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Convolution with odd kernels*

by

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Dedicated to A. Zygmund

Abstract. Let $k(x)$ be an odd kernel which is positive and monotone decreasing for $x > 0$. Let $g(x)$ be the convolution of $f(x)$ with $k(x)$. Then under certain conditions we have the inequality,

$$g^{-}(s) \leqslant 4 \int_0^\infty f^{-}(t) k(t) \sinh^{-1}\left(\frac{t}{s}\right) dt.$$

where f^{-} denotes the Hardy–Littlewood maximal function of f^*, the rearrangement of f onto $(0, \infty)$. This inequality may be used to derive a number of results particularly in the theory of Orlicz spaces.

The Hilbert transform, g, of a function, f, is given by convolution on $(-\infty, \infty)$ with the odd singular kernel, $H(x) = 1/\pi x$.

$$g(x) = \frac{1}{\pi} \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} \frac{f(x-t)}{t} dt.$$

If given a function f on $(-\infty, \infty)$ we let f^* denote the non-increasing rearrangement of $|f|$ onto $(0, \infty)$ and let f^{-} be the Hardy–Littlewood maximal function of f^*; for $x > 0$,

$$f^{-}(x) = \frac{1}{x} \int_0^x f^*(t) dt$$

(see [2], particularly Theorem 1 on p. 191 for the result below), then if g is the Hilbert transform of f, then f^{-} and g^{-} are related by the following inequality: for $s > 0$,

$$g^{-}(s) \leqslant \frac{2}{\pi} \int_0^\infty \frac{f^{-}(t)}{\sqrt{s^2+t^2}} dt. \tag{1}$$

Our object is to derive a similar inequality in case we replace the Hilbert transform by convolution with an odd kernel whose singularity at 0 is milder than the singularity of $1/\pi x$.

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The principal result is:

THEOREM 1. *Let $K(t)$ be an odd kernel and let $L(t) = -tK'(t)$. Suppose that*

(i) *For $t>0$, both $K(t)$ and $L(t)$ are positive and monotone decreasing tending to 0 as $t\to\infty$.*

(ii) *For $t>0$, both $tK(t)$ and $tL(t)$ are monotone increasing and both tend to 0 as $t\to 0$.*

(iii) $\int_1^\infty \frac{K(t)}{t}\,dt<\infty$.

Then $K(t)$ is the Hilbert transform of $k(t)$, an even kernel which is positive, locally integrable and decreasing for $t>0$. Moreover if

$$g(x)=\lim_{\varepsilon\to 0}\int_{|t|>\varepsilon} f(x-t)K(t)\,dt$$

then for $s>0$,

$$g^-(s)\leqslant 4\int_0^\infty \big(f^-(t)-f^*(t)\big)K(t)\sinh^{-1}\left(\frac{t}{s}\right)dt.$$

We make two remarks. First, if in the above formula we formally let $K(t)=1/\pi t$ and then replace $\sinh^{-1}(t/s)$ by $\int_0^t \frac{dy}{\sqrt{s^2+y^2}}$, change orders of integration and observe that $f^-(y)=\int_y^\infty \frac{f^-(t)-f^*(t)}{t}\,dt$, we obtain a formula which is essentially (1). Secondly we remark that for most applications the following formula suffices,

$$g^-(s)\leqslant 4\int_0^\infty f^-(t)K(t)\sinh^{-1}\left(\frac{t}{s}\right)dt.$$

We shall state and prove a series of lemmas which will lead us to the proof of Theorem 1 and another similar theorem. We will then discuss some applications to certain special Orlicz spaces.

The principal idea of the proof is contained in the following easy lemma.

LEMMA 2. *If $k(x)$ is even, non-negative and for $x>0$, non-increasing, and if*

$$K(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{k(x-t)}{t}\,dt=\lim_{\varepsilon\to 0}\int_{|t|>\varepsilon}$$

then for $x>0$,

$$2\pi K(x)\geqslant k^-(x)-k^*(x)\geqslant 0$$

where

$$k^*(x)=k\left(\frac{x}{2}\right)$$

and

$$k^-(x)=\frac{1}{x}\int_0^x k^*(t)\,dt.$$

Proof. For $x>0$,

$$\pi K(x)=\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{k(x-t)-k(x+t)}{t}\,dt$$

$$=\int_0^x \frac{k(x-t)-k(x+t)}{t}\,dt+\int_x^\infty \frac{k(t-x)-k(t+x)}{t}\,dt.$$

Both integrals are non-negative so that $K(x)\geqslant 0$.

$$2\pi K(x)\geqslant 2\int_{x/2}^x \frac{k(x-t)-k(x+t)}{t}\,dt\geqslant 2x^{-1}\int_{x/2}^x \big(k(x-t)-k(x+t)\big)\,dt$$

$$\geqslant 2x^{-1}\int_{x/2}^x \big(k(x-t)-k(x/2)\big)\,dt=2x^{-1}\int_0^{x/2} k(t)\,dt-2x^{-1}(x/2)k(x/2)$$

$$=2x^{-1}\int_0^{x/2} k^*(2t)\,dt-k^*(x)=x^{-1}\int_0^x k^*(t)\,dt-k^*(x)=k^-(x)-k^*(x).$$

LEMMA 3. *Let h be the Hilbert transform of the convolution of two functions f and k. Then for $s>0$,*

$$h^-(s)\leqslant \frac{2}{\pi}\int_0^\infty \big(f^-(t)-f^*(t)\big)\big(k^-(t)-k^*(t)\big)\sinh^{-1}\left(\frac{t}{s}\right)dt.$$

Proof. If $g=f*k$ then for $t>0$,

$$g^-(t)\leqslant tf^-(t)k^-(t)+\int_t^\infty f^*(y)k^*(y)\,dy.$$

(See [1], Theorem 1.7 p. 134 and for the following inequality see [2], Theorem 1, p. 191.) Thus if h is the Hilbert transform of g,

$$h^-(s)\leqslant\frac{2}{\pi}\int_0^\infty \frac{g^-(t)}{\sqrt{s^2+t^2}}\,dt\leqslant\frac{2}{\pi}\int_0^\infty \frac{tf^-(t)k^-(t)}{\sqrt{s^2+t^2}}\,dt+\frac{2}{\pi}\int_0^\infty \frac{dt}{\sqrt{s^2+t^2}}\int_t^\infty f^*(t)k^*(t)\,dt.$$

In the first integral we integrate by parts observing that the integral of $(\sqrt{s^2+t^2})^{-1}dt$ is $\sinh^{-1}t/s$ and that the derivative of $tf^-(t)k^-(t)$ is

$-f^-(t)k^-(t)+f^-(t)k^*(t)+f^*(t)k^-(t)$ and in the second integral we change orders of integration. Combining integrals gives the desired result.

The basic contents of the next lemma is that if $k(x)$ is the Hilbert transform of $-K(x)$ then $xk'(x)$ is the Hilbert transform of $-xK'(x)$.

Lemma 4. *If $K(t)$ is an odd kernel which for $t>0$ is positive, convex and decreasing with $K(t)\to 0$ as $t\to\infty$ and if $L(t) = -tK'(t)$ and if for $x>0$ define $k(\delta,\varepsilon\colon x)$ by*

$$-\pi k(\delta,\varepsilon\colon x) = \Big(\int_{-\infty}^{-\delta}+\int_{\delta}^{x-\varepsilon}+\int_{x+\varepsilon}^{\infty}\Big)\frac{K(x-t)}{t}\,dt$$

then the derivative $k'(\delta;\varepsilon\colon x) = \dfrac{d}{dx}k(\delta,\varepsilon\colon x)$ satisfies

$$\pi xk'(\delta,\varepsilon\colon x) = \Big(\int_{-\infty}^{-\delta}+\int_{\delta}^{x-\varepsilon}+\int_{x-\varepsilon}^{\infty}\Big)\frac{L(x-t)}{t}\,dt+\frac{2\varepsilon^2K(\varepsilon)}{x(x+\varepsilon)(x-\varepsilon)}+ +\big(K(x+\delta)-K(x-\delta)\big)/x.$$

Proof.

$$\pi k'(\delta,\varepsilon\colon x) = -\Big(\int_{-\infty}^{-\delta}+\int_{\delta}^{x-\varepsilon}+\int_{x+\varepsilon}^{\infty}\Big)\frac{K'(x-t)}{t}\,dt-\frac{K(\varepsilon)}{x-\varepsilon}+\frac{K(-\varepsilon)}{x+\varepsilon}.$$

Observe that $K(\varepsilon) = -\int_{-\infty}^{x-\varepsilon}K'(x-t)\,dt = -\int_{x+\varepsilon}^{\infty}K'(x-t)\,dt$.

Thus for example

$$\int_{x+\varepsilon}^{\infty}\frac{K'(x-t)}{t}\,dt+K(\varepsilon)/x = \int_{x+\varepsilon}^{\infty}K'(x-t)\Big[\frac{1}{t}-\frac{1}{x}\Big]dt$$

$$= \frac{1}{x}\int_{x+\varepsilon}^{\infty}\frac{(x-t)K'(x-t)}{t}\,dt = -\frac{1}{x}\int_{x+\varepsilon}^{\infty}\frac{L(x-t)}{t}\,dt.$$

We leave the completion of the proof to the reader.

Lemma 5. *If $K(t)$ is an odd function which for $t>0$ is monotone non-increasing tending to 0 as $t\to\infty$ and if there exists $b>0$ such that for $0<t<b$, $tK(t)$ is monotone non-decreasing and $tK(t)\to 0$ as $t\to 0$ and if $K(t)$ is not identically zero then*

$$\pi k(x) = -\int_{-\infty}^{\infty}\frac{K(x-t)}{t}\,dt$$

tends to ∞ as $x\to 0$.

Proof. Given $R>1$ and $\varepsilon>0$, choose c so that $\log b/2c>R$. Choose $\delta>0$, $\delta<c$ so that $\delta K(\delta)/c<\varepsilon$. Given any x, $0<x<\delta$ choose $\lambda=\lambda(x)$ by the equation $\lambda/x = x/2c$. Observe $\lambda<x/2$. Break $(-\infty,\infty)$ into the subintervals $\big(-\infty,-(c-x/2)\big)$, $\big(-(c-x/2),-\lambda\big)$, $(-\lambda,\lambda)$, $(\lambda,x-\lambda)$, $(x-\lambda,x+\lambda)$, $(x+\lambda,2x-\lambda)$, $(2x-\lambda,c+x/2)$, $(c+x/2,b+x)$, $(b+x,\infty)$.

$-K(x-t)/t\geqslant 0$ if $t\in\big(-\infty,-(c-x/2)\big)$ so that the integral over that interval is non-negative. Similarly the integral over $(b+x,\infty)$ is non-negative. The principal value of the integral over $(-\lambda,\lambda)$ equals

$$-\int_0^{\lambda}\frac{K(x-t)-K(x+t)}{t}\,dt$$

and since $(x-t)K(x-t)\leqslant(x+t)K(x+t)$ the absolute value of the above integral is dominated by

$$\int_0^{\lambda}\frac{K(x-t)2t}{x+t}\,dt = \int_0^{\lambda}\frac{(x-t)K(x-t)2t}{(x+t)(x-t)}\,dt$$

$$\leqslant 2xK(x)\int_0^{\lambda}\frac{2t}{(x+t)(x-t)}\,dt$$

$$\leqslant\frac{2\lambda xK(x)}{x^2-\lambda^2}<\frac{8}{3}\frac{\lambda}{x}K(x) = \frac{8}{3}\frac{x}{2c}K(x)<\frac{4\varepsilon}{3}.$$

Similarly the absolute value of the principal value of the integral over $(x-\lambda,x+\lambda)$ equals

$$\int_0^{\lambda}K(t)\Big(\frac{1}{x-t}-\frac{1}{x+t}\Big)dt\leqslant 2\lambda xK(x)/(x^2-\lambda^2)<\frac{8}{3}\frac{\lambda}{x}K(x)<4\varepsilon/3.$$

The integral over $\big(-(c-x/2),-\lambda\big)$ equals

$$\int_{\lambda}^{c-x/2}\frac{K(x+t)}{t}\,dt\geqslant(x+\lambda)K(x+\lambda)\int_{\lambda}^{c-x/2}\frac{dt}{t(x+t)}$$

$$\geqslant\frac{(x-\lambda)K(x-\lambda)}{x}\log\frac{c-x/2}{c+x/2}\,\frac{x+\lambda}{\lambda}$$

$$=\frac{(x-\lambda)K(x-\lambda)}{x}\log\frac{x-\lambda}{\lambda}$$

(observe $(c-x/2)/(c+x/2) = (x-\lambda)/(x+\lambda)$).

The sum of the integrals over $(\lambda, x-\lambda)$ and $(x+\lambda, 2x-\lambda)$ equals

$$-\int_{\lambda}^{x+\lambda} K(t)\left(\frac{1}{x-t}-\frac{1}{x+t}\right) dt$$

which in absolute value is dominated by

$$2(x-\lambda)K(x-\lambda)\int_{\lambda}^{x-\lambda}\frac{dt}{(x-t)(x+t)} = \frac{(x-\lambda)K(x-\lambda)}{x}\log\frac{2x-\lambda}{\lambda}\,\frac{x-\lambda}{x+\lambda}.$$

The integral over $(2x-\lambda, c+x/2)$ equals

$$\int_{2x-\lambda}^{c+x/2}\frac{K(t-x)}{t}\,dt \geqslant (x-\lambda)K(x-\lambda)\int_{2x-\lambda}^{c+x/2}\frac{dt}{t(t-x)}$$

$$= \frac{(x-\lambda)K(x-\lambda)}{x}\log\frac{2x-\lambda}{x+\lambda}.$$

Combining terms we see that the sum of the integrals over $(-(c-x/2), -\lambda)$, $(\lambda, x-\lambda)$, $(x+\lambda, 2x-\lambda)$ and $(2x-\lambda, c+x/2)$ is non-negative.

The integral over $(c+x/2, b+x)$ equals

$$\int_{c+x/2}^{b+x}\frac{K(t-x)}{t}\,dt \geqslant K(b)\int_{c+x/2}^{b+x}\frac{dt}{t} = K(b)\log(b+x)/(c+x/2)$$

$$\geqslant K(b)\log b/2c > K(b)R.$$

Thus for $|x| < \delta$,

$$\pi k(x) > RK(b) - 8\varepsilon/3.$$

A modification and simplification of the above proof yields the following lemma.

LEMMA 6. *If $K(t)$ is an odd function which is monotone non-increasing tending to zero as $t\to\infty$ and if $tK(t)$ is monotone non-decreasing and $tK(t)\to 0$ as $t\to 0$ and if $K(t)$ is not identically zero then*

$$k(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{K(x-t)}{t}\,dt$$

is non-negative and tends to ∞ as $x\to 0$.

We now prove Theorem 1. Let $K_n(t)\leqslant K(t)$ for $t>0$ be a kernel which coincides with $K(t)$ if $1/n\leqslant t\leqslant n$ and which is modified in such a way on $(0, 1/n)$ and (n, ∞) that hypotheses (i) and (ii) of the theorem hold for $K_n(t)$ but in such a way that $K_n(t)\in L^2$. Then $K_n(t)$ is the Hilbert transform of $k_n(t)\in L^2$. By Lemma 4 and Lemma 6 (with $L_n(t) = -tK_n'(t)$

replacing $K_n(t)$), $k_n(t)$ is positive and decreasing for $t>0$ (its derivative is negative). It is clear that if

$$k(x) = -\frac{1}{\pi}\int\frac{K(x-t)}{t}\,dt,$$

then $k_n(x)\to k(x)$ except possibly at $x=0$. We shall show that the k_n are uniformly absolutely continuous. Integrating by parts

$$xk_n^-(x) = \int_0^x k_n^*(t)\,dt = 2\int_0^{x/2} k_n(t)\,dt = \big(2k_n(x/2)x/2\big) + 2\int_0^{x/2}\big(-tk_n'(t)\big)\,dt$$

$$= xk_n^*(x) + 2\int_0^{x/2}\big(-tk_n'(t)\big)\,dt$$

so that

$$2\int_0^{x/2}\big(-tk_n'(t)\big)\,dt = xk_n^-(x) - xk_n^*(x) \leqslant 2\pi x K_n(x).$$

If we invent new odd kernels for $x>0$,

$$R_n(x) = \int_x^{\infty}\frac{K_n(t)}{t}\,dt \leqslant \int_x^{\infty}\frac{K(t)}{t}\,dt = R(x)$$

it is easily verified that R and R_n are odd monotone decreasing kernels; $xR_n(x)$ and $xR(x)$ are increasing and $xR(x)\to 0$ as $x\to 0$. But $k_n(x)$ bears the same relationship to $R_n(x)$ as $-xk_n'(x)$ bears to $K_n(x)$ so that

$$xk_n^-(x) = 2\int_0^{x/2} k_n(t)\,dt \leqslant 2xR_n(x) \leqslant 2xR(x).$$

Since the $k_n(x)$ are monotone for $x>0$, this inequality is sufficient to show the uniform absolute continuity. But the uniform absolute continuity of $k_n(x)$ implies $k(x)$ is locally integrable and

$$K(x) = \lim_{n\to\infty} K_n(x) = \lim_{n\to\infty}\frac{1}{\pi}\int\frac{k_n(x-t)}{t}\,dt = \frac{1}{\pi}\int\frac{k(x-t)}{t}\,dt.$$

To prove the inequality of the theorem let $H(f)$ denote the Hilbert transform of f. Then,

$$g = K*f = H(k)*f = H(k*f).$$

Apply Lemma 2 and Lemma 3.

THEOREM 7. *If $K(t)$ is an odd kernel on $(-\pi, \pi)$, monotone decreasing on $(0, \pi)$ such that there exists $\varepsilon>0$ such that for $0<t<\varepsilon$, $K(t)$ and $-tK'(t)$ are monotone decreasing and $tK(t)$ and $-t^2K'(t)$ are monotone increasing and $tK(t)\to 0$ and $-t^2K'(t)\to 0$ as $t\to 0$ then $K(t)$ is the conjugate function* *of $k(t)$ which is even and in L^1. Moreover there exists a constant A such that if f is periodic and*

$$g(x) = \int_{-\pi}^{\pi} K(t) f(x-t)\,dt$$

then for $0 < s < 2\pi$,

$$g^-(s) \leqslant A\,\|f\|_1\big(1+(\log 2\pi/s)^2\big) + A\int_0^{2\pi} \big(f^-(t)-f^*(t)\big)\,K(t/2)\sinh^{-1}\left(\frac{t}{s}\right)dt.$$

Proof. Extend the domain of $K(t)$ to $(-\infty, \infty)$ by setting $K(t) = 0$ for $|t| > \pi$. Let $K(t) = K_1(t)+K_2(t)$ where $K_1(t) = K(t)$ for $0 < t < \varepsilon/2$ and $K_1(t)$ satisfies the hypotheses of Theorem 1 and where $K_2(t)$ is uniformly bounded. Let

$$f_1(t) = \begin{cases} f(t) & \text{if} \quad |t| \leqslant 2\pi, \\ 0 & \text{if} \quad |t| > 2\pi, \end{cases}$$

then $g_1 = K*f_1$ coincides with g on $(-\pi, \pi)$ so that for $s > 0$, $g^-(s) \leqslant g_1^-(s)$, $g_1 = K*f_1 = K_1*f_1+K_2*f_1$.

$$\|K_2*f_1\|_\infty \leqslant \|K_2\|_\infty\,\|f_1\|_1 = 2\,\|K_2\|_\infty\,\|f\|_1.$$

If $g_2 = K_1*f_1$ then by Theorem 1,

$$g_2^-(s) \leqslant 4\int_0^\infty \big(f_1^-(t)-f_1^*(t)\big)K_1(t)\sinh^{-1}\left(\frac{t}{s}\right)dt.$$

It we observe that $f_1^-(t) = f^-(t/2)$ and $f_1^*(t) = f^*(t/2)$, that $tf^-(t) \leqslant \|f\|_1$ that $K_1(t) \leqslant K(t)+K(\varepsilon/2)$ and that $\sinh^{-1}(t/s)$ is the order of magnitude of t/s if t/s is small and $\log t/s$ if t/s is large then we see that

$$\int_{2\pi}^\infty [f^-(t)-f^*(t)]K_1(t)\sinh^{-1}t/s\,dt \leqslant c_1\|f\|_1\log 2\pi/s + c_2\|f\|_1\int_{2\pi}^\infty t^{-1}K_1(t)\log t\,dt,$$

$$\int_0^{2\pi} \big(f^-(t)-f^*(t)\big)\sinh^{-1}\left(\frac{t}{s}\right)dt \leqslant \|f\|_1\left(1+\int_s^{2\pi}\sinh^{-1}\left(\frac{t}{s}\right)\frac{dt}{t}\right)$$
$$\leqslant \|f\|_1\big(1+(\log 2\pi/s)^2\big).$$

We may combine these facts to obtain the inequality of the theorem.

We may apply Theorem 7 to obtain a theorem on Orlicz spaces over $(-\pi, \pi)$.

THEOREM 8. *Let A be a Young's function such that both A and its complement A^- satisfy the Δ_2-condition of Orlicz* (see [3]). *Let $b(t) = (\log^+ t)^\alpha$, $\alpha > 0$. Let $B(t)$ be a Young's function which for large values of t is the order of magnitude of $A\big(tb(t)\big)$. It is easily seen that B and its complement both satisfy the Δ_2-condition. Let $K(t)$ be an odd monotone kernel which equals $1/tb(1/t)$ for small positive t. Then if $f \in L_A$ then*

$$g(x) = \int_{-\pi}^{\pi} f(x-t)K(t)\,dt$$

is in L_B.

The following remarks should enable the reader to complete the proof. Recall that A, A^-, B and B^- all satisfy the Δ_2-condition.

1. The integral operator which transforms $f \in L_B$ by

$$g(s) = \int_0^{2\pi} f(t)\sinh^{-1}\left(\frac{t}{s}\right)\frac{dt}{t}$$

is a bounded operator from L_B to L_B.

2. $f \in L_B$ if and only if $f^* \in L_B$ if and only if $f^- \in L_B$.

3. If $f \in L_B$ then there exists $a > 0$ such that $f^-(t) \leqslant c_1/t^a$.

4. For $f^-(t)$ big enough,

$$B\big(f^-(t)/b(1/t)\big) = A\left(\frac{f^-(t)}{b(1/t)}\,b\left(\frac{f^-(t)}{b(1/t)}\right)\right) \leqslant A\left(\frac{f^-(t)}{b(1/t)}\,b(c_1/t^a)\right) \leqslant A\big(c_2 f^-(t)\big).$$

Thus if $f \in L_A$ then $f^-(t)/b(1/t) \in L_B$.

EXAMPLE 9. As a special case of Theorem 8 we see that if $K(t) = 1/t(\log 1/t)^\alpha$, $\alpha > 0$ for small positive t and $f \in L^p(-\pi, \pi)$ then $g = K*F$ is in $L^p(\log^+ L)^{\alpha p}$.

EXAMPLE 10. If $K(t) = 1/t(\log 1/t)^\alpha$, $0 < \alpha < 1$, for small positive t and if $f \in L(\log^+ L)^\beta$, $\beta \geqslant 1-\alpha$ then $g \in L(\log^+ L)^{\alpha+\beta-1}$.

EXAMPLE 11. Zygmund [4] has shown that of the Orlicz spaces close to L^1, $L\log^+ L$ is the most interesting. Thus it is no surprise that in the present theory perhaps the most interesting singular odd monotone kernel is $K(t)$ equal to $1/t(\log 1/t)$ for small positive t. For this choice of $K(t)$ our theorems show that if $f \in L(\log^+ L)^\beta$, $\beta > 0$ then $K*f \in L(\log^+ L)^\beta$. This is not true if $\beta = 0$, however we have the substitute result that if $f \in L(\log^+\log^+ L)$ then $K*f \in L$.

We remark that we may choose $b(t)$ in Theorem 8 more generally than a power of a logarithm. Indeed the only facts needed about $b(t)$ is that $K(t) = 1/tb(1/t)$ satisfies the hypotheses of Theorem 7, that $b(t)$ increases to ∞, that $b(t)$ is slowly varying and that there exists a constant c such that $b(t^2) \leqslant cb(t)$ for t sufficiently large. This last condition is a strong constraint, indeed $\exp(\sqrt{\log t})$ is a slowly varying function which fails to satisfy it and which indeed would not work in Theorem 7.

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Distribution function inequalities for the area integral*

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Abstract. Let A be the area integral of a function u harmonic in the Euclidean half-space $\mathbf{R}^n \times (0, \infty)$. Information about the distribution function of a localized version of A is obtained that leads to a general integral inequality between A and the nontangential maximal function of u and provides a convenient approach to the study of the pointwise behavior of u near the boundary. In addition, the general integral inequality of [2] between the nontangential maximal function of u and that of a properly chosen conjugate is shown to hold also in the case $n > 1$.

Our object here is to prove some partial distribution function inequalities for the area integral and to show how these inequalities can be used to study both the local and the global behavior of harmonic functions. Before describing our approach in detail, we consider a few of its applications.

Let u be harmonic in the Euclidean half-space

$$\mathbf{R}_+^{n+1} = \{(x, y): x \in \mathbf{R}^n, \ y > 0\}.$$

The area integral of u is the nonnegative function $A = A_a(u)$ defined on $\mathbf{R}^n$ by

$$A^2(x) = A_a^2(u, x) = \iint_{\Gamma(x)} y^{1-n} |\nabla u(s, y)|^2 \, ds\, dy$$

where a is a positive real number,

$$\Gamma(x) = \Gamma(x; a) = \{(s, y): |x-s| < ay\},$$

and $\nabla u = (\partial u/\partial y, \partial u/\partial x_1, \ldots, \partial u/\partial x_n)$. The nontangential maximal function $N = N_a(u)$ is defined by

(1) $$N(x) = \sup_{(s,y)\in\Gamma(x)} |u(s, y)|.$$

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