function of $B(z, x)$ whose square is $A(z, x)$, aside from a constant factor (depending on x). Hence C is a singular cocycle whose square is a function of x times A. This implies $C^2 = A$, and the theorem is proved.

5. The restriction to countable Γ and separable K is not essential. Without any restriction, a cocycle $A(t, x)$ is continuous as a mapping from R to $L^2(K)$ ([2], p. 186). Hence A_t takes its values in a separable subspace of $L^2(K)$, so the non-null Fourier coefficients of all the functions A_t lie in a countable subgroup of Γ. Thus A can be studied on a separable quotient group of K.

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Weak integrals defined on Euclidean n-space

by

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Abstract. The following representation for Banach-valued measurable weakly integrable functions on Euclidean n-space is established: $f = \sum_{i=1}^{\infty} x_i \xi_{I_i}$, where the x_i are elements of the given Banach space and the ξ_{I_i} are characteristic functions of intervals I_i; the convergence is absolute a.e. The weak integral of f is given by the equality $\int f d\lambda = \sum_{i=1}^{\infty} x_i \lambda(I_i)$, where the convergence is unconditional. This approach avoids entirely the use of functionals.

1. Introduction. In this paper we establish a representation theorem (Theorem 1) for Banach-valued measurable weakly integrable functions defined on Euclidean n-space, where the underlying measure is Lebesgue measure. The representation is given in terms of intervals and unconditionally convergent series. As a result, our approach avoids the use of the conjugate space and the theory of Lebesgue measure, except for the concept of almost everywhere convergence.

We also present a construction of Lebesgue measurable sets which seems to be an effective tool for examining measurable sets in terms of intervals (Theorem 2).

2. Definitions. $\mathfrak{X}$ is a Banach space over the complex numbers with conjugate space $\mathfrak{X}^*$. $|x|$ is the norm of an element $x \in \mathfrak{X}$. $(R^n, \mathscr{L}, \lambda)$ denotes the measure space consisting of the Lebesgue measurable subsets of R^n, with n-dimensional Lebesgue measure λ. $\int g$ or $\int g\,d\lambda$ denotes $\int_{R^n} g\,d\lambda$. $f\colon R^n \to \mathfrak{X}$ is said to be Gelfand–Pettis integrable [6], or weakly integrable with respect to λ if:

(1) $x^* f$ is λ-integrable for every $x^* \in \mathfrak{X}^*$;

(2) For every $E \in \mathscr{L}$ there exists an element $x_E \in \mathfrak{X}$ such that $x^*(x_E) = \int_E x^* f\,d\lambda$ for every $x^* \in \mathfrak{X}^*$.

In this case we define x_E to be the weak integral of f over E; in symbols: $x_E = \int_E f\,d\lambda$. f is measurable if it is the almost everywhere (a.e.)

limit of a sequence of simple functions. $f\colon R^n \to \mathfrak{X}$ is said to be Bochner integrable [2] if it is measurable and $|f|$ is integrable. ξ_E denotes the characteristic function of the set E.

3. The main results.

THEOREM 1. *Let $f\colon R^n \to \mathfrak{X}$ be measurable and weakly integrable. Then there exist elements $x_i \in \mathfrak{X}$ and intervals I_i, $i = 1, 2, \ldots$ such that*

1. *$f = \sum_{i=1}^{\infty} x_i \xi_{I_j}$ a.e., where the convergence is absolute;*

2. *$\int f d\lambda = \sum_{i=1}^{\infty} x_i \lambda(I_i)$, where the convergence is unconditional.*

Remark 1. We note that any function which satisfies the above two conditions is necessarily measurable and weakly integrable.

Remark 2. Suppose that the series $\sum_{i=1}^{\infty} x_i \lambda(I_i)$ converges absolutely. Then from the theory of the Bochner integral, it follows that $\sum_{i=1}^{\infty} x_i \xi_{I_i}$ converges absolutely a.e. It is natural to ask if an analogous result holds for the weak integral. The following shows that it does not; we give an example in which $\sum_{i=1}^{\infty} x_i \lambda(I_i)$ converges unconditionally and $\sum_{i=1}^{\infty} x_i \xi_{I_i}$ diverges on a set of positive measure. Let $\mathfrak{X} = l_2$; x_n denotes the element in l_2 with 1 in the nth place and zeros elsewhere. Construct intervals I_n in $[0, 1]$ such that $\lambda(I_n) = 1/n$ and $x \in [0, 1]$ implies that x belongs to infinitely many I_n. Note that $\sum_{i=1}^{\infty} x_n \lambda(I_n)$ converges unconditionally in l_2, but $\sum_{i=1}^{\infty} x_n \xi_{I_n}$ diverges everywhere on $[0, 1]$.

THEOREM 2. *Let S be a measurable set of finite measure. If ε is a positive number, then there exist numbers $a_n = \pm 1$ and intervals I_n such that*

1. *$\xi_S = \sum_{n=1}^{\infty} a_n \xi_{I_n}$ a.e.;*

2. *$\sum_{n=1}^{\infty} \lambda(I_n) < \lambda(S) + \varepsilon$.*

Proof. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon/2$. We shall construct a sequence $\{S_n\}$ of measurable sets. In the sequel, ξ_n will denote ξ_{S_n}; ξ_0 denotes the characteristic function of S. Define successively the sets S_n such that for each n

1° S_n is a union of non-overlapping intervals;

2° $\sum_{i=0}^{n} (-1)^{n-i} \xi_i \geqslant 0$;

3° $\sum_{i=0}^{n} (-1)^{n-i} \int \xi_i < \varepsilon_n$.

Because of 2°, the inequality in 3° can be written as

$$\int \Big| \xi_0 - \sum_{i=1}^{n} (-1)^{i-1} \xi_i \Big| < \varepsilon_n.$$

Since $\varepsilon_n \to 0$, we deduce that

$$\xi_S = \sum_{i=1}^{\infty} (-1)^{i-1} \xi_i \text{ a.e.}.$$

Moreover, we obtain from 3° that

$$\int \xi_n < \sum_{i=0}^{n-1} (-1)^{n-i-1} \int \xi_i + \varepsilon_n < \varepsilon_{n-1} + \varepsilon_n, \qquad n = 1, 2, \ldots,$$

where $\varepsilon_0 = \int \xi_0$. Hence

$$\sum_{i=1}^{\infty} \int \xi_i < \int \xi_0 + 2 \sum_{n=1}^{\infty} \varepsilon_n < \int \xi_0 + \varepsilon. \tag{1}$$

By 1°, $S_i = \bigcup_{j=1}^{\infty} J_j^i$, where for each fixed i the intervals $J_1^i, J_2^i, \ldots$ are non-overlapping. Thus $\xi_i = \sum_{j=1}^{\infty} \xi_{J_j^i}$ and $\int \xi_i = \sum_{j=1}^{\infty} \lambda(J_j^i)$. We now order the pairs (i, j) into a sequence $\{p_n\}$. Let $I_n = J_{p_n}$ and $a_n = (-1)^{i-1}$, where i is the first element in the pair p_n. Then $\xi_S = \xi_0 = \sum_{n=1}^{\infty} a_n \xi_{I_n}$ a.e., and by (1), $\sum_{n=1}^{\infty} \lambda(I_n) = \sum_{i=1}^{\infty} \int \xi_i < \lambda(S) + \varepsilon$.

4. Preliminary lemmas. We now present some lemmas that will be used in the proof of Theorem 1.

LEMMA 1. (cf. [5]). *Let $f\colon R^n \to \mathfrak{X}$ be Bochner integrable. Then there exist elements $x_n \in \mathfrak{X}$ and intervals I_n such that:*

(a) *$f = \sum_{n=1}^{\infty} x_n \xi_{I_n}$ a.e., where the convergence is absolute a.e.;*

(b) *$\sum_{n=1}^{\infty} x_n \lambda(I_n)$ converges absolutely.*

The next lemma is a special case of Theorem 1 in [3].

LEMMA 2. *Let $f\colon R^n \to \mathfrak{X}$ be a measurable weakly integrable function. Then f can be represented in the form $f = g + h$ a.e., where g is a bounded Bochner integrable function and h assumes at most countably many values in $\mathfrak{X}$. If one writes h in the form $h = \sum_{i=1}^{\infty} x_i \xi_{E_i}$, where the measurable sets E_i are disjoint, then $\int_E f d\lambda = \int_E g d\lambda + \sum_{i=1}^{\infty} x_i \lambda(E \cap E_i)$, where the series converges unconditionally for every measurable set E.*

We shall need the inequality stated in the lemma below ([1], Th. 5 and [4]).

LEMMA 3. *Let $\{\lambda_i\}_{i=1}^n$ and $\{a_i\}_{i=1}^n$ be sets of complex numbers and elements of $\mathfrak{X}$ respectively. Then there exists a subset $\Delta \subset \{1, 2, \ldots, n\}$ such that*

$$|\lambda_1 a_1 + \ldots + \lambda_n a_n| \leqslant 4(\max_i |\lambda_i|)\left(\left|\sum_{i\in\Delta} a_i\right|\right).$$

LEMMA 4. *Let $\{a_i\}$ be a sequence of elements from $\mathfrak{X}$ such that $\sum_{i=1}^{\infty} a_i$ converges unconditionally. Let $\{\lambda_{ij}\}$ be a sequence of complex numbers such that $\sum_{j=1}^{\infty} |\lambda_{ij}| < M$, $i = 1, 2, \ldots$, for some number M. Then $\sum_{i,j} \lambda_{ij} a_i$ converges unconditionally.*

Proof. Assume that the $a_i \neq 0$. Let $\varepsilon > 0$ be given. There is an index i_0 such that $\left|\sum_{i\in T} a_i\right| < \frac{\varepsilon}{8M}$ for every finite set T of integers such that $i \in T$ implies that $i > i_0$. There is an index j_0 such that $\sum_{j=j_0}^{\infty} |\lambda_{ij}| < \frac{\varepsilon}{2i_0 |a_i|}$ for $i = 1, 2, \ldots, i_0$.

Let I_r be the set of ordered pairs of integers (i, j) such that $(i, j) \in I_r$ if and only if $i \leqslant r$ and $j \leqslant r$. Let $r_0 = \max\{i_0, j_0\}$. It suffices to show that for $r \geqslant r_0$, we have $\left|\sum_{(i,j)\in S} \lambda_{ij} a_i\right| < \varepsilon$, for every finite set S of ordered pairs of integers such that $S \cap I_{r_0} = \emptyset$. In fact, let J_i denote the set of all j such that $(i, j) \in S$. Since S is finite, the J_i are empty for large i, say $i > k > i_0$. We then have

$$\left|\sum_{(i,j)\in S} \lambda_{ij} a_j\right| \leqslant \left|\left(\sum_{i=1}^{i_0} a_i\right)\left(\sum_{j\in J_i} \lambda_{ij}\right)\right| + \left|\sum_{i=i_0+1}^{k} a_i \left(\sum_{j\in J_i} \lambda_{ij}\right)\right| = U + V < \varepsilon,$$

since $U < \left(\sum_{i=1}^{i_0} |a_i|\right) \left(\sum_{j=j_0}^{k} \lambda_{ij}\right) \leqslant \varepsilon/2$, and by Lemma 3, $V \leqslant 4M \left|\sum_{i\in\Delta} a_i\right| \leqslant \varepsilon/2$, for some $\Delta \subset \{i_0+1, \ldots, k\}$.

5. The proof of Theorem 1. By virtue of Lemmas 1 and 2, we may assume that our function f has the form $f = \sum_{i=1}^{\infty} x_i \xi_{E_i}$, where the sets E_i are pairwise disjoint. By Theorem 2, for each i there exist $a_j^i = \pm 1$ and intervals I_j^i, $j = 1, 2, \ldots$, such that

$$\xi_{E_i} = \sum_{j=1}^{\infty} a_j^i \xi_{I_j^i} \quad \text{a.e.}$$

and

$$\sum_{j=1}^{\infty} \lambda(I_j^i) \leqslant \lambda(E_i) + \min\{\lambda(I_i), 2^{-i}\}, \quad i = 1, 2, \ldots$$

Let Q be any bounded interval in R^n. First of all,

$$\sum_j \lambda(I_j^i \cap Q) = \sum_j \lambda(I_j^i) - \sum_j \lambda(I_j^i - Q) \leqslant \sum_j \lambda(I_j^i) - \lambda\Big(\big(\bigcup_{j=1}^{\infty} I_j^i\big) - Q\Big)$$
$$\leqslant \lambda(E_i) + \min\{\lambda(E_i), 2^{-i}\} - \lambda(E_i - Q) = \lambda(E_i \cap Q) + \min\{\lambda(E_i), 2^{-i}\}.$$

Hence

$$\sum_{i,j} \lambda(I_j^i \cap Q) \leqslant \sum_i \lambda(E_i \cap Q) + \sum_i 2^{-i} \leqslant \lambda(Q) + 1 < \infty.$$

One can show, for example, by means of the Borel–Cantelli lemma, that $\lambda(\overline{\lim_{i,j}} I_j^i) = 0$. This means that except for a set of measure zero, if $s \in Q$, then s belongs to at most finitely many I_j^i, $i, j = 1, 2, \ldots$ Thus, since R^n is a countable union of bounded intervals, we conclude that

$$f = \sum_{i,j} a_j^i x_i \xi_{I_j^i} \quad \text{a.e.},$$

where the convergence is absolute a.e.. Consider now the series

$$\sum_{i,j} a_j^i x_i \lambda(I_j^i).$$

This series converges unconditionally by Lemma 4, where we put $\lambda_{ij} = \frac{a_j^i \lambda(I_j^i)}{\lambda(E_i)}$ and $a_i = x_i \lambda(E_i)$. This completes the proof of the theorem.

Remark 3. In a later paper, the first author will extend Lemma 2 to spaces which are not necessairly σ-finite, and will present a representation theorem similar to Theorem 1 in a different setting.

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