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Structure of Blaschke cocycles

by

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Abstract. Analytic cocycles have been shown to play the same role on a group K dual to a subgroup of the discrete real line as innerfunctions on the circle group. In a previous paper, an arbitrary analytic cocycle was shown to be the product of a Blaschke and a singular cocycle (unless already of one of these types). Now a necessary and sufficient condition is given for a subset of $R^+ \times K$ (R^+ is the set of positive real numbers) to be the "zero set" of a Blaschke cocycle. The main difficulty is measure-theoretic; it arises in constructing a cocycle having a given zero set. The result is used to prove that every (non-constant) analytic cocycle is the product of two other.

1. Let Γ be a countable subgroup of R_d, the discrete real line, and K the separable compact group dual to Γ. We shall assume that Γ is not cyclic, so K is not a circle. Harmonic analysis on K has been studied in several previous papers [3], [4], [6], particularly the aspect related to the order defined in Γ. The basic problem is to describe certain functions defined on $R \times K$ (R is the real line) called *analytic cocycles*. Each analytic cocycle is the product of a Blaschke and a singular cocycle. The purpose of this paper is to show that an arbitrary Borel set subject to some obvious necessary conditions is the zero set for a Blaschke cocycle. Thus Blaschke cocycles are parametrized by their zero sets almost as effectively as ordinary Blaschke products. Although no similar analysis of singular cocycles has been made ([1]), the method of proof can be used to show that all analytic cocycles can be properly factored. This means that $H^2(K)$ has no maximal invariant subspaces except the trivial one $H_0^2(K)$. (The relation of cocycles to harmonic analysis is explained in [3]; but here we shall discuss the cocycles themselves. An easy result from [4] will be required later. However Theorem 1 below implies the main result of [4] and the proof will be independent.)

The elements of K are $x, y, \ldots$; those of Γ are $\gamma, \lambda, \ldots$. K contains a distinguished subgroup K_0 consisting of all characters on Γ of the form $e_t(\lambda) = e^{it\lambda}$, where t is a real number. The mapping of R into K that carries t to e_t is one-one and continuous, and K_0 is dense in K.

([1]) The author stated in [5] that all singular cocyles are trivial. That assertion is hereby retracted.

A *cocycle* on K is a Borel function $A(t, x)$ on $R \times K$ such that for all real t, u and x in K we have

$$|A(t, x)| = 1, \quad A(t+u, x) = A(t, x)A(u, x+e_t). \tag{1}$$

(The definition in this form, without reference to exceptional null sets, was given by Gamelin [2]. Once certain measure-theoretic difficulties have been met it works more smoothly than older definitions [3], [6].)

A cocycle is called *analytic* if for almost every x the function $A(t, x)$ defined on R is the boundary function of $A(z, x)$ analytic and bounded in the upper half-plane, but not constant in z. That is, $A(z, x)$ is an *inner function*. If this inner function is a Blaschke product for almost all x the cocycle is a *Blaschke cocycle*; if it is a singular inner function for almost all x, the cocycle is called *singular*. In [4] it was shown that every analytic cocycle is the product of a Blaschke and a singular cocycle, unless it is already of one type.

With these definitions we can go on to our problem.

2. Let A be a Blaschke cocycle on K, and E_x the set of zeros of $A(z, x)$ in the upper half-plane. For the null set of x on which $A(z, x)$ is not an inner function E_x is not yet defined. In order to obtain a definite Borel set for every x we can define $A(z, x)$ by the Poisson integral of the bounded function $A(t, x)$, obtaining an inner function for almost all x but a harmonic function everywhere. Let E_x be the zeros of this function.

The sets E_x are related by the functional equation of a cocycle. The complex version of (1)

$$A(t+z, x) = A(t, x)A(z, x+e_t) \tag{2}$$

shows that $E_{x+e_t} = E_x - t$ for all t and x. Let E be the subset of $R^+ \times K$ (R^+ is the set of positive real numbers) consisting of all (u, x) such that iu is a zero of $A(z, x)$. Then E describes the zeros of the cocycle completely: $A(t+iu, x) = 0$ just if $(u, x+e_t)$ is in E. And E is a Borel set.

For (u, x) in E let $q(u, x)$ be the multiplicity of the zero of $A(z, x)$ at $z = iu$. For x in the Borel null set where $A(z, x)$ is not inner we set $q = 0$. Then q is a Borel function on E. For the derivative of $A(z, x)$ with respect to z is a Borel function whose zeros have the same translation property as those of A, and the zeros of the derivative are associated with the points of E where $q \geqslant 2$. Similarly the points where $q \geqslant n$ form a Borel subset of E for each integer n, so q is a Borel function.

Let $X = x + K_0$, the coset of K_0 containing x. Denote by E_X the set of all (u, y) in E such that y is in X. We can visualize a half-plane erected over the line X; E_X is the set of points in E lying in this half-plane. But it is important to note that this half-plane has no natural horizontal coordinate, because X has no origin.

For any choice of origin in X, E_X is (for almost all x) the set of zeros of a bounded analytic function, and therefore satisfies the Blaschke condition. Each point of E_X is to be repeated, of course, according to its multiplicity.

Our main theorem now asserts that any set E in $R^+ \times K$ with multiplicity function q having all these properties arises from a Blaschke cocycle.

3. THEOREM 1. *Let E be a Borel set in $R^+ \times K$ and q a Borel function on E taking positive integral values. Suppose that q, interpreted as a multiplicity function, satisfies the Blaschke condition on E_X for each $X = x + K_0$, except possibly in a null set of x. Assume that E_X is not empty for almost all x. Then there is a Blaschke cocycle whose zero set matches E_X, and whose multiplicity function matches q, for almost all x.*

A set E such that E_X is empty for almost every x will be called *negligible*. A union of countably many negligible sets is negligible. Two cocycles are customarily identified if they differ only on a null set of (t, x). Then it is obvious, if we grant the theorem, that two Borel sets arise from the same cocycle if and only if they differ by a negligible set.

If E is not negligible, then E_X is infinite for almost every x [4]. Thus there is no analogue for cocycles of finite Blaschke products.

The main difficulty in the proof, met in the following lemma, is measure-theoretic. This is not surprising, because on each coset X there is a Blaschke inner function having the right zeros, and the problem is to fit them together measurably to form a cocycle. We shall need some theorems about Borel structures, now accessible in a new expository paper by Arveson [1].

LEMMA. *The function*

$$m(x) = \sum 2u_n(u_n^2 + t_n^2)^{-1}, \tag{3}$$

where the summation extends over the elements $(u_n, x+e_{t_n})$ of E_X, $X = x + K_0$, is absolutely measurable on K.

In words: we sum the Poisson kernel over the points of E in the half-plane over X, with x chosen as origin. The result should depend measurably on x. (A function is called *absolutely measurable* if is measurable for every Borel measure.)

Choose and fix a positive integer r. We construct the Polish space $D = K \times (R^+ \times K)^r$, whose elements are sequences $(x, u_1, x_1, \ldots, u_r, x_r)$ where each x is in K and each u in R^+. Denote by F the subset of points such that each x_j belongs to $x + K_0$. These are the sequences

$$(x, u_1, x+e_{t_1}, \ldots, u_r, x+e_{t_r}) \tag{4}$$

for arbitrary real $t_1, \ldots, t_r$. F is a Borel subset of D. Indeed the set F_k of points in F such that $|t_j| \leqslant k$ (each j) is closed, and F is the countable union of F_k.

The subset of F consisting of points (4) such that some two pairs (u_j, t_j) are identical is relatively closed in F. Hence the set G of all points in F such that all these pairs are distinct is a Borel subset of D.

Let f_k be the function defined on F by

$$\sum_{j=1}^{r} 2u_j(u_j^2+t_j^2)^{-1} \tag{5}$$

at each point (4) belonging to F_k, and equal to 0 at other points of F. This function is continuous on F_k, and therefore is a Borel function on F. Hence $f = \lim f_k$, defined by (5) for all points of F, is a Borel function.

Denote by H the subset of G consisting of those points (4) in G such that $(u_j, x+e_{t_j})$ is in E for each j. H is another Borel subset of D, and f restricted to H is a Borel function.

For any real k let H_k be the subset of H on which $f > k$. Let P be the function that carries each element of D to its first component x. P is continuous and H_k is a Borel subset of D; therefore the image $P(H_k)$ is an analytic set in K. But this image is just the set of x such that *some* r terms in (3) have sum exceeding k. The set where (3) itself is greater than k is the union for $r = 1, 2, \ldots$ of analytic sets $P(H_k)$. Since analytic sets are absolutely measurable the set where $m(x) > k$ is absolutely measurable for every k. Hence m is absolutely measurable.

The lemma remains true if we incorporate the multiplicity function q in (3):

$$m(x) = \sum 2u_n d_n(u_n^2+t_n^2)^{-1}, \tag{6}$$

where $d_n = q(u_n, x+e_{t_n})$, is still absolutely measurable. For the subset of E on which q has a given value is a Borel set to which the lemma is applicable, and (6) is a weighted sum of these functions.

The sum (6) is finite for almost every x because E_X (with q) satisfies the Blaschke condition. In order to make m smooth on cosets of K_0 we assume for the moment that all the points (u, x) of E satisfy $u \geqslant \delta > 0$. A straightforward argument shows in this case that $m(x+e_t)$ is continuous as a function of t for almost every x. We set

$$A(t, x) = e^{i\int_0^t m(x+e_s)ds}, \tag{7}$$

a formula used previously to produce cocycles [3]. Then identically in t

$$\frac{1}{i}\frac{d}{dt}\log A(t, x) = m(x+e_t). \tag{8}$$

By definition

$$m(x+e_t) = \sum 2u_n d_n(u_n^2+(t_n-t)^2)^{-1}, \quad (u_n, x+e_{t_n}) \text{ in } E. \tag{9}$$

The Blaschke product on zeros t_n+iu_n with multiplicity d_n

$$B(t) = \prod \varepsilon_n (t-(t_n+iu_n))^{d_n}(t-(t_n-iu_n))^{-d_n}, \quad |\varepsilon_n| = 1, \tag{10}$$

has the same logarithmic derivative (9). Therefore $A(t, x)$ is a constant multiple of $B(t)$, where the constant depends of course on x. If $A(t, x)$ is a Borel function, it is a Blaschke cocycle with exactly the right zeros.

Now $m(x)$ was shown to be absolutely measurable. It is possible to change its values on a Haar null set so as to be a Borel function. Then $A(t, x)$ is a Borel function, and for almost all x differs from the old function only on a null set of t. This completes the proof of the theorem under the special hypothesis on E.

In the general case let E_δ be the set of (u, x) in E such that $u \geqslant \delta$. For each positive δ we obtain a cocycle A_δ with zeros on E_δ of the proper multiplicity. It is unfortunately not possible to conclude that A_δ tends to a limiting cocycle with zeros on E as δ decreases to 0; for Blaschke products converge when they are normalized to be positive at a fixed point of the upper half-plane, whereas a cocycle equals 1 at the origin. We are forced to a less direct argument.

Let $A'_\delta(z, x)$ be $A_\delta(z, x)$ multiplied by a constant of modulus 1 so that it or its first non-vanishing derivative is positive at $z = i$. We may set $A'_\delta(z, x) = 1$ for those x in a fixed Borel null set, invariant under translations from K_0, where $A(z, x)$ is not an inner function. Then $A'_\delta(z, x)$ is a Borel function of (z, x). The limit

$$B(z, x) = \lim_{\delta\to 0} A'_\delta(z, x) \tag{11}$$

exists for all z in the upper half-plane and all x, and is a Blaschke product with zeros of the proper multiplicity on E for almost all x. B is not a cocycle, but its zeros have the translation property of cocycles. Since a Blaschke product is determined by its zeros up to a multiplicative constant,

$$B(t+z, x) = L(t, x)B(z, x+e_t) \tag{12}$$

for some number L depending on t and x. It is easy to verify that L is a cocycle.

Since B is inner for fixed x, $B(t+iu, x)$ has a limit $B(t, x)$ for almost all t as u decreases to 0. The exceptional set of t depends on x, but there is one t at least, by the Fubini theorem, such that the limit exists for almost every x. For this t and any positive numbers u, v we have

$$(13)\quad \int |B(t+iu, x)-B(t+iv, x)|^2 dx = \int |B(iu, x+e_t)-B(iv, x+e_t)|^2 dx = \int |B(iu, x)-B(iv, x)|^2 dx.$$

The first integral tends to 0 as u and v decrease independently to 0. Hence $B(iu, x)$ has a limit in the norm of $L^2(K)$ as u tends to 0. The same argument shows that $B(t+iu, x)$ converges in $L^2(K)$ for *every* real t. We denote this limit by B_t, defined only modulo null sets of x.

In (12) replace z by iu and let u decrease to 0. We obtain

$$(14)\qquad B_t = L_t \cdot T_t B_0,$$

an equality in $L^2(K)$, where T_t is translation: $T_t f(x) = f(x+e_t)$ for any f. Choose a definite Borel representative for the element B_0 of $L^2(K)$, of modulus 1 everywhere. The right side of (14) is now a Borel function of (t, x). Thus B_t can be chosen so as to make the equation true for all t and x. Dividing by B_0 exhibits B_t/B_0 as a product of the cocycle L with a coboundary. Hence $A_t = B_t/B_0$ is a cocycle. For almost every x, $A(t, x)$ is the limit of $B(t+iu, x)/B(0, x)$ for almost every t, because a pointwise limit must agree with the Lebesgue limit almost everywhere. Hence A is a Blaschke cocycle with the desired zeros, and the proof is finished.

4. An ordinary inner function can be factored as the product of inner functions, unless it is a simple Blaschke factor. This is obvious if the inner function is not a pure Blaschke or singular function. For Blaschke products it is proved by detaching one factor from the product; a singular inner function is the square of its square root. A Blaschke cocycle has infinitely many zeros above almost every coset of K_0 [4], so there is no trivially excluded case, and we ask whether analytic cocycles can always be factored.

THEOREM 2. *Every analytic cocycle is the product of two others.*

Let A be an analytic cocycle. If it is not of Blaschke or singular type, then it is the product of two pure cocycles [4]. Thus we only have to study cocycles of pure type.

First suppose A is a Blaschke cocycle. It suffices to express its zero set E (which is non-negligible by definition of an analytic cocycle) as the disjoint union of two non-negligible Borel subsets. Then Theorem 1 provides Blaschke cocycles B and C whose zeros together match the zeros of A. It follows that $A = BC$.

Such a decomposition is immediate except in the case that u has the same value for every (u, x) in E, except for a negligible subset of E. Thus we may suppress u and consider E as a subset of K.

For any positive element γ of Γ let G_γ be the closed subgroup of K consisting of all x such that $x(\gamma) = 1$. Every x in K can be written uniquely in the form $y_\gamma + e_t$, where y_γ is in G_γ and $0 \leqslant t < 2\pi/\gamma$. The natural correspondence between K and $G_\gamma \times [0, 2\pi/\gamma)$ preserves Borel sets; without changing notation we view E as a subset of the product space.

Let E_t be the set of elements (y, u) of E with $u < t$. Denote by t_0 the upper bound of t such that E_t is negligible. (If E_t is non-negligible for each positive t, set $t_0 = 0$.) Then E_{t_0} itself is negligible, which implies that $t_0 < 2\pi/\gamma$. Similarly let t_1 be the lower bound of t such that the set of (y, u) in E with $u > t$ is negligible, or $t_1 = 2\pi/\gamma$ if this set is non-negligible for every smaller t. If $t_0 < t_1$, and if t is any number between them, then E_t and its complement in E are both non-negligible, and the separation has been accomplished.

Otherwise $t_0 = t_1$, and this number may be 0 but cannot be $2\pi/\gamma$. The value of (y, u) as a character on γ is $e^{iu\gamma}$. Now $u < t_0$ only on a negligible subset of E; and $u > t_0$ on another negligible subset. Hence for all x in E except a negligible subset we have $x(\gamma) = e^{it_0\gamma}$.

The theorem is proved if t_0 turns out less than t_1 for any positive γ in Γ. Otherwise the elements of E have the same value as characters on γ except in a negligible subset of E; and this is true for every positive γ. The exceptional set depends on γ, but since Γ is countable the statement holds for all γ at once except on a grand negligible set. Since an element of K is determined by its values as a character on the positive elements of Γ, E can have only one element outside the negligible set. This is absurd, and the proof for Blaschke cocycles is finished.

Let A be a singular cocycle. Define $A'(z, x)$ to be $A(z, x)$ multiplied by a constant of modulus 1 so that $A'(i, x) > 0$. On the Borel null set of x, invariant under translations from K_0, on which $A(z, x)$ fails to be a singular inner function, let $A'(z, x) = 1$. Then A' is a Borel function in (z, x). Set $B(z, x) = A'(z, x)^{1/2}$, choosing the square root so as to be positive at $z = i$. Then B is a Borel function. Indeed the values of A and its derivatives at i are Borel functions of x, in terms of which we can compute $B(z, x)$ for $|z-i| < 1$. Hence B is a Borel function on the product of this circle with K. If B is redefined to be 0 for z outside this circle, we have a Borel function on the whole product space. By analytic continuation we obtain a sequence of Borel functions whose limit is the original function $B(z, x)$.

B will not be a cocycle, but its modulus satisfies

$$(15)\qquad |B(t+z, x)| = |B(z, x+e_t)|$$

because B^2 has the same modulus as A. Hence (12) holds for a function $L(t, x)$ that is again a cocycle. The argument goes on through (13) and (14) to show that $C_t = B_t/B_0$ is a cocycle. Now $B(t, x)$ is the boundary function of $B(z, x)$ whose square is $A(z, x)$, aside from a constant factor (depending on x). Hence C is a singular cocycle whose square is a function of x times A. This implies $C^2 = A$, and the theorem is proved.

5. The restriction to countable Γ and separable K is not essential. Without any restriction, a cocycle $A(t, x)$ is continuous as a mapping from R to $L^2(K)$ ([2], p. 186). Hence A_t takes its values in a separable subspace of $L^2(K)$, so the non-null Fourier coefficients of all the functions A_t lie in a countable subgroup of Γ. Thus A can be studied on a separable quotient group of K.

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Weak integrals defined on Euclidean n-space

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Abstract. The following representation for Banach-valued measurable weakly integrable functions on Euclidean n-space is established: $f = \sum_{i=1}^{\infty} x_i \xi_{I_i}$, where the x_i are elements of the given Banach space and the ξ_{I_i} are characteristic functions of intervals I_i; the convergence is absolute a.e. The weak integral of f is given by the equality $\int f d\lambda = \sum_{i=1}^{\infty} x_i \lambda(I_i)$, where the convergence is unconditional. This approach avoids entirely the use of functionals.

1. Introduction. In this paper we establish a representation theorem (Theorem 1) for Banach-valued measurable weakly integrable functions defined on Euclidean n-space, where the underlying measure is Lebesgue measure. The representation is given in terms of intervals and unconditionally convergent series. As a result, our approach avoids the use of the conjugate space and the theory of Lebesgue measure, except for the concept of almost everywhere convergence.

We also present a construction of Lebesgue measurable sets which seems to be an effective tool for examining measurable sets in terms of intervals (Theorem 2).

2. Definitions. $\mathfrak{X}$ is a Banach space over the complex numbers with conjugate space $\mathfrak{X}^*$. $|x|$ is the norm of an element $x \in \mathfrak{X}$. $(R^n, \mathscr{L}, \lambda)$ denotes the measure space consisting of the Lebesgue measurable subsets of R^n, with n-dimensional Lebesgue measure λ. $\int g$ or $\int g\, d\lambda$ denotes $\int_{R^n} g\, d\lambda$. $f: R^n \to \mathfrak{X}$ is said to be Gelfand–Pettis integrable [6], or weakly integrable with respect to λ if:

(1) $x^* f$ is λ-integrable for every $x^* \in \mathfrak{X}^*$;

(2) For every $E \in \mathscr{L}$ there exists an element $x_E \in \mathfrak{X}$ such that $x^*(x_E) = \int_E x^* f d\lambda$ for every $x^* \in \mathfrak{X}^*$.

In this case we define x_E to be the weak integral of f over E; in symbols: $x_E = \int_E f\, d\lambda$. f is measurable if it is the almost everywhere (a.e.)