Next consider (ii).

$$\|f_1^*(x, y+t)-f_1^*(x, y)\|_p = \Big\| \int_{-\pi}^{\pi} \{f(x+u, y+t)-f(x+u, y)\} \frac{du}{u} \Big\|_p = \|\tilde{g_y}(x)\|_p,$$

where $g_y(x+u) = f(x+u, y+t)-f(x+u, y)$. Clearly $g_y(x)$ is in $L^p(T^2)$, the problem is to show that the constant "C" does not depend on the parameter y.

$$\int |\tilde{g_y}(x)|^p dx \leqslant A_p^p \int |g_y(x)|^p dx.$$

This follows from the Riesz theorem for $1 < p < \infty$. The constant A_p depends on the p-only, and not on the particular function in that class. Thus,

$$\int dy \int |\tilde{g_y}(x)|^p dx \leqslant A_p^p \int\int |g_y(x)|^p dx\, dy \leqslant A_p^p C^p t^{2p},$$

and putting all the inequalities together we obtain the desired inequality, with a constant independent of t.

COROLLARY 1. *If $f(x, y) \in \Lambda_\alpha^p(T^2)$, then $f_3(x, y) \in \Lambda_\alpha^p(T^2)$.*

Proof. If $f(x, y) \in \Lambda_\alpha^p(T^2)$, then $f_1(x, y) \in \Lambda_\alpha^p(T^2)$ by Theorem 1. Then $(f_1)_2 \in \Lambda_\alpha^p(T^2)$ by the same argument. Finally $(f_1)_2 = f_3$ a.e. by Fubini's theorem.

Remark 1. The classes Λ_*, Λ_*^p of Zygmund [4] do not seem as natural for iterated operators as they were for the usual conjugate operator. The technique of Theorem 1 can be used to show $f_1(x, y)$ in $\Lambda_*(T)$ when the y variable is held fixed or when the x variable is fixed. But this does not tell us about $\Lambda_*^p(T^2)$.

Remark 2. The results are stated for T^2, but they can be generalized to T^m by repeating the argument.

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Estimates for translation-invariant operators on spaces with mixed norms (*)

by

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Abstract. Let Γ be a locally compact group, H an amenable group. Set $G = \Gamma H$ their semidirect product (in particular their cartesian product). Let T be a translation invariant operator bounded in $L_p(G) = L_p(\Gamma, L_p(H))$ for, say, $p \leqslant 2$ (or $2 \leqslant p$). The main result of this paper shows that with the same or smaller norm T is a bounded operator on the mixed-norm spaces $L_p(\Gamma, L_q(H))$ for $p \leqslant q \leqslant 2$ (or $2 \leqslant q \leqslant p$).

Let $G = \boldsymbol{R}^d$ or any locally compact group whatsoever. We write $\mathrm{CONV}_p(G)$, $1 \leqslant p < \infty$, for the Banach algebra of bounded linear operators T: $L_p(G) \to L_p(G)$ which commute with right-translations where $L_p(G)$ is the Lebesgue space formed with respect to the left-invariant Haar measure on G. The norm is $|||T|||_p = \sup \|Tu\|_p$ where $u \in L_p(G)$ and $\|u\|_p \leqslant 1$.

We are concerned with the situation in which we have a semi-direct product $G = \Gamma H$ of an amenable locally compact group H by an arbitrary locally compact group Γ. The basic example is $\Gamma = \boldsymbol{R}^m$ acting trivially on $H = \boldsymbol{R}^n$ so that $G = \boldsymbol{R}^{m+n}$.

THEOREM. *Suppose $T \in \mathrm{CONV}_p(G)$ where $G = \Gamma H$, the semi-direct product of an amenable group H by a locally compact group Γ. Take q such that $p \leqslant q \leqslant 2$ or $p \geqslant q \geqslant 2$. Then for any complex-valued continuous function f of compact support on G we have*

$$\int_\Gamma \Big\{ \int_H |Tf(\sigma, x)|^q dx \Big\}^{p/q} d\sigma \leqslant |||T|||_p^p \int_\Gamma \Big\{ \int_H |f(\sigma, x)|^q dx \Big\}^{p/q} d\sigma.$$

In other words, the translation-invariant operator T, which is assumed to be bounded on $L_p(G) = L_p(\Gamma;\ L_p(H))$, has the same or smaller bound on the mixed-norm space $L_p(\Gamma;\ L_q(H))$.

If in the Theorem we take Γ trivial we obtain

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Corollary. *Let G be an amenable group and $1 \leqslant p \leqslant q \leqslant 2$ or $\infty > p \geqslant q \geqslant 2$. If $T \in CONV_p(G)$ then also $T \in CONV_q(G)$, and $|||T|||_q \leqslant |||T|||_p$.*

This statement occurs in [3], and the essential idea of the proof indicated there carries over to the present Theorem. Of course, if G is commutative then the Corollary is a banal consequence of the Riesz Convexity Theorem. On the other hand, we do not know how to prove by interpolation methods that elements of $\mathrm{CONV}_p(\boldsymbol{R}^{m+n})$ give bounded operators on $L_p(\boldsymbol{R}^m; L_q(\boldsymbol{R}^n))$. Thus our main result appears to say something genuinely new for classical convolution operators. In particular, it shows that the boundedness of singular integral operators on certain spaces with mixed norms (see [1], [4], and [6]) is nothing but an immediate consequence of their L_p-boundedness.

Write $\mathscr{K}(G)$ for the space of continuous complex-valued functions of compact support on G. We shall call an element $T \in CONV_p(G)$ a *special* convolution operator if there exists a kernel $k \in L_p(G)$ such that for each $f \in \mathscr{K}(G)$ we have

$$Tf(x) = \int_G k(y) f(y^{-1}x)\, dy = k * f(x).$$

(The phrase "almost everywhere" is suppressed throughout.)

We observe first that every operator which commutes with right-translations can be approximated in the bounded strong operator topology by special convolution operators.

Lemma 1. *Let $\{N\}$ be a complete system of compact neighbourhoods of the identity in G. For each N, choose a positive continuous function k_N such that $\operatorname{supp} k_N \subset N$ and $\int_G k_N(x)\, dx = 1$. Suppose $T \in CONV_p(G)$. Then*

(i) *Each Tk_N is the kernel of a special convolution operator T_N, i.e. $T_N f = (Tk_N) * f$, with $|||T_N|||_p \leqslant |||T|||_p$;*

(ii) $\lim \|Tf - T_N f\|_p = 0$ *for each $f \in L_p(G)$.*

Proof. If $k \in L_p(G)$ and $f \in \mathscr{K}(G)$ then we may write

$$k * f(x) = \int_G k(xy) f(y^{-1})\, dy.$$

Hence, if T is an operator on $L_p(G)$ which commutes with right-translations we obtain

$$(Tk) * f = T(k * f).$$

In particular, $(Tk_N) * f = T(k_N * f)$, and so

$$\|(Tk_N) * f\|_p \leqslant |||T|||_p \|k_N * f\|_p \leqslant |||T|||_p \|f\|_p.$$

This proves that Tk_N, which is an element of $L_p(G)$, is the kernel of an operator $T_N \in \mathrm{CONV}_p(G)$ with $|||T_N|||_p \leqslant |||T|||_p$. Since $\lim \|f - k_N * f\|_p = 0$

for each $f \in L_p(G)$ and T is bounded, it follows that $T_N f = T(k_N * f)$ converges in norm to Tf.

Lemma 1 having been established, it clearly suffices to prove the Theorem in the case where T is a special convolution operator.

The decisive point in the proof of the Theorem depends on the

Lemma 2. *Let G and H be arbitrary measure spaces and $T: L_p(G) \to L_p(G)$ a bounded linear operator. Then if $p \leqslant q \leqslant 2$ or $p \geqslant q \geqslant 2$ there exists a bounded linear operator $\tilde{T}: L_p(G; L_q(H)) \to L_p(G; L_q(H))$ with $|||\tilde{T}||| \leqslant |||T|||_p$ such that for $g \in L_p(G; L_q(H))$ of the form $g(\xi, x) = f(\xi) u(x)$ where $f \in L_p(G)$ and $u \in L_q(H)$ we have*

$$(\tilde{T}g)(\xi, x) = (Tf)(\xi) u(x).$$

Lemma 2, for real-valued functions, is essentially due to Marcinkiewicz and Zygmund [5]. Complete details including the extension to complex-valued functions will be found in [3]. It is worth noting that if G and H are non-trivial groups then Lemma 2 is false for $q < p \leqslant 2$ or $q > p \geqslant 2$ even if we suppose $T \in \mathrm{CONV}_p(G)$(¹).

Let H be a topological group whose operation is written as $+$ and Γ a topological group, written multiplicatively, such that there is a continuous map $\Gamma \times H \to H$, notation $(\sigma, x) \to x\sigma$ with $(x+y)\sigma = (x\sigma) + (y\sigma)$ and $(x\sigma)\tau = x(\sigma\tau)$. The semi-direct product ΓH in the topological space $\Gamma \times H$ with the group operation

$$(\sigma, x) \cdot (\tau, y) = (\sigma\tau, x\tau + y).$$

If Γ and H are locally compact then so is $G = \Gamma H$, and the left-invariant Haar measure of G is the product measure of the left-invariant Haar measures of Γ and H. The direct product is the semi-direct product where Γ acts trivially on H, i.e. $x\sigma \equiv x$.

Commutative (more generally, solvable) groups and compact groups are amenable; non-compact semi-simple Lie groups with finite center are non-amenable. The role of amenability in this work depends on the next.

Lemma 3. *Let H be an amenable group. Given a compact $K \subset H$ and $\varepsilon > 0$ there exist $u \in L_p(H)$ and $v \in L_{p'}(H)$ with $\|u\|_p \leqslant 1$ and $\|v\|_{p'} < 1 + \varepsilon$ such that for the function*

$$\lambda(x) = \int u(-x+y) v(y)\, dy$$

we have $\lambda = 1$ on K.

Proof. A characteristic property of amenable groups is that given a compact K and $\delta > 0$ there exists a bounded open set U such that $\mu(K + U) < (1+\delta)\mu(U)$ where μ is the left-invariant Haar measure on H.

(¹) Added in proof: Charles McCarthy has shown that the extension of Lemma 2 for translation invariant operators is also false for $p \leqslant 2 < q$ or $p \geqslant 2 > q$.

Define u and v by $u(x) = [\mu(U)]^{-1/p}$ for $x \in U$, $u = 0$ elsewhere, and $v(x) = [\mu(U)]^{-1/p'}$ for $x \in K+U$, $v = 0$ elsewhere. The assertions of the Lemma hold with $1+\varepsilon = (1+\delta)^{1/p'}$.

Remark. The measure-theoretic property of amenable groups used above is not easy to prove in complete generality, see [2]. At the expense of slightly complicating the proof of the Theorem we could replace the trapezoidal function λ of Lemma 3 by a positive-definite function $\varkappa$ such that $|1-\varkappa| < \varepsilon$ on K, and the existence of such positive-definite functions on amenable groups is fairly easy to establish.

Proof of Theorem. Lemma 1 shows that if suffices to consider special convolution operators, i.e. we may suppose that T has the form

$$(Tf)(\sigma, x) = \int_{\Gamma}\int_{H} \big(k(\tau, t) f(\tau^{-1}\sigma, -t\tau^{-1}\sigma) + x\big)\, d\tau\, dt,$$

where $k \in L_p(G)$.

By virtue of Lemma 2, the operator $T^{\sim}$ defined on functions $g \in \mathscr{K}(G\times H)$ by

$$(T^{\sim} g)(\sigma, x, y) = \int_{\Gamma}\int_{H} k(\tau, t) g(\tau^{-1}\sigma, -t\tau^{-1}\sigma + x, y)\, d\tau\, dt$$

is a bounded operator on $L_p\big(G; L_q(H)\big)$ with norm $\leqslant |||T|||_p$. On the other hand $\pi\colon \mathscr{K}(G\times H) \to \mathscr{K}(G\times H)$ defined by $(\pi g)(\sigma, x, y) = g(\sigma, x, x+y)$ extends to an isometric automorphism of $L_p\big(G; L_q(H)\big)$. We therefore have

$$|\langle T^{\sim}\pi g, \pi\psi\rangle| \leqslant |||T|||_p \|g\| \|\psi\|,$$

where $\|g\|$ is the $L_p\big(G; L_q(H)\big)$ norm of $g \in \mathscr{K}(G\times H)$ and $\|\psi\|$ is the $L_{p'}\big(G; L_{q'}(H)\big)$ norm of $\psi \in \mathscr{K}(G\times H)$. We take g and ψ in the special forms

$$g(\sigma, x, y) = f(\sigma, y) u(x), \quad \psi(\sigma, x, y) = \varphi(\sigma, y) v(x),$$

where $f, \varphi \in \mathscr{K}(G)$ and $u, v \in \mathscr{K}(H)$. Here $\|g\| = \|f\|\, \|u\|_p$ and $\|\psi\| = \|\varphi\|\, \|v\|_{p'}$ where $\|f\|$ is the $L_p\big(\Gamma; L_q(H)\big)$-norm and $\|\varphi\|$ is the $L_{p'}\big(\Gamma; L_{q'}(H)\big)$-norm. On writing $\lambda(s) = \int_H u(-s+x) v(x)\, dx$ we get

$$\langle T^{\sim}\pi g, \pi\psi\rangle = \int_{G\times G} k(\tau, t)\lambda(t\tau^{-1}\sigma) f(\tau^{-1}\sigma, -t\tau^{-1}\sigma + y)\varphi(\sigma, y)\, d\tau\, dt\, d\sigma\, dy,$$

with the integral being absolutely convergent. Indeed, since $f(\cdot, y) = 0$ for $y \notin K_1$ and $\varphi(\cdot, y) = 0$ for $y \notin K_2$ where K_1 and K_2 are compacts in H, if we choose u, v as in Lemma 3 so that $\lambda = 1$ on $K_2 - K_1$ we obtain

$$\langle T^{\sim}\pi g, \pi\psi\rangle = \langle Tf, \varphi\rangle.$$

This yields the estimate

$$|\langle Tf, \varphi\rangle| = |\langle T^{\sim}\pi g, \pi\psi\rangle| \leqslant |||T|||_p \|g\|\, \|\psi\| = |||T|||_p \|f\|\, \|\varphi\|\, \|u\|_p\, \|v\|_{p'}.$$

Given $\varepsilon > 0$, we could have chosen u, v so that $\|u\|_p \|v\|_{p'} < 1+\varepsilon$. Hence we have $|\langle Tf, \varphi\rangle| \leqslant |||T|||_p \|f\|\, \|\varphi\|$ which is tantamount to the assertion of the Theorem.

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ERRATA

Page, line	For	Read
484^2	$x =$	$X =$
487^8	$1/t$	$1/\|t\|$
488^9	$y = t$	$y - t$