Structure of Blaschke cocycles

by

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Abstract. Analytic cocycles have been shown to play the same role on a group $K$ dual to a subgroup of the discrete real line as innerfunctions on the circle group. In a previous paper, an arbitrary analytic cocycle was shown to be the product of a Blaschke and a singular cocycle (unless already of one of these types). Now a necessary and sufficient condition is given for a subset of $B^+ \times K$ ($B^+$ is the set of positive real numbers) to be the "zero set" of a Blaschke cocycle. The main difficulty is measure-theoretic: it arises in constructing a cocycle having a given zero set. The result is used to prove that every (non-constant) analytic cocycle is the product of two other.

1. Let $\Gamma$ be a countable subgroup of $B_\mu$, the discrete real line, and $\Gamma$ the separable compact group dual to $\Gamma$. We shall assume that $\Gamma$ is not cyclic, so $K$ is not a circle. Harmonic analysis on $K$ has been studied in several previous papers [3], [4], [6], particularly the aspect related to the order defined in $\Gamma$. The basic problem is to describe certain functions defined on $R \times K$ ($R$ is the real line) called analytic cocycles. Each analytic cocycle is the product of a Blaschke and a singular cocycle. The purpose of this paper is to show that an arbitrary Borel set subject to some obvious necessary conditions is the zero set for a Blaschke cocycle. Thus Blaschke cocycles are parametrized by their zero sets almost as effectively as ordinary Blaschke products. Although no similar analysis of singular cocycles has been made (1), the method of proof can be used to show that all analytic cocycles can be properly factored. This means that $H^2(K)$ has no maximal invariant subspaces except the trivial one $H^2(K)$. (The relation of cocycles to harmonic analysis is explained in [3]; but here we shall discuss the cocycles themselves. An easy result from [4] will be required later. However Theorem 1 below implies the main result of [4] and the proof will be independent.)

The elements of $K$ are $x, y, \ldots$; those of $\Gamma$ are $\gamma, \lambda, \ldots$. $\Gamma$ contains a distinguished subgroup $\Gamma_\mu$ consisting of all characters on $\Gamma$ of the form $\eta(x) = e^{itx}$, where $t$ is a real number. The mapping of $R$ into $K$ that carries $t$ to $\eta$ is one-one and continuous, and $\Gamma_\mu$ is dense in $K$.

(1) The author stated in [5] that all singular cocycles are trivial. That assertion is hereby retracted.
A cocycle on $K$ is a Borel function $A(t, x)$ on $R \times K$ such that for all real $t$, $u$ and $z$ in $K$ we have

$$A(t, u, w) = A(t, w + t, A(t, w)),$$

(1)

(The definition in this form, without reference to exceptional null sets, was given by Gamelin [2].) Once certain measure-theoretic difficulties have been met it works more smoothly than older definitions [3], [6].

A cocycle is called analytic if for almost every $x$ the function $A(t, x)$ defined on $E$ is the boundary function of $A(z, x)$ analytic and bounded in the upper half-plane, but not constant in $z$. That is, $A(z, x)$ is an inner function. If this inner function is a Blaschke product for almost all $x$ the cocycle is a Blaschke cocycle; if it is a singular inner function for almost all $x$, the cocycle is called singular. In [4] it was shown that every analytic cocycle is the product of a Blaschke and a singular cocycle, unless it is already of one type.

With these definitions we can go on to our problem.

2. Let $A$ be a Blaschke cocycle on $K$ and $E$ the set of zeros of $A(z, x)$ in the upper half-plane. For the null set of $x$ on which $A(z, x)$ is not an inner function $E_x$ is not yet defined. In order to obtain a definite Borel set for every $x$ we can define $A(z, x)$ by the Poisson integral of the bounded function $A(t, x)$, obtaining an inner function for almost all $x$ a harmonic function everywhere. Let $E_x$ be the zeros of this function.

The sets $E_x$ are related by the functional equation of the cocycle. The complex version of (1)

$$A((t + i, x) = A(t, x)A(x, x + t)$$

(2)

shows that $E_{x + x} = E_x + x$ for all $t$ and $x$. Let $E$ be the subset of $R^+ \times K$ ($R^+$ is the set of positive reals) consisting of all $(u, x)$ such that $iu$ is a zero of $A(z, x)$. Then $E$ describes the zeros of the cocycle completely: $A((t + iu, x) = 0$ just if $(u, u + x)$ is in $E$. And $E$ is a Borel set.

For $(u, x)$ in $E$ let $q(u, x)$ be the multiplicity of the zero of $A(z, x)$ at $z = iu$. For $x$ in the Borel null set where $A(z, x)$ is not inner we set $q = 0$. Then $q$ is a Borel function on $E$. For the derivative of $A(z, x)$ with respect to $z$ is a Borel function whose zeros have the same translation property as those of $A$, and the zeros of the derivative are associated with the points of $E$ where $q \geq 2$. Similarly the points where $q \geq n$ form a Borel subset of $E$ for each integer $n$, so $q$ is a Borel function.

Let $X = x + K_x$, the coset of $K_x$ containing $x$. Denote by $E_x$ the set of all $(u, y)$ in $E$ such that $y$ is in $X$. We can visualize a half-plane erected over the line $X$; $E_x$ is the set of points in $E$ lying in this half-plane. But it is important to note that this half-plane has no natural horizontal coordinate, because $X$ has no origin.

For any choice of origin in $X$, $E_x$ is (for almost all $x$) the set of zeros of a bounded analytic function, and therefore satisfies the Blaschke condition. Each point of $E_x$ is to be repeated, of course, according to its multiplicity.

Our main theorem now asserts that any set $E$ in $R^+ \times K$ with multiplicity function $q$ having all these properties arises from a Blaschke cocycle.

3. Theorem 1. Let $E$ be a Borel set in $R^+ \times K$ and $q$ a Borel function on $E$ taking positive integral values. Suppose that $q$, interpreted as a multiplicity function, satisfies the Blaschke condition on $E_x$ for each $x = x + K_x$, except possibly in a null set of $x$. Assume that $E_x$ is not empty for almost all $x$. Then there is a Blaschke cocycle whose zero set matches $E_x$, and whose multiplicity function matches $q$, for almost all $x$.

A set $E$ such that $E_x$ is empty for almost all $x$ will be called negligible. A union of countably many negligible sets is negligible. Two cocycles are customarily identified if they differ only on a null set of $(t, x)$. Then it is obvious, if we grant the theorem, that two Borel sets arise from the same cocycle if and only if they differ by a negligible set.

If $E$ is not negligible, then $E_x$ is infinite for almost every $x$ [4]. Thus there is no analogue for cocycles of finite Blaschke products.

The main difficulty in the proof, met in the following lemma, is measure-theoretic. This is not surprising, because on each coset $X$ there is a Blaschke inner function having the right zeros, and the problem is to fit them together measurably to form a cocycle. We shall need some theorems about Borel structures, now accessible in a new expository paper by Arveson [1].

**Lemma.**

$$m(x) = \sum u(u_x) \gamma_u,$$

where the summation extends over the elements $(u_x, x + e_x)$ of $E_x$, $X = x + K_x$, is absolutely measurable on $E$.

In words: we sum the Poisson kernel over the points of $E$ in the half-plane over $X$, with $x$ chosen as origin. The result should depend measurably on $x$. (A function is called absolutely measurable if is measurable for every Borel measure.)

Choose and fix a positive integer $e$. We construct the Polish space $D = K \times (R^+ \times K'$, whose elements are sequences $(x, u_1, x + e_1, \ldots, u_n, x)$ where each $x$ is in $K$ and each $u$ in $R^+$. Denote by $E$ the subset of points such that each $x$ belongs to $x + K_x$. These are the sequences

$$(x, u_1, x + e_1, \ldots, u_n, x + e_n)$$
for arbitrary real $t_1, \ldots, t_n$. $F$ is a Borel subset of $D$. Indeed the set $F_n$ of points in $F$ such that $|t_j| < k$ (each $j$) is closed, and $F$ is the countable union of $F_n$.

The subset of $F$ consisting of points (4) such that some two pairs $(u_j, t_j)$ are identical is relatively closed in $F$. Hence the set $\Theta$ of all points in $F$ such that all these pairs are distinct is a Borel subset of $D$.

Let $f_\epsilon$ be the function defined on $F$ by

$$
(5) \quad f_\epsilon(x) = \sum_{j=1}^{n} 2u_j (u_j^2 + \epsilon^2)^{-1}
$$

at each point (4) belonging to $F_n$, and equal to 0 at other points of $F$. This function is continuous on $F_n$, and therefore is a Borel function on $F$. Hence $f = \lim f_\epsilon$, defined by (5) for all points of $F$, is a Borel function.

Denote by $H$ the subset of $\Theta$ consisting of those points (4) in $\Theta$ such that $(u_j, x - \epsilon k_j)$ is in $F$ for each $j$. $H$ is another Borel subset of $D$, and if restricted to $H$ is a Borel function.

For any real $k$ let $H_k$ be the subset of $H$ on which $f > k$. Let $P$ be the function that carries each element of $D$ to its first component $x$. $P$ is continuous and $H_k$ is a Borel subset of $D$; therefore the image $P(H_k)$ is an analytic set in $K$. But this image is just the set of $x$ such that $\sum x_r$ in (3) has sum exceeding $k$. The set (3) itself is greater than $k$ is the union for $r = 1, 2, \ldots$ of analytic sets $P(H_k)$. Since analytic sets are absolutely measurable, the set where $m(x) > k$ is absolutely measurable for every $k$. Hence $m$ is absolutely measurable.

The lemma remains true if we incorporate the multiplicity function $\eta$ in (3):

$$
(6) \quad m(x) = \sum_{j=1}^{n} 2u_j \eta_j (u_j^2 + \epsilon^2)^{-1},
$$

where $\eta_j = \eta(u_j, x - \epsilon k_j)$, is still absolutely measurable. For the subset of $E$ on which $\eta$ has a given value is a Borel set to which the lemma is applicable, and (6) is a weighted sum of these functions.

The sum (6) is finite for almost every $x$ because $E_x$ (with $\eta$) satisfies the Blaschke condition. In order to make $m$ smooth on cosets of $K_x$ we assume for the moment that all the points $(u_j, x)$ of $E$ satisfy $u \geq \epsilon > 0$. A straightforward argument shows in this case that $m(x + \epsilon)$ is continuous as a function of $t$ for almost every $x$. We set

$$
(7) \quad A(t, x) = \int_n m(x+\epsilon) \eta_j dt
$$

a formula used previously to produce cocycles [3]. Then identically in $t$

$$
(8) \quad \frac{1}{i} \frac{d}{dt} \log A(t, x) = m(x + \epsilon).
$$

By definition

$$
(9) \quad m(x + \epsilon) = \sum_{k=1}^{n} 2u_k \eta_k (u_k^2 - (x_k - \epsilon)^2)^{-1}, \quad (u_k, x - \epsilon k_k) \in E.
$$

The Blaschke product on zeros $t_k - in_k$ with multiplicity $\eta_k$

$$
(10) \quad B(t) = \prod_{k=1}^{n} \left( t_k - (t_k - in_k)^{-\eta_k} \right)^{\eta_k}, \quad \eta_k = 1,
$$

has the same logarithmic derivative (9). Therefore $A(t, x)$ is a constant multiple of $B(t)$, where the constant depends of course on $x$. If $A(t, x)$ is a Borel function, it is a Blaschke cocycle with exactly the zero riesz.

Now $m(x)$ was shown to be absolutely measurable. It is possible to change its values on a Haar null set so as to be a Borel function. Then $A(t, x)$ is a Borel function, and for almost all $x$ differs from the old function only on a null set of $t$. This completes the proof of the theorem under the special hypothesis on $E$.

In the general case let $E_\delta$ be the set of $(u, x) \in E$ such that $u \geq \delta$. For each positive $\delta$ we obtain a cocycle $A_\delta$ with zeros on $E_\delta$ of the proper multiplicity. It is unfortunately not possible to conclude that $A_\delta$ tends to a limiting cocycle with zeros on $E$ as $\delta$ decreases to 0; for Blaschke products converge when they are normalized to be positive at a fixed point of the upper half-plane, whereas a cocycle equals 1 at the origin. We are forced to a less direct argument.

Let $A_\delta(z, x)$ be $A_\delta(z, x)$ multiplied by a constant of modulus 1 so that it or its first non-vanishing derivative is positive at $z = 1$. We may set $A_\delta(z, x) = 0$ for those $x$ in a fixed Borel null set, invariant under translations from $K_x$, where $A(z, x)$ is not an inner function. Then $A_\delta(z, x)$ is a Borel function of $(z, x)$. The limit

$$
(11) \quad B(z, x) = \lim_{\delta \to 0} A_\delta(z, x)
$$

exists for all $z$ in the upper half-plane and all $x$, and is a Blaschke product with zeros of the proper multiplicity on $E$ for almost all $x$. $B$ is not a cocycle, but its zeros have the translation property of cocycles. Since a Blaschke product is determined by its zeros up to a multiplicative constant,

$$
(12) \quad B(t + z, x) = L(t, x)B(z, x + \epsilon)
$$

for some number $L$ depending on $t$ and $x$. It is easy to verify that $L$ is a cocycle.

Since $B$ is inner for fixed $x$, $B(t + iz, x)$ has a limit $B(t, x)$ for almost all $t$ as $u$ decreases to 0. The exceptional set of $t$ depends on $x$, but there is one $t$ at least, by the Fubini theorem, such that the limit exists for
almost every $x$. For this $t$ and any positive numbers $u, v$, we have

$$
(13) \int |B(t+ix, x) - B(t+ix, x)|^2 \, dx = \int |B(iu, x + \epsilon) - B(i\epsilon, x + \epsilon)|^2 \, dx = \int |B(iu, x) - B(i\epsilon, x)|^2 \, dx.
$$

The first integral tends to 0 as $u$ and $v$ decrease independently to 0. Hence $B(is, x)$ has a limit in the norm of $L^2(K)$ as $s$ tends to 0. The same argument shows that $B(t+is, x)$ converges in $L^2(K)$ for every real $t$. We denote this limit by $B_t$, defined only modulo null sets of $x$.

In (12) replace $z$ by $is$ and let $u$ decrease to 0. We obtain

$$
(14) \quad B_t = T \cap B_0,
$$

an equality in $L^2(K)$, where $T$ is translation: $Tf(s) = f(s + \epsilon)$ for any $f$. Choose a definite Borel representative for the element $B_0$ of $L^2(K)$; of modulus 1 everywhere. The right side of (14) is now a Borel function of $(t, x)$. Thus $B_t$ can be chosen so as to make the equation true for all $x$ and $t$. Dividing by $B_0$ exhibits $B_t/B_0$ as a product of the cocycle $J_t$ with a coboundary. Hence $A_0 = B_t/B_0$ is a cocycle. For almost every $x$, $A(t, x)$ is the limit of $B(t+is, x)/B_0(x)$ for almost every $t$, because a pointwise limit must agree with the Lebesgue limit almost everywhere. Hence $A$ is a Blaschke cocycle with the desired zeros, and the proof is finished.

4. An ordinary inner function can be factored as the product of inner functions, unless it is a simple Blaschke factor. This is obvious if the inner function is not a pure Blaschke or singular function. For Blaschke products, it is proved by detaching one factor from the product; a singular inner function is the square of its square root. A Blaschke cocycle has infinitely many zeros above almost every coset of $K$ ([4]), so there is no trivially excluded case, and we ask whether analytic cocycles can always be factored.

**Theorem 2.** Every analytic cocycle is the product of two others.

Let $A$ be an analytic cocycle. If it is not of Blaschke or singular type, then it is the product of two pure cocycles [4]. Thus we only have to study cocycles of pure type.

First suppose $A$ is a Blaschke cocycle. It suffices to express its zero set $E$ (which is non-negligible by definition of analytic cocycle) as the disjoint union of two non-negligible Borel subsets. Then Theorem 1 provides Blaschke cocycles $B$ and $C$ whose zeros together match the zeros of $A$. It follows that $A = BC$.

Such a decomposition is immediate except in the case that $w$ has the same value for every $(u, v)$ in $E$ for a negligible subset of $E$. Thus we may suppose $u$ and consider $E$ as a subset of $K$.

For any positive element $y$ of $\Gamma$ let $G_y$ be the closed subgroup of $K$ consisting of all $x$ such that $x(y) = 1$. Every $x$ in $K$ can be written uniquely in the form $x + \epsilon$, where $y_\epsilon$ is in $G_{\epsilon}$ and $0 < \epsilon < 2\pi/y$. The natural correspondence between $K$ and $G_{\epsilon} \times [0, 2\pi/y]$ preserves Borel sets; without changing notation we view $E$ as a subset of the product space.

Let $B_0$ be the set of elements $(y, u, v)$ of $E$ with $u < v$. Denote by $t_0$ the upper bound of $u$ such that $E_{t_0}$ is negligible. If $E_{t_0}$ is non-negligible for each positive $t_0$, set $t_0 = 0$. Then $E_{t_0}$ itself is negligible, which implies that $t_0 < 2\pi/y$. Similarly let $t_0$ be the lower bound of $t$ such that the set of $(y, u, v)$ in $E$ with $u > v$ is negligible, or $t_0 = 2\pi/y$ if this set is non-negligible for every smaller $t$. If $t_0 < t_0$, and if $t$ is any number between them, then $E_{t}$ and its complement in $E$ are both non-negligible, and the separation has been accomplished.

Otherwise $t_0 = t_0$, and this number may be 0 but cannot be $2\pi/y$. The value of $(y, u, v)$ as a character on $\gamma$ is $e^{uy}$. Now $u < t_0$ only on a negligible subset of $E$; and $v > t_0$ on another negligible subset. Hence for all $x$ in $E$ except a negligible subset we have $x(y) = e^{uy}$. The theorem is proved if $t_0$ turns out less than $t_0$ for any positive $y$ in $\Gamma$. Otherwise the elements of $E$ have the same value as characters on $\gamma$ except in a negligible subset of $E$; and this is true for every positive $y$.

The exceptional set depends on $y$, but since $\Gamma$ is countable the statement holds for all $y$ at once except on a grand negligible set. Since an element of $K$ is determined by its values as a character on the positive elements of $\Gamma$, $E$ can have only one element outside the negligible set. This is absurd, and the proof for Blaschke cocycles is finished.

Let $A$ be a singular cocycle. Define $A'(x, y)$ to be $A(x, y)$ multiplied by a constant of modulus 1 so that $A'(x, t) > 0$. On the Borel null set of $x$, invariant under translations from $K$, on which $A(x, y)$ fails to be a singular inner function, let $A'(x, y) = 1$. Then $A'$ is a Borel function in $(x, y)$. Set $B_0(x, y) = A'(x, y)^{1/2}$, choosing the square root so as to be positive at $x = y$. Then $B$ is a Borel function. Indeed the values of $A$ and its derivatives at 0 are Borel functions of $x$ in terms of which we can compute $B_0(x, y)$ for $|x - y| < 1$. Hence $B$ is a Borel function on the product of this circle with $K$. If $B$ is redefined to be 0 for $x$ outside this circle, we have a Borel function on the whole product space. By analytic continuation we obtain a sequence of Borel functions whose limit is the original function $B(x, y)$.

$B$ will not be a cocycle, but its modulus satisfies

$$
|B(t+i, x)|^2 = |B(t, x+\epsilon)|
$$

because $B^2$ has the same modulus as $A$. Hence (13) holds for a function $L(t, x)$ that is again a cocycle. The argument goes on through (13) and (14) to show that $C = B_0/B_0$ is a cocycle. Now $B(t, x)$ is the boundary
function of $B(t, z)$ whose square is $A(t, z)$, aside from a constant factor (depending on $z$). Hence $G$ is a singular cocycle whose square is a function of $z$ times $A$. This implies $G = A$, and the theorem is proved.

5. The restriction to countable $\Gamma$ and separable $K$ is not essential. Without any restriction, a cocycle $A(t, z)$ is continuous as a mapping from $R$ to $U(K)$ ([2], p. 186). Hence $A_t$ takes its values in a separable subspace of $U(K)$, so the non-null Fourier coefficients of all the functions $A_t$ lie in a countable subgroup of $G$. Thus $A$ can be studied on a separable quotient group of $K$.

References


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Weak integrals defined on Euclidean n-space

by

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Abstract. The following representation for Banach-valued measurable weakly integrable functions on Euclidean $n$-space is established: $f = \sum_{i=1}^{\infty} x_i I_{I_i}$, where the $x_i$ are elements of the given Banach space and the $I_i$ are characteristic functions of intervals $I_i$; the convergence is absolute a.e. The weak integral of $f$ is given by the equality $\int f d\lambda = \sum_{i=1}^{\infty} x_i \lambda(I_i)$, where the convergence is unconditional. This approach avoids entirely the use of functionals.

1. Introduction. In this paper we establish a representation theorem (Theorem 1) for Banach-valued measurable weakly integrable functions defined on Euclidean $n$-space, where the underlying measure is Lebesgue measure. The representation is given in terms of intervals and unconditionally convergent series. As a result, our approach avoids the use of the conjugate space and the theory of Lebesgue measure, except for the concept of almost everywhere convergence.

We also present a construction of Lebesgue measurable sets which seems to be an effective tool for examining measurable sets in terms of intervals (Theorem 2).

2. Definitions. $X$ is a Banach space with conjugate space $X'$. $|x|$ is the norm of an element $x \in X$. $(R^n, L, \lambda)$ denotes the measure space consisting of the Lebesgue measurable subsets of $R^n$, with $n$-dimensional Lebesgue measure $\lambda$. $fg$ or $\int g d\lambda$ denotes $\int g d\lambda; f: R^n \to X$ is said to be Gelfand-Pettis integrable [6], or weakly integrable with respect to $\lambda$ if:

1. $\alpha f$ is $\lambda$-integrable for every $\alpha \in \mathbb{R}$;

2. For every $E \in \mathcal{L}$ there exists an element $x_E \in X$ such that $\alpha \in (x_E) = \int \alpha f d\lambda$ for every $\alpha \in \mathbb{R}$.

In this case we define $x_E$ to be the weak integral of $f$ over $E$; in symbols: $x_E = \int f d\lambda$. $f$ is measurable if it is the almost everywhere (a.e.)