Note our choice of τ in (5), and infer from (14) that

(15) $$|\mu^{\wedge}(\chi)-\tau\beta(\chi)| \leqslant \tfrac{1}{2}\tau \text{ for all } \chi \in \Delta\cup\Delta^{-1},$$

in Case (4). In like fashion, (12_2) leads to

(15_2) $$|\mu^{\wedge}(\chi)-2\tau\beta(\chi)| \leqslant \tfrac{1}{2}\tau \text{ for all } \chi \in \Delta\cup\Delta^{-1},$$

in Case (4_2). From (15) it is evident that the measure $\nu = \frac{1}{\tau}\mu$ satisfies (3) with $d = \frac{1}{2}$. From (15_2) it is evident that $\nu = \frac{1}{2\tau}\mu$ satisfies (3) with $d = \frac{1}{4}$. ■

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A divergent multiple Fourier series of power series type

by

J. MARSHALL ASH and LAWRENCE GLUCK (Chicago, Ill.)

We present this paper to honor a great mathematician. The first co-author wishes to thank Professor Zygmund for the personal interest he has taken and the encouragement he has given over the pre- and post-doctoral years.

Abstract A continuous complex-valued function on the torus whose (double) Fourier series diverges restrictedly rectangularly at every point has been constructed by Charles Fefferman. The present paper presents a function which has the above properties and whose Fourier series is of power series type ($a_{mn} = 0$ if $m < 0$ or $n < 0$).

Charles Fefferman [2] has given an example of a continuous function $F(x, y)$ defined on the torus T^2 with the property that the double Fourier series $\Sigma a_{mn}\exp(i(mx+ny))$ of F is everywhere restrictedly rectangularly divergent. This means that for each point (x, y) and $E > 1$,

$$S_{MN}(x, y) = \sum_{\substack{|m|\leqslant M\\ |n|\leqslant N}} a_{mn} e^{i(mx+ny)}$$

fails to tend to a limit as M and N tend to infinity with $E^{-1} \leqslant M/N \leqslant E$.

In this paper we extend Fefferman's result by proving the following.

THEOREM 1. *There is a continuous complex-valued function $H(x, y)$ on the torus whose double Fourier series is of power series type ($a_{mn} = 0$ if $m < 0$ or $n < 0$) and is restrictedly rectangularly divergent everywhere.*

On $[0, 2\pi]\times[0, 2\pi]$ set $g_0(x, y) = g_0(x, y;\ \lambda) = \varphi(x)\varphi(y)e^{i\lambda xy}$ where φ is a C^∞ function equal to 0 *if* $0 \leqslant t \leqslant 1/40$ or if $2\pi - \frac{1}{40} \leqslant t \leqslant 2\pi$ and to 1 *if* $\frac{1}{20} \leqslant t \leqslant 2\pi - \frac{1}{20}$ with $0 \leqslant \varphi \leqslant 1$ elsewhere on $[0, 2\pi]$. The real parameter λ is greater then 1. Then clearly g_0 is *a* C^∞ function on the torus T^2 obtained from $[0, 2\pi]\times[0, 2\pi]$ by identifications, and $\|g_0\|_\infty = \sup|g_0(x, y)| = 1$.

We consider the three conjugate functions of g_0,

$$g_1(x, y) = g_1(x, y; \lambda) = \frac{1}{\pi}\int_0^{2\pi} \frac{g_0(s, y)}{2\tan\frac{x-s}{2}}\,ds,$$

$$g_2(x, y) = g_2(x, y; \lambda) = \frac{1}{\pi}\int_0^{2\pi} \frac{g_0(x, t)}{2\tan\frac{y-t}{2}}\,dt,$$

and

$$g_3(x, y) = g_3(x, y; \lambda) = \frac{1}{\pi}\int_0^{2\pi} \frac{g_1(x, t)}{2\tan\frac{y-t}{2}}\,dt = \frac{1}{\pi}\int_0^{2\pi} \frac{g_2(s, y)}{2\tan\frac{x-s}{2}}\,ds$$

([5], vol. I, p. 51).

The equality of the two representations of g_3 follows from the facts that as functions on the torus they both have the same double Fourier series and both are continuous. Actually, all these integrals are C^∞ functions on the torus. Consider

$$g_1(x, y) = (2\pi)^{-1}\int_0^{2\pi} g_0(s, y)\cot\big((x-s)/2\big)\,ds = (2\pi)^{-1}\int_0^{2\pi} g_0(x-s, y)\cot(s/2)\,ds$$
$$= (2\pi)^{-1}\int_0^{\pi} [g_0(x-s, y) - g_0(x+s, y)]\cot(s/2)\,ds.$$

Although, in principle, g_1 is defined only in the principal value sense, it follows from the infinite differentiability of g_0 that the integrand together with all its derivatives are continuous functions on the torus. It follows from Leibnitz's rule for differentiation under the integral sign that $g_1 \in C^\infty(T^2)$. Since g_1 and g_2 are C^∞, the two representations of g_3 may be shown by the above argument also to be C^∞.

Remark. We shall often differentiate under principal value integrals without explicitly repeating this argument.

We form the function

$$h(x, y) = h_\lambda(x, y) = g_0(x, y) - g_3(x, y) + i[g_1(x, y) + g_2(x, y)].$$

Then h is C^∞ and has a double Fourier series of power series type. We shall devote the main body of the paper to proving that h has the following four properties:

LEMMA 1. *The function h is continuous on T^2 and bounded, independent of λ.*

LEMMA 2. *If λ is sufficiently large, $|S_{MN}(h)| \leqslant A\log\lambda$.*

LEMMA 3. *If $\lambda \geqslant 400\min\{M, N\}$, $|S_{MN}(h)| \leqslant A$.*

LEMMA 4. *If $(x, y) \in Q = [.1, 2\pi-.1]\times[.1, 2\pi-.1]$ and M = the greatest integer in $\lambda y = [\lambda y]$, $N = [\lambda x]$, then $|S_{MN}(h)(x, y)| \geqslant B\log\lambda$ for λ sufficiently large.*

(*Constants will be independent of all parameters unless otherwise indicated.*)

We temporarily postpone the proofs of these lemmas in order to show how Theorem 1 follows from them.

Proof of Theorem 1. Arrange the countable set $S = \{(x, y) \in [0, 2\pi) \times [0, 2\pi) \mid x, y \text{ rational}\}$ into a sequence, $S = \{S_k\}_{k=1,2,\ldots}$. For $k = 1, 2, \ldots$ let T_k be the translation of the torus with the property that $T_k(S_k) = (\pi, \pi)$. Then,

$$H(x, y) = \sum_{k=1}^{\infty} c_k h_{\lambda(k)}\big(T_k(x, y)\big),$$

with $c_k = 2^{-2^k}$ and $\lambda(k) = \exp(4^{2^k})$, is the function of Theorem 1.

The choice of $\{c_k\}$ and Lemma 1 assure that this series converges uniformly to a continuous function on T^2. Since the Fourier series of each $h_{\lambda(k)}\big(T_k(x, y)\big)$ is of power series type, so also is that of H. It remains only to demonstrate the divergence of the Fourier series of H. Fix a point $(x, y) \in T^2$ and an $E > 1$. Set $T_k(x, y) = \big(x(k), y(k)\big)$. Since S is dense and $\lambda(k)$ tends rapidly to infinity, there are infinitely many k with the following properties:

(i) $4 \geqslant x(k) \geqslant 1$, $4 \geqslant y(k) \geqslant 1$.

(ii) $E^{-1} \leqslant \dfrac{[\lambda(k)y(k)]}{[\lambda(k)x(k)]} \leqslant E$ (this will hold if S_k is close to (x, y) and if $\lambda(k)$ is large relative to $1/(E-1)$).

(iii) $\lambda(k)$ is so large that Lemmas 2 and 4 are valid with $\lambda = \lambda(k)$.

(iv) $\lambda(k)$ is so large that $\lambda(k+1) \geqslant 400\cdot\lambda(k)\cdot 4$.

For such a k we have

$$|S_{[\lambda(k)y(k)],[\lambda(k)x(k)]}(H)(x, y)| \geqslant -\sum_{j=1}^{k-1} |c_j h_{\lambda(j)} T_j(x, y)| +$$
$$+ |c_k h_{\lambda(k)}\big(T_k(x, y)\big)| - \sum_{j=k+1}^{\infty} |c_j h_{\lambda(j)}\big(T_j(x, y)\big)|$$
$$\geqslant -\sum_{j=1}^{k-1} c_j \log\lambda(j) A + Bc_k\log\lambda(k) - A\sum_{j=k+1}^{\infty} c_j,$$

by Lemmas 2, 4, and 3 respectively. Hence,

$$S_E^*(x, y) = \sup_{\substack{\min\{M,N\}\to\infty \\ E^{-1}\leqslant M/N\leqslant E}} |S_{MN}(x, y)| \geqslant B2^{2^k} - A - A\sum_{j=1}^{k-1} 2^{2^j}$$

for infinitely many k. Since the right side $\to\infty$ as $k\to\infty$, $S_E^*(x,y)=\infty$, so that the Fourier series of H diverges restrictedly at (x,y). Since (x,y) and E were arbitrary, Theorem 1 is proved.

Remark. A theorem of Plessner states that the everywhere convergence of a trigonometric series of one variable forces the almost everywhere convergence of its conjugate ([5], vol. II, p. 216). If Plessner's theorem were extendable to the two variable case, one could easily deduce from it and Fefferman's result a slightly diluted version of Theorem 1. However, such an extension is not possible, as the following counterexample shows. Consider the series $S=M(x)\delta(y)=\Sigma a_m e^{i(mx+ny)}$ where $\delta(y)=\Sigma e^{iny}$ and $M(x)\,(=\Sigma a_m e^{imx})$, the Fourier–Stieltjes series of a singular measure, converges to zero almost everywhere. Although S is convergent almost everywhere, $S_1=\widetilde{M(x)}\,\delta(y)=\sum(-i\operatorname{sgn} m)\,a_m e^{i(mx+ny)}$ diverges almost everywhere. For details, see [1]. (Also see [3].)

To prove Lemmas 1 through 4 we will need five technical lemmas. Let

$$C_1(x,y)=C_1(x,y;\lambda)=\varphi(y)\int_0^{2\pi}e^{i\lambda(s-x)y}\varphi(s)\cdot\tfrac12\cot\big((x-s)/2\big)\,ds,$$

$$C_2(x,y)=C_2(x,y;\lambda)=\varphi(x)\int_0^{2\pi}e^{i\lambda(t-y)x}\varphi(t)\cdot\tfrac12\cot\big((y-t)/2\big)\,dt,$$

$$C_3(x,y)=C_3(x,y;\lambda)=\int_0^{2\pi}\int_0^{2\pi}e^{i\lambda(st-xy)}\varphi(s)\varphi(t)\cdot\tfrac14\cot\big((x-s)/2\big)\times$$
$$\times\cot\big((y-t)/2\big)\,dt.$$

From $g_1(x,y)=e^{i\lambda xy}C_1(x,y)/\pi$, $g_2(x,y)=e^{i\lambda xy}C_2(x,y)/\pi$, and $g_3(x,y)=e^{i\lambda xy}C_3(x,y)/\pi^2$, it follows that the $C_j\in C^\infty([0,2\pi]\times[0,2\pi])$. (Differentiation is one-sided on the boundary.) But the C_j, like $e^{i\lambda xy}$, are not periodic, and hence not continuous on T^2.

LEMMA 5. *For $j=1,2,3$, $|C_j|\leqslant A$, $|D_1C_j|\leqslant A$, $|D_2C_j|\leqslant A$, and for all (x,y) satisfying $d\big((x,y),\partial T^2\big)=$ distance from (x,y) to boundary $([0,2\pi]\times\times[0,2\pi])>1/160$, $|D_1D_2C_j(x,y)|\leqslant A$.*

Here D_l denotes partial differentiation with respect to the l-th variable.

Proof. We write

$$(1)\qquad C_1(x,y)=\varphi(y)\int_{x-2\pi}^{x}\frac{e^{-i\lambda ys}}{2\tan(s/2)}\varphi(x-s)\,ds$$
$$=\varphi(y)\varphi(x)\int_{x-2\pi}^{x}\frac{e^{-i\lambda ys}}{s}ds-\varphi(y)\int_{x-2\pi}^{x}e^{-i\lambda ys}\Big[\varphi(x-\xi)\frac{\xi}{2\tan(\xi/2)}\Big]'ds$$

where ξ, $0<|\xi|<|s|$ is determined by the mean value theorem. If $d(x,\partial T^1)$ $=$ distance from x to boundary $([0,2\pi])\leqslant 1/80$, then $\varphi(x-s)\equiv 0$ in a neighbourhood of the singularities of $\cot(s/2)$. Hence, it follows from the first line of (1) that C_1 is bounded for such x. In the contrary case, the boundedness of the last integral follows from the fact that $[x-2\pi,x]\subset\subset[-2\pi+1/80,2\pi-1/80]$. For the first term we note that the integrand is bounded if $|s|>1/80$ so that one need only show that

$$G=\int_{-1/80}^{1/80}\frac{e^{-i\lambda ys}}{s}ds$$

is bounded. The process which achieves this we shall call *folding*. This involves writting $G=\int_{-1/80}^{0}+\int_{0}^{1/80}=A+B$. Substitute $s=-s$ in A and then combine. We have

$$G=-2i\int_0^{1/80}\frac{\sin\lambda ys}{s}ds=-2i\int_0^{\lambda y/80}\frac{\sin s}{s}ds=0(1)$$

([5]; vol. I, p. 57). This proves that C_1 is bounded on T^2. The boundedness of C_2 follows by symmetry.

We consider D_1C_1. Since the integrand vanishes identically near the endpoints, we have

$$(2)\qquad D_1C_1(x,y)=\varphi(y)\int_{x-2\pi}^{x}e^{-i\lambda ys}\cdot\tfrac12\cot(s/2)\varphi'(x-s)\,ds.$$

This and all subsequent differentiations under the principal value integral signs are justifiable by arguments similar to the one that showed $g_1(x,y)$ to be C^∞. Since this expression differs from C_1 only in that $\varphi'(x-s)$ replaces $\varphi(x-s)$, the same argument that showed C_1 to be bounded applies to D_1C_1. By symmetry, D_2C_2 is also bounded.

For D_2C_1, we have

$$(3)\qquad D_2C_1(x,y)=\varphi'(y)\int_{x-2\pi}^{x}e^{-i\lambda ys}\cdot\tfrac12\cot(s/2)\varphi(x-s)\,ds$$
$$+\big(\varphi(y)/y\big)\int_{x-2\pi}^{x}(-i\lambda ye^{-i\lambda ys})\{s\cdot\tfrac12\cot(s/2)\varphi(x-s)\}\,ds=A+B.$$

Clearly A is bounded. (See the argument following equation (1).) An integration by parts yields

$$B=\big(\varphi(y)/y\big)\Big\{\big(s\cdot\tfrac12\cot(s/2)\big)e^{-i\lambda ys}\big|_{x-2\pi}^{x}-\int_{x-2\pi}^{x}e^{-i\lambda ys}\big(s\cdot\tfrac12\cot(s/2)\,\varphi(x-s)\big)'\,ds\Big\}.$$

The integrated terms vanish and the last integral is bounded. (If $d(x, \partial T^1) \geqslant 1/80$, s stays away from $\pm 2\pi$; if $d(x, \partial T^1) \leqslant 1/80$, $\varphi(x-s) = \varphi'(x-s) = 0$ whenever s is within $1/80$ of $\pm 2\pi$.) Since $\varphi(y)/y \leqslant 40$, $D_2 C_1$ is bounded. By symmetry, $D_1 C_2$ is bounded.

The discussion of C_3 will require the boundedness of $D_2 D_1 C_1$, $D_2 D_2 C_1$, and $D_2 D_2 D_1 C_1$. The investigation of the behavior of the higher derivatives of C_1 (and, symmetrically, C_2) merely involves iterating the above techniques. We leave the details to the reader.

We now turn to C_3. Since

$$g_3(x, y) = (2\pi)^{-} \int_{y-2\pi}^{y} g_1(x, y-t)\cot(t/2)\,dt,$$

we may write

$$C_3(x, y) = \int_{y-2\pi}^{y} e^{-i\lambda xt}\cdot \tfrac{1}{2}\cot(t/2)\, C_1(x, y-t)\,dt. \tag{4}$$

To show that C_3 is bounded, we essentially repeat the argument that showed $C_1(x, y)$ to be bounded. One need only observe that the factor $C_1(x, y-t)$ in equation (4) has the two properties that were required of the corresponding factor $\varphi(x-s)$ in equation (1); first $C_1(x, \tau) = 0$ if $d(\tau, \partial T^1) \leqslant 1/40$ and second that $D_2 C_1$ is bounded.

As in equation (2), we have

$$D_2 C_3(x, y) = \int_{y-2\pi}^{y} \frac{e^{-i\lambda xt}}{2\tan\frac{t}{2}} D_2 C_1(x, y-t)\,dt. \tag{5}$$

This is bounded exactly as was C_3 itself since $D_2 C_1(x, \tau)$ again has the properties required of $\varphi(x-s)$ in equation (1).

We may rewrite

$$C_3(x, y) = \pi^2 e^{-i\lambda xy} g_3(x, y) = \int_{x-2\pi}^{x} e^{-i\lambda xs}\cdot \tfrac{1}{2}\cot(s/2)\, C_2(x-s, y)\,ds$$

by using the second integral representation of g_3. From this representation it is clear by symmetry that $D_1 C_3(x, y)$ is bounded.

Finally, restrict x to $[1/160, 2\pi]$ and differentiate (5) to obtain

$$D_1 D_2 C_3(x, y) = \int_{y-2\pi}^{y} \frac{e^{-i\lambda xt}}{2\tan\frac{t}{2}} D_1 D_2 C_1(x, y-t)\,dt$$
$$+\frac{1}{x}\int_{y-2\pi}^{y} (-i\lambda x e^{-i\lambda xt})\left\{\frac{t}{2\tan\frac{t}{2}} D_2 C_1(x, y-t)\right\} dt.$$

The factor $\dfrac{1}{x}$ is bounded by 160. The boundedness of the second term now follows from integration by parts. (The factor in curly brackets plays the role of the factor in curly brackets in equation (3).) The first integral is bounded since

$$D_1 D_2 C_1(x,\tau) = D_2 D_1 C_1(x, \tau) \qquad \text{(recall } C_1 \text{ is } C^\infty)$$
$$= \varphi'(\tau)\int_{x-2\pi}^{x} e^{-i\lambda\tau s}\cdot \tfrac{1}{2}\cot(s/2)\,\varphi'(x-s)\,ds +$$
$$+\big(\varphi(\tau)/\tau\big)\int_{x-2\pi}^{x} (e^{-i\lambda\tau s})' s\cdot \tfrac{1}{2}\cot(s/2)\,\varphi'(x-s)\,ds$$

again has the properties required of $\varphi(x-s)$ in equation (1).

LEMMA 6. (LOCALIZATION) *Let $C(x, y)$ satisfy the smoothness properties specified in Lemma 5. Then for any δ, ε and (x, y) with $|\delta| \leqslant 1$, $|\varepsilon| \leqslant 1$ and $d((x, y), \partial T^2) \geqslant 1/160$, we have*

$$S_{MN}\big(e^{i\lambda xy} C(x, y)\big) = T_{M+\delta, N+\varepsilon} - T_{M+\delta, -(N+\varepsilon)} - T_{-(M+\delta), N+\varepsilon} + \tag{6}$$
$$+ T_{-(M+\delta), -(N+\varepsilon)} + E(x, y;\ M, N;\ \delta, \varepsilon;\ \lambda)$$

where $S_{MN}\big(f(x, y)\big)$ denotes the M, N-th partial sum of the double Fourier series of f,

$$T_{\alpha,\beta} = T_{\alpha,\beta}\big(f(x, y)\big) = \frac{-1}{4\pi^2}\int_{x-1/160}^{x+1/160}\int_{y-1/160}^{y+1/160} \frac{f(s, t)\, e^{-i(\alpha s+\beta t)}}{(x-s)(y-t)}\,ds\,dt,$$

and E is bounded independently of all its parameters.

Remark. Since localization is not true for continuous functions of two variables in general ([5], vol. II, p. 304), the conclusions of this lemma do not follow from general considerations.

Proof. We have ([5], vol. II, p. 302 and vol. I, p. 49)

$$S_{MN} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{i\lambda st} C(s, t)\, \frac{\sin(M+\frac{1}{2})(x-s)\sin(N+\frac{1}{2})(y-t)}{2\sin\frac{x-s}{2}\cdot 2\sin\frac{y-t}{2}}\,ds\,dt$$
$$= \pi^{-2}\iint_X + \pi^{-2}\iint_{T^2-X}$$

where

$$X=\left(\left[x-\frac{1}{160},x+\frac{1}{160}\right]\times\left[y+\frac{1}{160},2\pi\right]\right)\cup\left(\left[x-\frac{1}{160},x+\frac{1}{160}\right]\times\right.$$
$$\left.\times\left[0,y-\frac{1}{160}\right]\right)\cup\left(\left[x+\frac{1}{160},2\pi\right]\times\left[y-\frac{1}{160},y+\frac{1}{160}\right]\right)\cup$$
$$\cup\left(\left[0,x-\frac{1}{160}\right]\times\left[y-\frac{1}{160},y+\frac{1}{160}\right]\right)\cup\left(\left[x-\frac{1}{160},x+\frac{1}{160}\right]\times\right.$$
$$\left.\times\left[y-\frac{1}{160},y+\frac{1}{160}\right]\right)=\bigcup_{i=1}^{5}A_i$$

is a cross neighbourhood of (x,y). The integrand is bounded off X so $\pi^{-2}\iint\limits_{T^2-X}$ may be absorbed by E. Now

(7)

$$\iint\limits_{A_1}=\int\limits_{y+1/160}^{2\pi}\frac{e^{i\lambda xt}\sin(N+\frac{1}{2})(y-t)}{2\sin\frac{y-t}{2}}\left(\int\limits_{-1/160}^{1/160}\frac{e^{-i\lambda st}\sin(M+\frac{1}{2})s\,C(x-s,t)}{2\sin\frac{s}{2}}\,ds\right)dt,$$

so $\left(\text{since } -2\pi+\frac{1}{160}\leqslant y-t\leqslant-\frac{1}{160}\right)$ to bound the contribution from A_1, it suffices to bound the inner integral, call it I, of equation (7). Since

$$\frac{1}{s}\left(C(x-s,t)\frac{s}{2\sin\frac{s}{2}}\right)$$
$$=\frac{C(x,t)}{s}+s\left\{\left(\frac{\sigma}{2\sin\frac{\sigma}{2}}\right)'C(x-\sigma,t)+\frac{\sigma}{2\sin\frac{\sigma}{2}}D_1C(x-\sigma,t)\right\}\Bigg|_{\sigma=\theta(x,t,s)s},$$

where the term in curly brackets is bounded, we need only fold to get that

$$I+O(1)=\int\limits_{-1/160}^{1/160}\frac{e^{-i\lambda st}\sin(M+\frac{1}{2})s}{s}\,ds$$
$$=\int\limits_{0}^{1/160}\frac{\sin(M+\frac{1}{2}-\lambda t)s+\sin(M+\frac{1}{2}+\lambda t)s}{s}\,ds$$
$$=\int\limits_{0}^{\frac{M+\frac{1}{2}-\lambda t}{160}}\frac{\sin u}{u}\,du+\int\limits_{0}^{\frac{M+\frac{1}{2}+\lambda t}{160}}\frac{\sin u}{u}\,du=O(1).$$

The integrals over the regions A_2, A_3, and A_4 behave symmetrically so that $\sum\limits_{i=1}^{4}\iint\limits_{A_i}$ may be absorbed by E. Several additions and subtractions give

(8)

$$\iint\limits_{A_5}=\int\limits_{x-1/160}^{x+1/160}\int\limits_{y-1/160}^{y+1/160}\frac{e^{i\lambda st}\sin(M+\delta)(x-s)\cdot\sin(N+\varepsilon)(y-t)\cdot C(s,t)}{(x-s)(y-t)}\,ds\,dt+$$
$$+\int\limits_{x-1/160}^{x+1/160}\sin(M+\delta)(x-s)\left\{\frac{1}{2\sin\frac{x-s}{2}}-\frac{1}{(x-s)}\right\}e^{i\lambda ys}\times$$
$$\times\left(\int\limits_{-1/160}^{1/160}\frac{e^{-i\lambda st}\sin(N+\varepsilon)t\cdot C(s,y-\tau)}{t}\,dt\right)ds+$$
$$+\int\limits_{y-1/160}^{y+1/160}\sin(N+\varepsilon)(y-t)\left\{\frac{1}{2\sin\frac{y-t}{2}}-\frac{1}{(y-t)}\right\}e^{i\lambda xt}\times$$
$$\times\left(\int\limits_{-1/160}^{1/160}\frac{e^{-i\lambda st}\sin(M+\delta)s\cdot C(x-s,t)}{2\sin\frac{s}{2}}\,ds\right)dt+$$
$$+\int\limits_{y-1/160}^{y+1/160}\left\{\frac{\sin(N+\frac{1}{2})(y-t)-\sin(N+\varepsilon)(y-t)}{2\sin\frac{y-t}{2}}\right\}e^{i\lambda xt}\times$$
$$\times\left(\int\limits_{-1/160}^{1/160}\frac{e^{-i\lambda st}\sin(M+\delta)s\cdot C(x-s,t)}{2\sin\frac{s}{2}}\,ds\right)dt+$$
$$+\int\limits_{x-1/160}^{x+1/160}\left\{\frac{\sin(M+\frac{1}{2})(x-s)-\sin(M+\delta)(x-s)}{2\sin\frac{x-s}{2}}\right\}e^{i\lambda ys}\times$$
$$\times\left(\int\limits_{-1/160}^{1/160}\frac{e^{-i\lambda st}\sin(N+\frac{1}{2})t\cdot C(s,y-t)}{2\sin\frac{t}{2}}\,dt\right)ds.$$

Euler's formula, $\sin\theta=(e^{i\theta}-e^{-i\theta})/2i$, shows that the first term on the right side of (8) is equal to the first four terms on the right side of equation (6), so Lemma 6 will be proved if each of the remaining four terms of (8) are bounded. All the inner integrals are bounded by arguments analogous to the one following equation (7). Furthermore, the first two terms in curly brackets are clearly bounded. The remaining two terms in curly brackets are bounded since the mean value theorem implies

$$\left|\frac{\sin(N+\frac{1}{2})\tau-\sin(N+\varepsilon)\tau}{2\sin\frac{\tau}{2}}\right|\leqslant\frac{|\frac{1}{2}-\varepsilon|\tau}{2\sin\frac{\tau}{2}}$$

which is bounded since $|\tau|\leqslant 1/160$ and $|\varepsilon|\leqslant 1$.

Lemma 7. *If* $d\big((x,y),\partial T^2\big)\leqslant 1/80$, $|S_{MN}(g_j)(x,y)|\leqslant A$, $j=0,1,2,3$.

Proof. Each $S_{MN}(g_j)$ may be written as

$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}e^{i\lambda st}\varphi(s)\varphi(t)E_M(x-s)E_N(y-t)\,ds\,dt \tag{9}$$

where $E_L(z)$ represents either the Dirichlet kernel $D_L(=\sin(L+\frac{1}{2})z/2\times\sin(z/2))$ or the conjugate Dirichlet kernel $D_L^{\sim}\left(=\frac{1}{2\tan\frac{z}{2}}-\frac{\cos(L+\frac{1}{2})z}{2\sin\frac{z}{2}}\right)$ depending on the value of j ([5], vol. I, p. 49).

Assume, without loss of generality, that $d(x,\partial T^1)\leqslant 1/80$. Then $E_M(x-s)\varphi(s)$ is bounded. If $d(y,\partial T^1)\leqslant 1/80$ also, the integrand of (9), and hence also (9), is bounded. If $d(y,\partial T^1)>1/80$, then write (9) as

$$\pi^{-2}\int_0^{2\pi}E_M(x-s)\varphi(s)e^{i\lambda sy}\left(\int_{y-2\pi}^{y}e^{-i\lambda st}\varphi(y-t)E_N(t)\,dt\right)ds.$$

Again it suffices to show the inner integral bounded. Since $\varphi(y-t)=\varphi(y)+O(t)$ and $E_N(t)$ is bounded for $d(t,\partial T^1)>1/80$, we need only bound

$$\int_{-1/80}^{1/80}e^{-i\lambda st}E_N(t)\,dt.$$

We fold obtaining

$$-i\int_0^{1/80}\frac{\sin\lambda st}{\tan(t/2)}dt+i\int_0^{1/80}\frac{\sin\lambda st\cos(N+\frac{1}{2})t}{\sin(t/2)}dt,\quad\text{or}\quad\int_0^{1/80}\frac{\cos\lambda st\sin(N+\frac{1}{2})t}{\sin(t/2)}dt$$

depending on whether $E_N=D_N^{\sim}$ or D_N. All three of the integrals are bounded. (See the argument following equation (1) for the first and the argument following equation (7) for the other two.)

Lemma 8. (Cf. [2].) *Set* $F_\lambda(\mu,\nu)=\int_{-1/160}^{1/160}\int_{-1/160}^{1/160}\frac{e^{i(\lambda st-\mu s-\nu t)}}{st}\,ds\,dt$. *Then* F *satisfies*

(10.1) $|F_\lambda(\mu,\nu)|\leqslant A\log\lambda$,

(10.2) $|F_\lambda(\mu,\nu)|\leqslant A$ *if* $\lambda\leqslant 100\max\{|\mu|,|\nu|\}$,

(10.3) $|F_\lambda(0,0)-2\pi i\log\lambda|\leqslant A$.

Proof. Let

$$g(t)=\int_0^{t/160}\frac{\sin u}{u}du$$

so that $|g(t)|\leqslant A$, $|g'(t)|\leqslant\min\{1/160,1/t\}$. We may write

(11)

$$\frac{1}{2i}F_\lambda(\mu,\nu)=\int_{-1/160}^{1/160}\frac{e^{-i\nu t}}{t}\left(\int_0^{1/160}\frac{\sin(\lambda t-\mu)s}{s}ds\right)dt=\int_{-1/160}^{1/160}\frac{e^{-i\nu t}}{t}g(\lambda t-\mu)\,dt.$$

Since the last integrand is bounded by A/t, we may replace the limits of integration by $-1/\lambda$ and $1/\lambda$ while making an error on the order of $\log\lambda$. Then an application of the mean value theorem yields

$$\frac{1}{2i}F_\lambda(\mu,\nu)=O(\log\lambda)+g(-\mu)\int_{-1/\lambda}^{1/\lambda}\frac{e^{-i\nu t}}{t}dt+\int_{-1/\lambda}^{1/\lambda}\frac{e^{-i\nu t}}{t}\lambda tg'(\theta\lambda t-\mu)\,dt$$

$$=O(\log\lambda),$$

since the first integral folds and the second integral is bounded by $\frac{2}{\lambda}\cdot\lambda\cdot\sup|g'|$.

For the proof of (10.2) we may assume (since $F_\lambda(\mu,\nu)=F_\lambda(\nu,\mu)$) that $|\mu|\geqslant\lambda/100$. Expanding g by the mean value theorem in (11) yields

$$\frac{1}{2i}F_\lambda(\mu,\nu)=g(-\mu)\int_{-1/160}^{1/160}\frac{e^{-i\nu t}}{t}dt+\int_{-1/160}^{1/160}\frac{e^{-i\nu t}}{t}\lambda tg'(\theta\lambda t-\mu)\,dt.$$

The first integral is again bounded by folding and the second integral is bounded since the integrand is $\leqslant\lambda\cdot|g'(\theta\lambda t-\mu)|\leqslant\frac{\lambda}{|\theta\lambda t-\mu|}\leqslant\frac{\lambda}{|\mu|-\lambda/160}\leqslant 800/3$.

For the proof of (10.3) set $\mu = \nu = 0$ in (11) and interchange the order of integration, obtaining

$$\frac{1}{2i}F(0,0) = 2\int_0^{1/160}\frac{1}{t}\left(\int_0^{\lambda t/160}\frac{\sin u}{u}\,du\right)dt = 2\int_0^{\lambda/(160)^2}\frac{\sin u}{u}\left(\int_{160u/\lambda}^{1/160}\frac{dt}{t}\right)du$$

$$= 2\log\lambda\int_0^{\lambda/(160)^2}\frac{\sin u}{u}\,du + 2\int_0^{\lambda/(160)^2}\frac{\sin u}{u}\log((160)^2 u)\,du$$

$$= \pi\log\lambda + O(1).$$

LEMMA 9. *Uniformly for* $(x, y)\in Q$, *we have*

$$\lim_{\lambda\to\infty} 1+\frac{i}{\pi}C_1(x, y;\ \lambda)+\frac{i}{\pi}C_2(x, y;\ \lambda)-\frac{1}{\pi^2}C_3(x, y;\ \lambda) = 4.$$

Proof. We have $C_3(x, y) = \int_{y-2\pi}^{y} e^{-i\lambda xt}h(t)t^{-1}dt = \int_{|t|>1/40} + \int_{|t|\leqslant 1/40} = A+B$ where $h(t) = h(t, \lambda, x, y) = t\cdot\frac{1}{2}\cot(t/2)C_1(x, y = t)$. We integrate A by parts:

$$A = \frac{e^{-i\lambda xt}}{-i\lambda x}\left(h(t)t^{-1}\right)\left(\Big|_{y-2\pi}^{-1/40}+\Big|_{1/40}^{y}\right)+\frac{1}{i\lambda x}\int_{|t|>1/40} e^{-i\lambda xt}\left(h(t)t^{-1}\right)'dt$$

$$= O\left(\frac{1}{\lambda x}\right) = O\left(\frac{1}{\lambda}\right) = o(1) \qquad \text{as } \lambda \text{ tends to infinity.}$$

We write

$$B = h(0)\int_{-1/40}^{1/40}\frac{e^{-i\lambda xt}}{t}\,dt+\int_{-1/40}^{1/40} e^{-i\lambda xt}k(t)\,dt = B_1+B_2$$

where $k(t) = [h(t)-h(0)]t^{-1}$. Because h has a continuous second t-derivative which is uniformly bounded (in t, λ, x, and y), $k'(t) = [th'(t)-h(t)+h(0)]t^{-2}$ $(t \neq 0)$ is seen to be bounded by expanding h and h' by Mac Laurin's formula with remainder. (If $t = 0$, first note that $k(0) = h'(0)$. Then an easy calculation shows $k'(0) = \frac{1}{2}h''(0)$.) Thus, an integration by parts yields $B_2 = O(1/x\lambda) = o(1)$ as λ tends to infinity. We note that $h(0) = h(0, \lambda, x, y) = C_1(\lambda, x, y)$ and that

$$\lim_{\lambda\to\infty}\int_{-1/40}^{1/40}\frac{e^{-i\lambda xt}}{t}\,dt = -i\pi \quad \text{uniformly}$$

(since $|x| \geqslant 1/10$) ([5], vol. I, p. 57). An argument exactly analogous to the preceding yields $\lim_{\lambda\to\infty} C_1(\lambda, x, y) = -i\pi$ and $\lim_{\lambda\to\infty} C_2(\lambda, x, y) = -i\pi$ uniformly for $(x, y)\in Q$. Hence,

$$\lim_{\lambda\to\infty} C_3(\lambda, x, y) = \lim_{\lambda\to\infty} B_1(\lambda, x, y) = (-i\pi)(-i\pi) = -\pi^2$$

uniformly for $(x, y)\in Q$. This completes Lemma 9.

The proofs of Lemmas 1 through 4 now follow easily. For Lemma 1: the boundedness of h_λ is immediate from Lemma 5 and the definitions of the C_i; while the continuity follows from $g_j\in C^\infty$, $j = 1, 2, 3$. Lemma 2 is a consequence of Lemma 7 if $d((x, y), \partial T^2)\leqslant 1/80$, and of Lemma 6 and (10.1) of Lemma 8 if $d((x, y), \partial T^2)\geqslant 1/80$.

Proof of Lemma 3. Lemma 3 is immediate from Lemma 7 if $d((x, y), \partial T^2)\leqslant 1/80$. Assume now that $d((x, y), \partial T^2)\geqslant 1/80$. Because of Lemma 6 (with $\delta = \varepsilon = 0$) we need only bound

$$4\pi^2 T_{\pm M, \pm N} = -\int_{x-1/160}^{x+1/160}\int_{y-1/160}^{y+1/160}\frac{e^{i(\lambda\sigma\tau\pm M\sigma\pm N\tau)}C(\sigma, \tau)}{(x-\sigma)(y-\tau)}\,d\sigma\,d\tau$$

where $C = 1+\frac{i}{\pi}C_1+\frac{i}{\pi}C_2-\frac{1}{\pi^2}C_3$. Setting $s = x-\sigma$, $t = y-\tau$ and taking absolute values produces

$$\left|\int_{-1/160}^{1/160}\int_{-1/160}^{1/160}\frac{e^{i(\lambda st-s\mu-t\nu)}}{st}\,C(x-s, y-t)\,ds\,dt\right| = |I|,$$

where $\mu = \lambda y\pm M$, $\nu = \lambda x\pm N$. If we could replace I by

$$J = \int_{-1/160}^{1/160}\int_{-1/160}^{1/160}\frac{e^{i(\lambda st-s\mu-t\nu)}C(x, y)}{st}\,ds\,dt,$$

we would be done by (10.2) since if $\mu = \min\{M, N\}$, $|\mu|\geqslant\frac{\lambda}{80}-\frac{\lambda}{400} = \frac{\lambda}{100}$. To see that we may, write $C(x-s, y-t)-C(x, y) = \{C(x-)s, y-t)-C(x, y-t)-C(x-s, y)+C(x, y)\}+\{C(x, y-t)-C(x, y)\}+\{C(x-s, y)-C(x, y)\} = D_1D_2C(x-\theta s, y-\varphi t)st+D_2C(x, y-\varphi_1 t)t+D_1C(x-\theta_1 s, y)s$ by the mean value theorem [4].

We have a corresponding decomposition of the difference between I and the desired integral into three integrals. The first integral has bounded integrand by Lemma 5 since $d((x-\theta s, y-\varphi t), \partial T^2)\geqslant 1/160$. The second is equal to

$$\int_{-1/160}^{1/160} D_2 C(x, y-\varphi_1 t)\left(\int_{-1/160}^{1/160} \frac{e^{i(\lambda t-\mu)s}}{s}\, ds\right) dt$$

and, hence, is bounded since the inner integral is bounded by folding and D_2C is bounded by Lemma 5. The third integral is bounded in a similar manner.

Proof of Lemma 4. Pick a point $(x, y)\in Q$ and a $\lambda > 10$. Define positive integers M, N and fractions δ, ε by $M+\delta = \lambda y$, $N+\varepsilon = \lambda x$. By Lemma 6, it suffices to study the four integrals

$$T_{\pm\lambda_y, \pm\lambda x}\big(e^{i\lambda xy} C(x, y)\big)$$

where $C = 1+\frac{i}{\pi}C_1+\frac{i}{\pi}C_2-\frac{1}{\pi^2}C_3$ is a function satisfying the conclusions of Lemma 5. We have

$$e^{-i\lambda xy}T_{\pm\lambda y, \pm\lambda x}\big(e^{i\lambda xy} C(x, y)\big) = \frac{-1}{4\pi^2}\iint\limits_{|s|,|t|\leqslant\frac{1}{160}} \frac{e^{i[\lambda st-(y\mp y)\lambda s-(x\mp x)\lambda t]}}{st}\times$$

$$\times C(x-s, y-t)\, ds\, dt.$$

As above, we may replace $C(x-s, y-t)$ by $C(x, y)$ with bounded error, obtaining

$$T_{\pm\lambda y, \pm\lambda x}\big(h_\lambda(x, y)\big) = \frac{-1}{4\pi^2} C(x, y, \lambda)\, e^{i\lambda xy}\iint\limits_{|s|,|t|\leqslant\frac{1}{160}} \frac{e^{i[\lambda st-(y\mp y)\lambda s-(x\mp x)\lambda t]}}{st}\, ds\, dt.$$

Unless the $(+, +)$ sign combination occurs, either $\mu = 2y\lambda \geqslant \frac{2\cdot\lambda}{160} \geqslant \frac{\lambda}{100}$ or $\nu = 2y\lambda \geqslant \frac{\lambda}{100}$ so from (10.2) we see that the corresponding three T's are bounded.

Thus, uniformly for $(x, y)\in Q$

$$S_{[\lambda y],[\lambda x]}\big(h_\lambda(x, y)\big) = T_{\lambda y,\lambda x}\big(h_\lambda(x, y)\big)+O(1)$$

$$= \frac{-1}{4\pi^2} C(x, y, \lambda)\, e^{i\lambda xy}(2\pi i \log\lambda)+O(1) \qquad \text{(by (10.3))}$$

$$= \frac{2}{\pi i}\log\lambda\, e^{i\lambda xy}+o(\log\lambda) \qquad \text{(by Lemma 9)}$$

from which Lemma 4 is immediate.

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DEPAUL UNIVERSITY
CHICAGO, ILL.

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