality and then to apply Corollary 3.4.

Remark 3. (due to A. Pełczyński) Proposition 3.1. may be strengthened as follows: Banach space X is isomorphic to a Hilbert space if for some $0 < p, q < +\infty$ there exists a constant $C > 0$ such that for each $x_1, x_2, \ldots, x_n$ in X

$$\frac{1}{C}\Big(E\Big\|\sum_{i=1}^{n} x_i\gamma_i\Big\|^p\Big)^{\frac{1}{p}} \leqslant \Big(\sum_{i=1}^{n} \|x_i\|^2\Big)^{\frac{1}{2}} \leqslant C\Big(E\Big\|\sum_{i=1}^{n} x_i\gamma_i\Big\|^q\Big)^{\frac{1}{q}}.$$

The same is true for the sequence (δ_i). This follows from the result of L. Shepp, J. Landau [7] which implies that for each $0 < p, q < +\infty$

$$\Big(E\Big\|\sum_{i=1}^{n} x_i\gamma_i\Big\|^p\Big)^{\frac{1}{p}} \text{ is equivalent to } \Big(E\Big\|\sum_{i=1}^{n} x_i\gamma_i\Big\|^q\Big)^{\frac{1}{q}}.$$

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