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Distribution function inequalities for the area integral*

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Dedicated to Antoni Zygmund

Abstract. Let A be the area integral of a function u harmonic in the Euclidean half-space $\boldsymbol{R}^n \times (0, \infty)$. Information about the distribution function of a localized version of A is obtained that leads to a general integral inequality between A and the nontangential maximal function of u and provides a convenient approach to the study of the pointwise behavior of u near the boundary. In addition, the general integral inequality of [2] between the nontangential maximal function of u and that of a properly chosen conjugate is shown to hold also in the case $n > 1$.

Our object here is to prove some partial distribution function inequalities for the area integral and to show how these inequalities can be used to study both the local and the global behavior of harmonic functions. Before describing our approach in detail, we consider a few of its applications.

Let u be harmonic in the Euclidean half-space

$$\boldsymbol{R}_+^{n+1} = \{(x, y):\ x \in \boldsymbol{R}^n,\ y > 0\}.$$

The area integral of u is the nonnegative function $A = A_a(u)$ defined on $\boldsymbol{R}^n$ by

$$A^2(x) = A_a^2(u, x) = \iint_{\Gamma(x)} y^{1-n} |\nabla u(s, y)|^2 \, ds\, dy$$

where a is a positive real number,

$$\Gamma(x) = \Gamma(x; a) = \{(s, y):\ |x-s| < ay\},$$

and $\nabla u = (\partial u/\partial y, \partial u/\partial x_1, \ldots, \partial u/\partial x_n)$. The nontangential maximal function $N = N_a(u)$ is defined by

$$N(x) = \sup_{(s,y)\in\Gamma(x)} |u(s, y)|. \tag{1}$$

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Now let Φ be any function on $[0, \infty]$ such that $0 < \Phi(1) < \infty$ and

$$\Phi(b) = \int_0^b \varphi(\lambda)\,d\lambda, \quad 0 \leqslant b \leqslant \infty,$$

for some nonnegative measurable function φ on $(0, \infty)$ satisfying the growth condition

$$(2) \qquad \varphi(2\lambda) \leqslant c\varphi(\lambda), \quad \lambda > 0.$$

Throughout the paper, the letter c denotes a positive real number, not always the same number in different lines. The growth condition on φ implies a comparable one on Φ and so $\Phi(\lambda)$ is finite and positive for all finite and positive λ. If $0 < p < \infty$, then $\Phi(b) = b^p$, $\Phi(b) = \log(b+1)$, and $\Phi(b) = (b+1)\log(b+1)$ are examples of such functions. The following theorem, one of our main results, shows that A and N are remarkably closely related.

THEOREM 1. *Under the above conditions*

$$(3) \qquad \int_{\mathbf{R}^n} \Phi(A)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi(N)\,dx.$$

If the left side of (3) *is finite, then* $\lim_{y\to\infty} u(x, y)$ *exists, and is finite and constant, for* $x \in \mathbf{R}^n$. *If u is normalized so that this limit is zero, then the converse inequality holds:*

$$(4) \qquad \int_{\mathbf{R}^n} \Phi(N)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi(A)\,dx.$$

The choice of $c_{(3)}$ *and* $c_{(4)}$ *depends only on* n, a, *and the growth constant* $c_{(2)}$.

The addition of a constant to u does not change A but ordinarily does change N; normalization is necessary to assure (4).

Theorem 1 can be used to prove inequalities of the form

$$(5) \qquad \int_{\mathbf{R}^n} \Phi\big(N(v)\big)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi\big(N(u)\big)\,dx$$

where v, normalized to vanish at $y = \infty$, is a conjugate of u in a generalized sense ([8], [15]). For the case $n = 1$, (5) is already known [2]. Here is another proof: For $n = 1$, v is conjugate to u if $u + iv$ is analytic in the upper half-plane. The Cauchy–Riemann equations imply that $|\nabla v| = |\nabla u|$, hence $A(v) = A(u)$. By applying (3) to $N(u)$, (4) to $N(v)$, and using $A(v) = A(u)$, we obtain (5). The case $n > 1$ is discussed in Section 4.

Bounds on the L^p norm of the area integral have been given by many authors. We mention especially the work, for the case $n = 1$, of Marcinkiewicz and Zygmund [9] and Calderón [5], and the work, for the case $n > 1$, of Stein [12], [14], Gasper [7], and Segovia [10]. Recently, Fefferman and Stein [6] obtained Theorem 1 for several special cases including Φ a power: $\Phi(b) = p$ $(0 < p < \infty)$. They also proved

$$\|A\|_p \leqslant c_{p,n,a} \|N_0\|_p, \quad 0 < p < \infty,$$

where N_0 is the radial maximal function:

$$N_0(x) = \sup_{y>0} |u(x, y)|.$$

It is interesting to note that if N is replaced by N_0 in (3), then (3) no longer holds for all Φ satisfying the requirements of Theorem 1; see [2], p. 152.

The Φ inequalities of Theorem 1 follow easily from the partial distribution function inequalities proved below. The latter also provide a convenient and illuminating approach to some of the results on the local behavior of harmonic functions due to Marcinkiewicz and Zygmund [9], Spencer [11], Calderón [3], [4], and Stein [13]. As an example, the following result of Calderón [3] can be mentioned: *If u is nontangentially bounded at every point of a measurable set $E \subset \mathbf{R}^n$, then u has a nontangential limit at almost every point of E.* We return to these questions in Section 3.

1. The basic inequalities. Keeping the notation already established, we consider local versions of the area integral and the nontangential maximal function. Let R be a measurable subset of $\mathbf{R}_+^{n+1}$ and A_R the nonnegative function on $\mathbf{R}^n$ defined by

$$(6) \qquad A_R^2(x) = \iint_{\Gamma(x)\cap R} y^{1-n} |\nabla u(s, y)|^2\,ds\,dy.$$

If $\Gamma(x) \cap R$ is nonempty, let

$$N_R(x) = \sup_{(s,y)\in\Gamma(x)\cap R} |u(s, y)|,$$

$$D_R(x) = \sup_{(s,y)\in\Gamma(x)\cap R} y\,|\nabla u(s, y)|;$$

otherwise, let $N_R(x) = D_R(x) = 0$. In the following theorem, $m(A_R > \lambda)$ denotes the Lebesgue measure of the set of $x \in \mathbf{R}^n$ satisfying $A_R(x) > \lambda$.

THEOREM 2. *Let G be a bounded open subset of $\mathbf{R}^n$ and R the interior of the complement of $\bigcup_{x\notin G} \Gamma(x)$. Let $a > 1$ and $\beta > 1$. Then*

$$(7) \qquad m(A_R > \lambda) \leqslant cm(cN_R > \lambda) + cm(cD_R > \lambda)$$

for all $\lambda > 0$ satisfying

$$(8) \qquad m(A_R > \lambda) < am(A_R > \beta\lambda).$$

The choice of c depends only on a, β, n and a.

In the proof, only the harmonicity of u in R is used; u need not be defined outside of R. Note that over each component of G the shape of R resembles that of a mountain range. The theorem is also true with G unbounded; this follows easily from the bounded case since A_R, N_R, and D_R increase as R increases.

In the following theorem, we need to use an interesting variant of the nontangential maximal function. If $\Gamma(x) \cap R$ is empty, let $N_R^0(x) = 0$; otherwise, let

$$N_R^0(x) = \sup_{(s,y)\in\Gamma(x)\cap R} |u(s,y)-u(s,y_s)|$$

where (s, y_s) is the point on the upper boundary of R directly above (s, y):

$$y_s = \sup\{y:\ (s,y)\in R\}.$$

THEOREM 3. *Let G be a bounded open subset of $\boldsymbol{R}^n$ and R the interior of the complement of $\bigcup_{x\notin G}\Gamma(x)$. Let $a > 1$ and $\beta > 1$. Then*

(9) $$m(N_R^0 > \lambda) \leqslant cm(cA_R > \lambda) + cm(cD_R > \lambda)$$

for all $\lambda > 0$ satisfying

(10) $$m(N_R^0 > \lambda) < am(N_R^0 > \beta\lambda).$$

The choice of c depends only on a, β, n and a.

We now use additional variants of N and A to simplify the right sides of (7) and (9). Consider the truncated cones

$$\Gamma(x;\ b, k) = \{(s, y):\ |x-s| < by,\ 0 < y < k\}$$

where b and k are positive real numbers. Let $N_{b,k}$ be defined by (1) with $\Gamma(x)$ replaced by $\Gamma(x;\ b, k)$. Define $A_{b,k}$ and $D_{b,k}$ analogously.

LEMMA 1. *Let G and R be as in Theorems 2 and 3 and let k be a positive number such that ak is not less than the diameter of G. Let $b = 2a$. Then*

(11) $$D_R \leqslant cN_{b,k},$$

(12) $$D_R \leqslant cA_{b,k},$$

and the choice of c depends only on n and a. Therefore, in Theorem 2, (7) can be replaced by

(7′) $$m(A_R > \lambda) \leqslant cm(cN_{b,k} > \lambda, G),$$

and, in Theorem 3, (9) can be replaced by

(9′) $$m(N_R^0 > \lambda) \leqslant cm(cA_{b,k} > \lambda, G).$$

Note that the comma is sometimes used to denote intersection.

Proof. The height of R does not exceed $h = \frac{1}{2}k$: if $(x, y)\in R$, then

$$ay \leqslant \inf\{|x-s|:\ s\notin G\} \leqslant ah.$$

Therefore, $D_R \leqslant D_{a,h}$. By Lemmas 4 and 5 of Stein [13], $D_{a,h} \leqslant cN_{b,k}$ and $D_{a,h} \leqslant cA_{b,k}$. Accordingly, (11) and (12) follow.

Because both N_R and D_R vanish off G and are dominated by $cN_{b,k}$, the right side of (7) is dominated by the right side of (7′). A similar comparison holds for (9) and (9′). Therefore, the last statement of Lemma 1 follows from (7) and (9).

Here is another basic inequality.

LEMMA 2. *Let $b > a > 0$. Then, for all $\lambda > 0$,*

(13) $$m(N_b > \lambda) \leqslant cm(N_a > \lambda).$$

The choice of c depends only on n and the ratio a/b.

Note that (13) implies

(14) $$m(N_{b,k} > \lambda, G) \leqslant m(N_b > \lambda) \leqslant cm(N_a > \lambda).$$

Furthermore, if Φ is as in Theorem 1, then

(15) $$\int_{\boldsymbol{R}^n} \Phi(N_b)\,dx \leqslant c\int_{\boldsymbol{R}^n} \Phi(N_a)\,dx.$$

To see this, use Fubini's theorem to obtain

$$\int_{\boldsymbol{R}^n} \Phi(N_b)\,dx = \int_0^\infty \varphi(\lambda)m(N_b > \lambda)\,d\lambda,$$

then use (13). Note that $c_{(15)} = c_{(13)}$.

Proof. Let $B(x, y) = \{s\in\boldsymbol{R}^n:\ |x-s| < y\}$. Then $B(x, ay) \subset \{N_a > \lambda\}$ for all $(x, y)\in\boldsymbol{R}_+^{n+1}$ satisfying $|u(x, y)| > \lambda$. In fact,

$$\{N_a > \lambda\} = \bigcup\{B(x, ay):\ |u(x, y)| > \lambda\}.$$

Let f be the characteristic function of this set and f^* the maximal function of f defined by

$$f^*(x) = \sup_{y>0} \int_{B(x,y)} f(s)\,ds/m\big(B(x, y)\big).$$

Then

(16) $$\{N_b > \lambda\} \subset \{f^* \geqslant \alpha\}$$

where $\alpha = a^n/(a+b)^n$. For, if $N_b(x) > \lambda$, then there is a point (s, y) satisfying $|u(s, y)| > \lambda$ and $x\in B(s, by)$. Note that $f = 1$ on $B(s, ay)$ since $B(s, ay) \subset \{N_a > \lambda\}$. Therefore,

$$\begin{aligned} f^*(x) &\geqslant m\big(B(s, ay) \cap B(x, ay+by)\big)/m\big(B(x, ay+by)\big) \\ &= m\big(B(s, ay)\big)/m\big(B(x, ay+by)\big) = \alpha, \end{aligned}$$

which proves (16). Using (16) and a variant of the Hardy–Littlewood maximal theorem (see, for example, Chapter I of [14]), we have

$$am(N_b > \lambda) \leqslant am(f^* \geqslant a) \leqslant c\|f\|_1 = cm(N_a > \lambda)$$

with the choice of c depending only on n. This completes the proof of Lemma 2.

As we have seen above, any "complete" distribution function inequality, such as (13), implies a corresponding Φ inequality. However, "partial" distribution function inequalities, such as those contained in Theorems 2 and 3, can also be used to obtain Φ inequalities as we now show.

2. Proof of Theorem 1. In addition to Theorems 2 and 3, which we prove below, we need the following elementary lemma.

LEMMA 3. *Let $f\colon \mathbf{R}^n \to [0, \infty]$ be measurable with compact support. Let Φ be as in Theorem 1 and suppose that $a > 1$, $\beta > 1$, $0 < \gamma < a/\beta$, and*

$$\varphi(\beta\lambda) \leqslant \gamma\varphi(\lambda), \qquad \lambda > 0.$$

Then

$$\int_{\mathbf{R}^n} \Phi(f)\,dx \leqslant \frac{a\beta\gamma}{a-\beta\gamma} \int_{\Lambda} \varphi(\lambda) m(f > \lambda)\,d\lambda$$

where

$$\Lambda = \{\lambda > 0\colon m(f > \lambda) < am(f > \beta\lambda)\}.$$

The easy proof is essentially contained in [1] and is omitted.

To prove (3), we apply the lemma to $f = A_R$ for R and G as in Theorem 2. Notice that A_R vanishes outside of G. Let $\beta = 2$, $\gamma = c_{(2)}$, and $a = 4\gamma$. Then, by Lemma 3,

$$\int_{\mathbf{R}^n} \Phi(A_R)\,dx \leqslant a \int_{\Lambda} \varphi(\lambda) m(A_R > \lambda)\,d\lambda$$

where Λ is the set of all $\lambda > 0$ satisfying (8). By (7′) and (14), the right side is no greater than

$$a\int_0^\infty \varphi(\lambda) cm(cN > \lambda)\,d\lambda = ca \int_{\mathbf{R}^n} \Phi(cN)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi(N)\,dx.$$

Therefore, (3) holds with A replaced by A_R. Now let $R \uparrow \mathbf{R}_+^{n+1}$. By the monotone convergence theorem, (3) follows.

We now consider the converse inequality and let $b = 2a$. Using the same pattern of reasoning as above, here in conjunction with Theorem 3, we obtain

$$\int_{\mathbf{R}^n} \Phi(N_R^0)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi(A_b)\,dx. \tag{17}$$

We assume from now on that the right side of (17) is finite and show first that $u(0, y)$ converges as $y \to \infty$. We restrict our attention to the regions $R = R_t$ corresponding to $G = B(0, at) = \{x\colon |x| < at\}$. Suppose that x is any point in $\mathbf{R}^n$ and that $|x| < ay < az < at$. Then $(0, y)$ and $(0, z)$ belong to $\Gamma(x) \cap R$ and it follows from the definition of N_R^0 that

$$|u(0, y) - u(0, z)| \leqslant |u(0, y) - u(0, t)| + |u(0, z) - u(0, t)| \leqslant 2N_R^0(x).$$

Therefore,

$$\delta = \tfrac{1}{2}\limsup_{y,z\to\infty} |u(0, y) - u(0, z)| \leqslant \liminf_{t\to\infty} N_R^0(x)$$

and, by Fatou's lemma and (17), we have

$$\int_{\mathbf{R}^n} \Phi(\delta)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi(A_b)\,dx < \infty,$$

which gives $\Phi(\delta) = 0$. Therefore, $\delta = 0$ and this implies that $u(0, y)$ converges as $y \to \infty$.

Using the mean value theorem and (12), we have that

$$|u(s, y) - u(0, y)| \leqslant \sup_{t>0} D_R(x_0)\,|s|\,y^{-1} \leqslant cA_b(x_0)\,|s|\,y^{-1}$$

provided $|x_0 - s| < ay$ and $|x_0| < ay$. Since $A_b(x_0)$ is finite for at least one x_0,

$$\lim_{y\to\infty} |u(s, y) - u(0, y)| = 0,$$

and the convergence is uniform for $s \in B(0, r)$. This proves the existence, finiteness, and constancy of the limit of $u(\cdot, y)$ as $y \to \infty$. From now on, assume this limit is 0.

Let

$$f_{r,R}(x) = \sup\{|u(s, y) - u(s, y_s)|\colon (s, y) \in \Gamma(x) \cap R,\ |s| < r\},$$
$$f_r(x) = \sup\{|u(s, y)|\colon (s, y) \in \Gamma(x),\ |s| < r\}.$$

As usual, if the sets are empty, $f_{r,R}(x) = f_r(x) = 0$. Then $f_{r,R} \leqslant N_R^0$ and

$$\lim_{t\to\infty} f_{r,R} = f_r, \quad \lim_{r\to\infty} f_r = N.$$

Using (17) and Fatou's lemma, we obtain

$$\int_{\mathbf{R}^n} \Phi(N)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi(A_b)\,dx.$$

In view of (15), this inequality also holds with N replaced by N_b. This completes the proof of Theorem 1.

3. Local behavior of harmonic functions. Our aim here is to illustrate how partial distribution function inequalities can be used to study problems of local behavior. First, consider the following result of Calderón [4]: *If u is nontangentially bounded at every point of a measurable set E, then $A_{a,k}$ is finite almost everywhere on E.* The first step of the proof is to show that

$$E \underset{\text{a.e.}}{\subseteq} \{N_{b,k} < \infty\}.$$

This is not difficult and rests on a familiar point of density argument (see Calderón [4]). The *key* step is to show that

$$\{N_{b,k} < \infty\} \underset{\text{a.e.}}{\subseteq} \{A_{a,k} < \infty\}. \tag{18}$$

We need to do this only for $b = 2a$. If (18) does not hold, then

$$m(A_{a,k} = \infty,\ cN_{b,k} \leqslant \lambda) > 0, \tag{19}$$

for some $\lambda > 0$, with c the constant appearing in (7′). Let E_0 be a measurable subset of the set in (19) with diameter not greater than $\frac{1}{2}ak$ and such that $m(E_0) > 0$. Let G_j be an open set containing E_0 with diameter not greater than ak and such that $m(G_j - E_0) \to 0$ as $j \to \infty$. Let R_j be the region, defined in Theorem 2, corresponding to G_j. Note that $A_{R_j}(x) = \infty$ for $x \in E_0$. Therefore,

$$m(E_0) \leqslant m(A_{R_j} > \beta\lambda) \leqslant m(A_{R_j} > \lambda) \leqslant m(G_j)$$

and, since $m(G_j) \to m(E_0)$, the number λ satisfies (8) for all large j. So, for such j, we can apply Theorem 2 and Lemma 1 to obtain

$$m(A_{R_j} > \lambda) \leqslant cm(cN_{b,k} > \lambda,\ G_j).$$

But the right side converges to $cm(cN_{b,k} > \lambda, E_0) = 0$ while the left side converges to $m(E_0)$. Accordingly, (19) cannot hold and the proof of the key step is complete.

Now consider Stein's result in the opposite direction (see [13], which also contains some remarks about the historical background): *If $A_{b,k}$ is finite at every point of a measurable set E, then u has a nontangential limit at almost every point of E.* Let $\varepsilon > 0$. We denote by M the set of all $x \in \boldsymbol{R}^n$ such that either

$$\limsup u(s, y) - \liminf u(s, y) > \varepsilon$$

or

$$\limsup |u(s, y)| = \infty$$

as $(s, y) \to (x, 0)$, $(s, y) \in \Gamma(x; a, k)$. Stein's result is a simple consequence of the fact that

$$m(E \cap M) = 0 \tag{20}$$

for all $\varepsilon > 0$, $a > 0$.

We show below that (20) holds for all $\varepsilon > 0$ with $a = \frac{1}{2}b$. This is enough! The special case implies that u is nontangentially bounded almost everywhere on E. By Calderón's result, we have that

$$E \underset{\text{a.e.}}{\subseteq} \bigcup_{b>0} \{A_{b,k} < \infty\}.$$

Therefore, by the special case, we have that (20) holds for all a.

Now fix $\varepsilon > 0$ and let $a = \frac{1}{2}b$. Let $\beta > 1$ as in Theorem 3 and choose λ to satisfy $2\beta\lambda = \varepsilon$. If (20) does not hold, then $m(E \cap M) > 0$ and, for h sufficiently small,

$$m(cA_{b,h} \leqslant \lambda,\ M) > 0. \tag{21}$$

Here we have used the fact that $A_{b,h}(x) \to 0$ as $h \to 0$ for all $x \in E$. Choose c to be the constant appearing in (9′). Let E_0 be a measurable subset of the set in (21) with diameter not greater than $\frac{1}{2}ah$ and such that $m(E_0) > 0$. Let G_j be an open set containing E_0 with diameter not greater than ah and such that $m(G_j - E_0) \to 0$ as $j \to \infty$. Let R_j be the region, defined in Theorem 3, corresponding to G_j. Let $x \in E_0$. Using the definition of M and λ, the continuity of u in a neighborhood of (x, y_x), and the fact that $|y_s - y_x| \leqslant a^{-1}|s - x|$, we obtain $N^0_{R_j}(x) > \beta\lambda$. Therefore,

$$m(E_0) \leqslant m(N^0_{R_j} > \beta\lambda) \leqslant m(N^0_{R_j} > \lambda) \leqslant m(G_j)$$

and λ satisfies (10) for all large j. Applying Theorem 3 and Lemma 1, we have

$$m(E_0) = \lim_{j\to\infty} m(N^0_{R_j} > \lambda) \leqslant \lim_{j\to\infty} cm(cA_{b,h} > \lambda, G_j) = cm(cA_{b,h} > \lambda, E_0) = 0,$$

a contradiction. This completes the proof of Stein's result.

4. Conjugate harmonic functions. Suppose $u, v_1, v_2, \ldots, v_n$ are harmonic in $\boldsymbol{R}^{n+1}_+$ and satisfy the generalized Cauchy–Riemann equations:

$$\frac{\partial u}{\partial y} + \frac{\partial v_1}{\partial x_1} + \ldots + \frac{\partial v_n}{\partial x_n} = 0,$$

$$\frac{\partial u}{\partial x_k} = \frac{\partial v_k}{\partial y}, \quad \frac{\partial v_k}{\partial x_j} = \frac{\partial v_j}{\partial x_k}, \qquad j, k = 1, \ldots, n.$$

Stein and Weiss [15] and Stein [13], [14] have made a deep study of such systems of conjugate harmonic functions. Here let $v = (v_1, \ldots, v_n)$,

$$|v| = \Big(\sum_{k=1}^{n} v_k^2\Big)^{\frac{1}{2}},$$

and define $N(v)$ by (1) with u replaced by v. Then we have the following generalization of Theorem 2 of [2]:

THEOREM 4. *Let u be harmonic in $\mathbf{R}_+^{n+1}$, let Φ be as in Theorem 1, and suppose the right side of* (22), *below, is finite. Then there is a $v = (v_1, \ldots, v_n)$, conjugate to u in the above sense, such that*

$$\int_{\mathbf{R}^n} \Phi\big(N(v)\big)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi\big(N(u)\big)\,dx. \tag{22}$$

The choice of c depends only on n, a, and the growth constant of φ.

Proof. There is a v, conjugate to u, such that v vanishes at $y = \infty$:

$$\lim_{y\to\infty} v(x, y) = 0. \tag{23}$$

For the case $n = 1$, the proof of this is contained in [2]; the proof of the general case is similar with Riesz transforms playing the role of the Hilbert transform. Now (23) implies that $\partial v_k/\partial x_j$ vanishes at $y = \infty$; using this, we obtain

$$\iint_{\Gamma(x;a)} y^{1-n}\left|\frac{\partial v_k}{\partial x_j}\right|^2 ds\,dy \leqslant c \iint_{\Gamma(x;b)} y^{1-n}\left|\frac{\partial v_k}{\partial y}\right|^2 ds\,dy$$

by slightly modifying the proof of Stein's Lemma 2.5.1 ([14], p. 213). Here $b = 2a$. Therefore, using $\partial v_k/\partial y = \partial u/\partial x_k$, we have

$$A(v_k) \leqslant cA_b(u). \tag{24}$$

Hence, by Theorem 1 and (15),

$$\int_{\mathbf{R}^n} \Phi\big(N(v_k)\big)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi\big(A(v_k)\big)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi\big(A_b(u)\big)\,dx$$
$$\leqslant c \int_{\mathbf{R}^n} \Phi\big(N_b(u)\big)\,dx \leqslant c \int_{\mathbf{R}^n} \Phi\big(N(u)\big)\,dx.$$

Theorem 4 now follows with the use of the following elementary facts:

$$N(v) \leqslant \sum_{k=1}^{n} N(v_k),$$

$$\Phi\Big(\sum_{k=1}^{n} \lambda_k\Big) \leqslant \sum_{k=1}^{n} \Phi(2^n\lambda_k) \leqslant c \sum_{k=1}^{n} \Phi(\lambda_k), \quad \lambda_k \geqslant 0,\ k = 1, \ldots, n.$$

5. Proof of Theorem 2. We need the following lemma.

LEMMA 4. *Suppose that G is an open bounded nonempty subset of $\mathbf{R}^n$ and that F is its complement. Let $a > 1$ and suppose that E is a measurable subset of G satisfying $m(G) \leqslant am(E)$. Then there is a ball $B \subset G$, with at least one of its boundary points in F, such that*

$$m(B) \leqslant cam(E \cap B). \tag{25}$$

The choice of c depends only on n.

Proof. There is a Whitney cube $Q \subset G$ (for example, see [14], p. 16) such that

$$m(Q) \leqslant am(E \cap Q), \tag{26}$$

$$q \leqslant d \leqslant 4q, \tag{27}$$

where q is the diameter of Q and d is the distance from Q to F. Now consider all balls centered at the center of Q, containing Q, and contained in G. By the left side of (27), this family is nonempty; let B denote its union. The maximality of B in the above family assures that at least one of its boundary points is in F. By (26) and the right side of (27), we obtain (25).

We now turn to the proof of Theorem 2. For the sake of simplicity and with no essential loss of generality, we assume that $a = 1$.

Suppose that λ satisfies (8). Let $R_\varepsilon = \{(x, y)\in R\colon y > \varepsilon\}$ for $\varepsilon > 0$. Since $A_{R_\varepsilon} \to A_R$ as $\varepsilon \to 0$, the inequality

$$m(A_{R_\varepsilon} > \lambda) < am(A_{R_\varepsilon} > \beta\lambda) \tag{28}$$

holds for all small ε. As we now show, this inequality implies that (7) holds with R replaced by R_ε. The result for R follows by letting $\varepsilon \to 0$.

Note that A_{R_ε} is a continuous function vanishing outside of G. Therefore, $G_0 = \{A_{R_\varepsilon} > \lambda\}$ is an open set whose closure is contained in G. Let

$$E = \{A_{R_\varepsilon} \geqslant \beta\lambda,\ N_{R_\varepsilon} \leqslant \gamma\lambda,\ D_{R_\varepsilon} \leqslant \delta\lambda\}$$

where γ and δ are positive numbers to be chosen later. Assume that (28) holds. Then

$$m(G_0) \leqslant am(E) + am(N_{R_\varepsilon} > \gamma\lambda) + am(D_{R_\varepsilon} > \delta\lambda).$$

The key step of the proof is to show that

$$am(E) \leqslant \tfrac{1}{2} m(G_0) \tag{29}$$

provided γ and δ are suitably chosen, the choice to depend only on a, β, and n. The desired inequality follows:

$$m(G_0) \leqslant 2am(N_{R_\varepsilon} > \gamma\lambda) + 2am(D_{R_\varepsilon} > \delta\lambda).$$

Suppose that (29) does not hold. Then $m(G_0) < 2am(E)$ and, by Lemma 4, there is a ball $B \subset G_0$, with at least one of its boundary points not in G_0, such that

$$m(B) \leqslant a_0 m(E \cap B),$$

where $a_0 = 2ac_{(25)}$. Without loss of generality, assume that B is centered at the origin and has unit radius. Let V be the interior of the cone with base B and vertex at $(0, 1)$. Then the closure of $V_\varepsilon = \{(x, y)\in V\colon y > \varepsilon\}$ is contained in R. Choose $0 < \eta < \frac{1}{2}$ so that the ball B_0 with center at the origin and radius $1-2\eta$ satisfies $m(B_0)/m(B) = 1-(2a_0)^{-1}$. Then, letting $E_0 = E \cap B_0$, we have that

$$(30) \qquad m(B) \leqslant 2a_0 m(E_0).$$

Finally, let

$$W = \bigcup_{x\in E_0} \Gamma(x) \cap V_\varepsilon .$$

Note that $W \subset V_\varepsilon$ and, except for its more "mountainous" lower boundary, W has roughly the same basic shape as V_ε. Certainly, W is a Lipschitz domain. Observe that

$$(31) \qquad |u(s,y)| \leqslant \gamma\lambda, \qquad y|\nabla u(s,y)| \leqslant \delta\lambda$$

for all $(s,y)\in W$, hence for all $(s,y)\in\partial W$; indeed, these inequalities hold for all (s,y) in $\Gamma(x)\cap R_\varepsilon$ if $x\in E_0$.

Now consider the area integral relative to the domain W; see (6). For a suitable δ, the choice of which depends only on a, β, and n, we have that

$$(32) \qquad A_W^2(x) \geqslant \tfrac{1}{2}(\beta^2-1)\lambda^2, \qquad x\in E_0.$$

To prove this, we fix $x\in E_0$ and observe first that

$$\beta^2\lambda^2 \leqslant A_{R_\varepsilon}^2(x) = A_W^2(x) + A_{U_1}^2(x) + A_{U_2}^2(x) + A_{U_3}^2(x)$$

where

$$U_1 = \{(s,y)\in R_\varepsilon\colon |s| < y-1,\ y>1\},$$
$$U_2 = \{(s,y)\in \Gamma(x)\cap R_\varepsilon\colon |s| > y-1,\ y>1\},$$
$$U_3 = \{(s,y)\in \Gamma(x)\cap R_\varepsilon\colon (s,y)\notin W,\ y<1\}.$$

Using now the fact that B has a boundary point, say x_0, not in G_0, we obtain $U_1 \subset \Gamma(x_0)\cap R_\varepsilon$ and

$$A_{U_1}^2(x) \leqslant A_{R_\varepsilon}^2(x_0) \leqslant \lambda^2.$$

Let v_n denote the volume of a ball of unit radius in $\boldsymbol{R}^n$. Then, by (31),

$$A_{U_2}^2(x) = \iint_{\Gamma(x)\cap U_2} y^{1-n}|\nabla u(s,y)|^2\,ds\,dy \leqslant \int_1^\infty \int_{\substack{|x-s|<y\\ |s|>y-1}} y^{1-n}\delta^2\lambda^2 y^{-2}\,ds\,dy$$
$$= \delta^2\lambda^2 \int_1^\infty [v_n y^n - v_n(y-1)^n]y^{-n-1}\,dy \leqslant v_n 2^n\delta^2\lambda^2 \int_1^\infty y^{-2}\,dy = c\,\delta^2\lambda^2.$$

We now use the fact that $x\in B_0$. This means that $|x| < 1-2\eta$ and if $(s,y)\in U_3$, then $\eta < y < 1$. Therefore,

$$A_{U_3}^2(x) \leqslant \int_\eta^1 \int_{|s|<y+1} y^{1-n}\delta^2\lambda^2 y^{-2}\,ds\,dy = c\delta^2\lambda^2.$$

These estimates imply the statement containing (32).

We now use Green's theorem assuming that the above region W is smooth enough for the theorem to apply. Whether or not this assumption is generally valid, W can always be approximated by a smoother region (see [13], [14]) to achieve the same end.

Let σ denote the measure of the surface area of W and let $\partial/\partial n$ denote the directional derivative along the inward normal. Then $\sigma(\partial W) \leqslant 4\,m(B)$ since $|\partial y/\partial n| \geqslant 2^{-\frac{1}{2}}$ almost everywhere with respect to σ. Now using (32), (31), (30), and the identity $\Delta u^2 = 2|\nabla u|^2$, we have that

$$(\beta^2-1)\lambda^2 m(E_0) \leqslant 2\int_{\boldsymbol{R}^n} A_W^2(x)\,dx = v_n \iint_W y\,\Delta u^2\,ds\,dy$$
$$= v_n \int_{\partial W} u^2\frac{\partial y}{\partial n}\,d\sigma - 2v_n\int_{\partial W} y\frac{\partial u}{\partial n}u\,d\sigma$$
$$\leqslant v_n\gamma^2\lambda^2\sigma(\partial W) + 2v_n\gamma\delta\lambda^2\sigma(\partial W)$$
$$\leqslant 4v_n(\gamma^2+2\gamma\delta)\lambda^2 m(B)$$
$$\leqslant 8a_0 v_n(\gamma^2+2\gamma\delta)\lambda^2 m(E_0).$$

This gives a contradiction for γ suitably small. This completes the proof of (29).

6. Proof of Theorem 3. Again, we set $a=1$ and let $R_\varepsilon = \{(x,y)\in R\colon y>\varepsilon\}$ for $\varepsilon>0$. Suppose that λ satisfies (10). Since $N_{R_\varepsilon}^0 \uparrow N_R^0$ as $\varepsilon\downarrow 0$, the inequality

$$(33) \qquad m(N_{R_\varepsilon}^0 > \lambda) < a m(N_{R_\varepsilon}^0 > \beta\lambda)$$

holds for all small ε. This inequality implies that (9) holds with R replaced by R_ε as we now show. The result for R follows by letting $\varepsilon\to 0$.

If x is a point of G within δ of the boundary of G, then $|y-y_\varepsilon| \leqslant \delta$ for $(s,y)\in\Gamma(x)\cap R_\varepsilon$. By the uniform continuity of u on R_ε, we have that $N_{R_\varepsilon}^0$ is a continuous function vanishing outside of G. Therefore, $G_0 = \{N_{R_\varepsilon}^0 > \lambda\}$ is an open set whose closure is contained in G.

Let f be the characteristic function of the set $\{A_{R_\varepsilon} > \gamma\lambda\}$ where γ is a positive number to be chosen later. Here, let f^* be the maximal function of f defined by

$$f^*(x) = \sup_{x\in B} \int_B f(s)\,ds/m(B)$$

where B is any ball containing x. Let

$$E = \{N_{R_\varepsilon}^0 \geqslant \beta\lambda,\ f^* \leqslant \tfrac{1}{2},\ D_R \leqslant \delta\lambda\}$$

where δ is a positive number to be chosen later. Assume that (33) holds. Then

$$m(G_0) \leqslant am(E) + am(f^* > \tfrac{1}{2}) + am(D_{R_\varepsilon} > \delta\lambda).$$

By the Hardy–Littlewood maximal theorem,

$$\tfrac{1}{2} m(f^* > \tfrac{1}{2}) \leqslant c\|f\|_1 = cm(A_{R_\varepsilon} > \gamma\lambda)$$

where the choice of c depends only on n. Therefore, if

$$am(E) \leqslant \tfrac{1}{2} m(G_0) \tag{34}$$

holds for suitable γ and δ, the choice of which depends only on a, β, and n, then the desired inequality also holds:

$$m(G_0) \leqslant cm(A_{R_\varepsilon} > \gamma\lambda) + 2am(D_{R_\varepsilon} > \delta\lambda).$$

So the key step is to prove (34). Suppose, on the contrary, that $m(G_0) < 2am(E)$. Then, by Lemma 4, there is a ball $B \subset G_0$, with at least one of its boundary points not in G_0, such that

$$m(B) \leqslant a_0 m(E \cap B),$$

where $a_0 = 2ac_{(25)}$. Without loss of generality, assume that B is centered at the origin and has unit radius. Let V be the interior of the cone with base B and vertex at $(0,1)$. Then the closure of $V_\varepsilon = \{(x,y) \in V\colon\ y > \varepsilon\}$ is contained in R. Choose $0 < \eta < \frac{1}{2}$ so that the ball B_0 with center at the origin and radius $1-2\eta$ satisfies $m(B_0)/m(B) = 1-(2a_0)^{-1}$. Then, letting $E_0 = E \cap B_0$, we have that

$$m(B) \leqslant 2a_0 m(E_0). \tag{35}$$

We need to consider several domains $W \subset W_0 \subset W_1$. Let

$$W_i = \bigcup_{x \in E_i} \Gamma(x) \cap V_\varepsilon, \quad i = 0, 1,$$

where $E_1 = \{f^* \leqslant \frac{1}{2}\} \cap B$. Notice first that

$$\int_{E_2} A^2_{W_1}(x)\,dx \leqslant \gamma^2\lambda^2 m(B),$$

where $E_2 = \{A_{R_\varepsilon} \leqslant \gamma\lambda\} \cap B$, since $A_{W_1} \leqslant A_{R_\varepsilon}$. Let $B(s,y) = \{x\colon\ |x-s| < y\}$. Then, by Fubini's theorem,

$$\int_{E_2} A^2_{W_1}(x)\,dx = \iint_{W_1} y^{1-n} |\nabla u(s,y)|^2 m\big(E_2 \cap B(s,y)\big)\,ds\,dy.$$

We can get a lower bound for this integral as follows. Suppose that $(s,y) \in W_1$, Then, for some $x \in E_1$, $(s,y) \in \Gamma(x) \cap V_\varepsilon$. Hence, $x \in B(s,y)$ and $f^*(x) \leqslant \frac{1}{2}$ so that

$$m\big(A_{R_\varepsilon} > \gamma\lambda,\ B(s,y)\big) \leqslant \tfrac{1}{2} m\big(B(s,y)\big)$$

by the definition of f^*. Since $B(s,y) \subset B$, this implies that

$$m\big(E_2 \cap B(s,y)\big) \geqslant \tfrac{1}{2} m\big(B(s,y)\big) = \tfrac{1}{2} v_n y^n$$

where v_n is the volume of a ball with unit radius. Combining these estimates, we obtain

$$2\gamma^2\lambda^2 m(B) \geqslant v_n \iint_{W_1} y\,|\nabla u|^2\,ds\,dy \tag{36}$$

and an even smaller integral if we replace W_1 by the smaller domain W, which we define below.

From now on, we assume that $u(0,1) = 0$. This we can do without loss of generality since $N^0_{R_\varepsilon}$ and ∇u are unchanged if u is replaced by $u - u(0,1)$. Using this assumption, we now show that

$$N_{W_0}(x) > \tfrac{1}{2}(\beta - 1)\lambda, \quad x \in E_0, \tag{37}$$

for a suitable choice of δ depending only on a, β, and n. We fix $x \in E_0$ and observe first that

$$\beta\lambda \leqslant N^0_{R_\varepsilon}(x) \leqslant N^0_{W_0}(x) + N^0_{U_1}(x) + N^0_{U_2}(x)$$

where

$$U_1 = \{(s,y) \in R_\varepsilon\colon\ |s| < y - 1\},$$

$$U_2 = \{(s,y) \in \Gamma(x) \cap R_\varepsilon\colon\ (s,y) \notin W_0 \cup U_1\}.$$

Note that the upper boundary of U_1 is part of the upper boundary of R_ε. We now use the fact that B has a boundary point, say x_0, not in G_0 to obtain $U_1 \subset \Gamma(x_0) \cap R_\varepsilon$ and

$$N^0_{U_1}(x) \leqslant N^0_{R_\varepsilon}(x_0) \leqslant \lambda.$$

If $(s,y) \in \Gamma(x) \cap R_\varepsilon$, then

$$y\,|\nabla u(s,y)| \leqslant \delta\lambda \tag{38}$$

by the definition of E_0. We know by the mean value theorem that if (38) is satisfied for every point (s,y) on the line segment joining the points (s_1,y_1) and (s_2,y_2) in $\boldsymbol{R}^{n+1}_+$, then

$$|u(s_1,y_1) - u(s_2,y_2)| \leqslant \delta\lambda\big(|s_1-s_2|^2 + (y_1-y_2)^2\big)^{\frac{1}{2}}/y_1 \wedge y_2. \tag{39}$$

Here $y_1 \wedge y_2$ denotes the minimum of y_1 and y_2. Now let (s,y_1) and (s,y_2) belong to U_2. Then $|s| \geqslant y_i - 1$; otherwise (s,y_i) would belong to U_1. Also, $|x-s| < y_i$, implying that $|s| < y_i + |x| < y_i + 1$. Therefore

$$|s| - 1 \leqslant y_i \leqslant |s| + 1, \quad i = 1, 2,$$

so that $|y_1 - y_2| \leqslant 2$. Using the fact that $x \in E_0 \subset B_0$, we obtain $y_1 \wedge y_2 > \eta$. Therefore, by (39), we have

$$N^0_{U_2}(x) \leqslant 2\delta\lambda\eta^{-1}.$$

Using our assumption that $u(0,1)=0$, we also have

(40) $$|u(s,y)| = |u(s,y)-u(0,1)| \leqslant 2\delta\lambda\eta^{-1}$$

for all upper boundary points (s,y) of W_0; this gives

$$N^0_{W_0}(x) \leqslant N_{W_0}(x) + 2\delta\lambda\eta^{-1}.$$

Combining these estimates, we obtain (37) for all positive δ satisfying

(41) $$\delta < 2^{-3}\eta(\beta-1).$$

We can now define W. Let $\Gamma(x,t)$ denote the cone $\Gamma(x)$ translated upward by t units; that is,

$$\Gamma(x,t) = \{(s,y)\colon |x-s| < y-t\}.$$

If $(x,t)\in W_0$, let

$$N(x,t) = \sup_{(s,y)\in\Gamma(x,t)\cap W_0} |u(s,y)|.$$

Finally, let $\theta = \frac{1}{2}(\beta-1)$ and

$$W = \{(x,t)\in W_0\colon N(x,t) < \theta\lambda\}.$$

By continuity and the fact that $u(0,1)=0$, W is a nonempty open set. If $(x,t)\in W$, let (x,t_x) be the point on the lower boundary of W directly below (x,t):

$$t_x = \inf\{t\colon (x,t)\in W\}.$$

Then $|t_x-t_s| \leqslant |x-s|$ for all x and s in the projection P_W of W on $\boldsymbol{R}^n$. For δ satisfying (41), which we assume, the upper boundary of W coincides with the upper boundary of W_0. Clearly, W is a Lipschitz domain.

By the definition of W,

(42) $$|u(s,y)| \leqslant \theta\lambda$$

for all $(s,y)\in W$, hence for all $(s,y)\in\partial W$. Our next step is to show that, in a sense, $|u|$ is near $\theta\lambda$ on a large part of the lower boundary of W. To be precise, we let

$$S = \{(x,t_x)\colon |u(x,t_x)| > \tfrac{1}{2}\theta\lambda\}$$

and denote by P_S the projection of S on $\boldsymbol{R}^n$. Then

(43) $$m(P_S) \geqslant cm(B)$$

with the choice of c depending only on α and n. To prove this, we let g be the characteristic function of P_S and g^* be the maximal function of g as defined in the proof of Lemma 2. Then

(44) $$E_0 \subset \{g^* \geqslant \xi\}$$

for a positive number ξ whose choice depends only on η and n, hence only on α and n. Inequality (43) easily follows from (44) and (35) since, by the Hardy–Littlewood theorem,

$$\xi m(E_0) \leqslant \xi m(g^*\geqslant\xi) \leqslant c\|g\|_1 = cm(P_S).$$

We now prove (44). Suppose that $x_0\in E_0$. Then, by (37), $N_{W_0}(x_0) > \theta\lambda$, so there is a point $(s,y)\in\Gamma(x_0)\cap W_0$ such that $|u(s,y)| > \theta\lambda$. Let (x,t_x) be a point on the lower boundary of W, in $\Gamma(s,y)\cap W_0$, and satisfying $|u(x,t_x)| = \theta\lambda$. By (40) and (41), such a point exists. If $x=x_0$, we have $g^*(x_0)=1$; if $x\neq x_0$, we proceed as follows. By (39), (41), and the fact that $u(0,1)=0$, we have $\Gamma(x_0)\cap V_\eta\subset W$. Therefore, $B(x_0,\eta)\subset P_W$, $|x-x_0| < t_x < \eta$, and

$$B(x_0,|x-x_0|)\subset P_W.$$

Let s be any point in

$$B\big(x,(1-\eta)|x-x_0|\big)\cap B(x_0,|x-x_0|).$$

Then $|t_x-t_s| \leqslant |x-s| < (1-\eta)|x-x_0| < |x-x_0|$,

$$t_s \geqslant t_x - |x-s| > |x-x_0| - (1-\eta)|x-x_0| = \eta|x-x_0|,$$

and, using (39) once more, we obtain

$$\begin{aligned}\theta\lambda - |u(s,t_s)| &\leqslant |u(x,t_x)-u(s,t_s)|\\ &\leqslant \delta\lambda\big(|x-s|^2+(t_x-t_s)^2\big)^{\frac{1}{2}}/t_x\wedge t_s\\ &\leqslant 2\delta\lambda\eta^{-1}.\end{aligned}$$

So, by (41), we have $|u(s,t_s)| > \frac{1}{2}\theta\lambda$ and s is in P_S. Therefore,

$$g^*(x_0) \geqslant m\big(P_S\cap B(x_0,|x-x_0|)\big)/m\big(B(x_0,|x-x_0|)\big) \geqslant \xi$$

where

$$\xi = m\Big(B\big(x,(1-\eta)|x-x_0|\big)\cap B(x_0,|x-x_0|)\Big)/m\big(B(x_0,|x-x_0|)\big),$$

which clearly depends only on η and n. This completes the proof of (44).

Keeping the same meaning for $\partial/\partial n$ and σ, we now apply Green's theorem with the same proviso as in Section 5. Denote the lower boundary of W by $(\partial W)^-$ and the upper boundary by $(\partial W)^+$. We have $\partial y/\partial n \geqslant 2^{-\frac{1}{2}}$ on $(\partial W)^-$, $\sigma(S)\geqslant m(P_S)$, and $\sigma(\partial W)\leqslant 4m(B)$. Using (38), (40), (42), and (43), we obtain

$$\begin{aligned}2\iint_W y|\nabla u|^2\,ds\,dy &= \iint_W y\Delta u^2\,ds\,dy\\ &= \int_{(\partial W)^-} u^2\frac{\partial y}{\partial n}\,d\sigma + \int_{(\partial W)^+} u^2\frac{\partial y}{\partial n}\,d\sigma - 2\int_{\partial W} y\frac{\partial u}{\partial n}u\,d\sigma\\ &\geqslant \int_S 2^{-3}\theta^2\lambda^2\,d\sigma - \int_{(\partial W)^+}(2\delta\lambda\eta^{-1})^2\,d\sigma - 2\int_{\partial W}\delta\theta\lambda^2\,d\sigma\\ &\geqslant \lambda^2 m(B)\,[c_{(43)}2^{-3}\theta^2 - 16\,\delta^2\eta^{-2} - 8\delta\theta].\end{aligned}$$

By (36), we have

$$4\gamma^2 \geqslant v_n[c_{(43)}2^{-3}\theta^2 - 16\delta^2\eta^{-2} - 8\delta\theta],$$

and we see that this leads to a contradiction if both δ and γ are chosen suitably small. Therefore, (34) must hold and the proof of Theorem 3 is complete.

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An inequality for the indefinite integral of a function in L^q

by

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To A. Zygmund, real analyst

Abstract. Let p, q be positive numbers with $1 = 1/p + 1/q$. Let F be continuously differentiable on the positive reals and zero at the origin. Let g denote the pth power of $|F|$. J. Moser has shown that if q is at least 2 and the q norm of F' is at most 1 then the integral of $\exp(g(x)-x)$ is bounded by a constant depending only on q. A new proof of this is given, and the result extended to all $q > 1$.

In his paper "*A sharp form of an inequality by N. Trudinger*", J. Moser ([2], Theorem 1) proves that if D is a bounded domain in R^n, $n \geqslant 2$, and u is a C^1 function with compact support in D such that $\int_D |\operatorname{grad} u(x)|^n dx \leqslant 1$ then $\int_D \exp \alpha_n |u(x)|^{n/(n-1)} dx \leqslant c_n$ for certain constants α_n, c_n independent of u.

Earlier N. Trudinger [3] proved this for some $\alpha > 0$.

Moser elegantly reduces the question to the following one-dimensional inequality which he proves for $q \geqslant 2$.

The present paper contains a new proof which incidentally works for $q > 1$.

THEOREM 1.1. *Let q and p denote positive numbers with $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ (Hölder conjugates). Let f be Lebesgue measurable on $(0, \infty)$, let $\int_0^\infty |f(x)|^q dx \leqslant 1$, and let $F(x) = \int_0^x f(t)\,dt$. Then there exists a number C_q depending only on q such that*

$$\int_0^\infty e^{|F(x)|^p} e^{-x} dx \leqslant C_q. \tag{1.2}$$

In what follows we consider only non-negative functions f. An equivalent theorem arises through use of the substitution $x = \log\frac{1}{u}$:

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