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Preservation of Lipschitz class by the iterated conjugate operator

by

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Abstract. The Lipschitz class of a function of two variables is not preserved pointwise under the action of the conjugate operator. It is shown here that this class is preserved in L^p measure for $1 < p < \infty$.

The idea of Lipschitz continuity for periodic functions can be expressed either as a pointwise condition, or as a condition in measure.

DEFINITION. a) $f \in \Lambda_\alpha(T)$ means that

$$\sup_{|s|\leqslant h} |f(x+s)-f(x)| < Ch^\alpha$$

for all $x \in [-\pi, \pi] = T$, with the C independent of the h and the x.

b) $f \in \Lambda_\alpha^p(T)$, $1 \leqslant p \leqslant \infty$, means that $f \in L^p(T)$ and

$$\sup_{|s|\leqslant h} \|f(\cdot+s)-f(\cdot)\|_p < Ch^\alpha$$

with the constant C independent of the h.

The functions in $\Lambda_\alpha(T)$ are continuous everywhere; the functions $\Lambda_\alpha^p(T)$ may have discontinuities, but only on a set of measure zero. If $\alpha p \leqslant 1$ the discontinuities may be of both types; if $\alpha p > 1$ the discontinuities must be non-essential. Thus, for example, $\Lambda_\alpha^\infty(T)$ consists of functions which can be "fixed-up" on a set of measure zero so that the "improved" function lies in Λ_α.

The definitions can be extended to several variables by changing the underlying space from T to $T^n = T \times T \times \ldots \times T$.

It has been shown that the conjugate operator defined by,

$$f^\sim(x) = \text{p. v.} \int_{-\pi}^{\pi} f(x+u)(2\operatorname{ctn} u/2)\,du,$$

preserves both classes. Thus, $f \in \Lambda_\alpha(T)$ implies $f^\sim \in \Lambda_\alpha(T)$; $f \in \Lambda_\alpha^p(T)$ implies $f^\sim \in \Lambda_\alpha^p(T)$ (see Zygmund [4] and Taibleson [2] for proofs). Unfortunately this result does not generalize to several variables with the same conciseness.

DEFINITION.

$$f_1(x,y) = \text{p. v.} \int_{-\pi}^{\pi} f(x+u,y)(2\operatorname{ctn} u/2)\,du,$$

$$f_2(x,y) = \text{p. v.} \int_{-\pi}^{\pi} f(x,y+v)(2\operatorname{ctn} v/2)\,dv,$$

$$f_3(x,y) = \text{p. v.} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v)(2\operatorname{ctn} u/2)(2\operatorname{ctn} v/2)\,du\,dv.$$

Cesari [1] showed that $f \in \Lambda_\alpha(T^2)$, then $f_3 \in \Lambda_{\alpha'}(T^2)$ for all $\alpha' < \alpha$. He also claimed that f_1 and f_2 are both in $\Lambda_\alpha(T^2)$, but this proved false. A counter-example was given by Landis in a paper of Zhak [3] which showed that the result for f_3 was the best possible and that no better result could be obtained for f_1 and f_2. (The reader should be warned that the definitions of Zhak do not agree with ours, but the example of Landis can be modified for our purposes.) Consequently, the conjugate operator will not map the class Λ_α into Λ_α. In spite of this, we have the following

THEOREM 1. *If $f(x,y) \in \Lambda_\alpha^p(T^2)$, $1 < p < \infty$, then $f_1(x,y) \in \Lambda_\alpha^p(T^2)$.*

Proof. Observe that the definition of $\Lambda_\alpha^p(T^2)$ is equivalent to the following

$$\sup_{\substack{|s|\leqslant h\\ |t|\leqslant k}} \Big(\int_{T^2} |f_1(x+s,y+t) - f_1(x,y)|^p\,dx\,dy\Big)^{1/p} \leqslant C(h^\alpha + k^\alpha).$$

This condition can be established by showing that

(i) $$\|f_1(x+s,y) - f_1(x,y)\|_p \leqslant Cs^\alpha$$

and

(ii) $$\|f_1(x,y+t) - f_1(x,y)\|_p \leqslant Ct^\alpha$$

where for convenience we assume $0 < s < 1$, $0 < t < 1$. The constant C must be independent of both s and t. For simplicity, let

$$f_1^*(x,y) = \text{p. v.} \int_{-\pi}^{\pi} f(x+u,y)u^{-1}\,du.$$

If $f_1^* \in \Lambda_\alpha^p(T^2)$ then $f_1 \in \Lambda_\alpha^p(T^2)$ also. To show this merely form the difference $f_1^* = f_1^* \mp f_1$, use the Minkowski inequality and observe that the difference of the kernels is a bounded function. Now to establish (i).

$$\|f_1^*(x+s,y) - f_1^*(x,y)\|_p$$
$$= \Big\|\int_0^{\pi} \{f(x+s+u,y) - f(x+s-u,y) - f(x+u,y) + f(x-u,y)\}\frac{du}{u}\Big\|_p$$
$$\leqslant \Big\|\int_0^{2s}\Big\|_p + \Big\|\int_{2s}^{\pi}\Big\|_p = A + B.$$

Consider the terms inside the curly brackets. By grouping the first two and the last two terms together and taking the p norm inside the integral of A (by Minkowski) we obtain

$$A \leqslant 2\int_0^{2s} Cu^{\alpha-1}\,du = \frac{2C}{\alpha}(2s)^\alpha = C's^\alpha.$$

$$B = \Big\|\int_{2s}^{\pi}\{f(x+u+s,y) - f(x,y)\}\frac{du}{u} - \int_{2s}^{\pi}\{f(x+u,y) - f(x,y)\}\frac{du}{u} -$$
$$- \int_{2s}^{\pi}\{f(x-u+s,y) - f(x,y)\}\frac{du}{u} + \int_{2s}^{\pi}\{f(x-u,y) - f(x,y)\}\frac{du}{u}\Big\|_p$$
$$\leqslant \Big\|\int_{2s}^{\pi} - \int_{2s}^{\pi}\Big\|_p + \Big\|\int_{2s}^{\pi} - \int_{2s}^{\pi}\Big\|_p = B_1 + B_2$$

where B_1 and B_2 are similar, so we consider only the first term B_1.

$$B_1 = \Big\|\int_{3s}^{\pi+s}\{f(x+u,y) - f(x,y)\}\frac{du}{u-s} - \int_{2s}^{\pi}\{f(x+u,y) - f(x,y)\}\frac{du}{u}\Big\|_p$$
$$\leqslant \Big\|\int_{2s}^{\pi}\{f(x+u,y) - f(x,y)\}\Big\{\frac{1}{u-s} - \frac{1}{u}\Big\}du\Big\|_p +$$
$$+ \Big\|\int_{2s}^{3s}\{f(x+u,y) - f(x,y)\}\frac{du}{u-s}\Big\|_p +$$
$$+ \Big\|\int_{\pi}^{\pi+s}\{f(x+u,y) - f(x,y)\}\frac{du}{u-s}\Big\|_p$$
$$= B_{11} + B_{12} + B_{13}.$$

Now by the Minkowski inequality for integrals, we have

$$B_{11} \leqslant \int_{2s}^{\pi} Cu^\alpha \frac{s}{u^2 - us}\,du \leqslant Cs\int_{2s}^{\pi}\frac{u^\alpha}{(u-s)^2}\,du$$
$$\leqslant C's\{(\pi)^{\alpha-1} - (s)^{\alpha-1}\} \leqslant C''(s + s^\alpha).$$

It should be noticed at this point that the argument here will fail for $\alpha = 1$.

$$B_{12} \leqslant \int_{2s}^{3s} Cu^\alpha \frac{du}{u-s} \leqslant C'\frac{1}{s}s^{\alpha+1} = C's^\alpha,$$

$$B_{13} \leqslant \int_{\pi}^{\pi+h} Cu^\alpha \frac{du}{u-s} \leqslant C's$$

and (i) is established for f_1^*.

Next consider (ii).

$$\|f_1^*(x, y+t)-f_1^*(x, y)\|_p = \left\| \int_{-\pi}^{\pi} \{f(x+u, y+t)-f(x+u, y)\} \frac{du}{u} \right\|_p = \|g_y^{\sim}(x)\|_p,$$

where $g_y(x+u) = f(x+u, y+t)-f(x+u, y)$. Clearly $g_y(x)$ is in $L^p(T^2)$, the problem is to show that the constant "C" does not depend on the parameter y.

$$\int |g_y^{\sim}(x)|^p dx \leqslant A_p^p \int |g_y(x)|^p dx.$$

This follows from the Riesz theorem for $1 < p < \infty$. The constant A_p depends on the p-only, and not on the particular function in that class.

Thus,

$$\int dy \int |g_y^{\sim}(x)|^p dx \leqslant A_p^p \int\int |g_y(x)|^p dx\, dy \leqslant A_p^p C^p t^{2p},$$

and putting all the inequalities together we obtain the desired inequality, with a constant independent of t.

COROLLARY 1. *If $f(x, y) \in \Lambda_\alpha^p(T^2)$, then $f_3(x, y) \in \Lambda_\alpha^p(T^2)$.*

Proof. If $f(x, y) \in \Lambda_\alpha^p(T^2)$, then $f_1(x, y) \in \Lambda_\alpha^p(T^2)$ by Theorem 1. Then $(f_1)_2 \in \Lambda_\alpha^p(T^2)$ by the same argument. Finally $(f_1)_2 = f_3$ a.e. by Fubini's theorem.

Remark 1. The classes Λ_*, Λ_*^p of Zygmund [4] do not seem as natural for iterated operators as they were for the usual conjugate operator. The technique of Theorem 1 can be used to show $f_1(x, y)$ in $\Lambda_*(T)$ when the y variable is held fixed or when the x variable is fixed. But this does not tell us about $\Lambda_*^p(T^2)$.

Remark 2. The results are stated for T^2, but they can be generalized to T^m by repeating the argument.

References

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Estimates for translation-invariant operators on spaces with mixed norms (*)

by

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Abstract. Let Γ be a locally compact group, H an amenable group. Set $G = \Gamma H$ their semidirect product (in particular their cartesian product). Let T be a translation invariant operator bounded in $L_p(G) = L_p(\Gamma, L_p(H))$ for, say, $p \leqslant 2$ (or $2 \leqslant p$). The main result of this paper shows that with the same or smaller norm T is a bounded operator on the mixed-norm spaces $L_p(\Gamma, L_q(H))$ for $p \leqslant q \leqslant 2$ (or $2 \leqslant q \leqslant p$).

Let $G = \boldsymbol{R}^d$ or any locally compact group whatsoever. We write $\mathrm{CONV}_p(G)$, $1 \leqslant p < \infty$, for the Banach algebra of bounded linear operators T: $L_p(G) \to L_p(G)$ which commute with right-translations where $L_p(G)$ is the Lebesgue space formed with respect to the left-invariant Haar measure on G. The norm is $|||T|||_p = \sup \|Tu\|_p$ where $u \in L_p(G)$ and $\|u\|_p \leqslant 1$.

We are concerned with the situation in which we have a semi-direct product $G = \Gamma H$ of an amenable locally compact group H by an arbitrary locally compact group Γ. The basic example is $\Gamma = \boldsymbol{R}^m$ acting trivially on $H = \boldsymbol{R}^n$ so that $G = \boldsymbol{R}^{m+n}$.

THEOREM. *Suppose $T \in \mathrm{CONV}_p(G)$ where $G = \Gamma H$, the semi-direct product of an amenable group H by a locally compact group Γ. Take q such that $p \leqslant q \leqslant 2$ or $p \geqslant q \geqslant 2$. Then for any complex-valued continuous function f of compact support on G we have*

$$\int_\Gamma \Big\{ \int_H |Tf(\sigma, x)|^q dx \Big\}^{p/q} d\sigma \leqslant |||T|||_p^p \int_\Gamma \Big\{ \int_H |f(\sigma, x)|^q dx \Big\}^{p/q} d\sigma.$$

In other words, the translation-invariant operator T, which is assumed to be bounded on $L_p(G) = L_p(\Gamma; L_p(H))$, has the same or smaller bound on the mixed-norm space $L_p(\Gamma; L_q(H))$.

If in the Theorem we take Γ trivial we obtain

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