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Lacunarity for compact groups, III

by

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Dedicated with deep homage to Professor Antoni Zygmund on the fiftieth anniversary of his scientific work

Abstract. Analogues and generalisations of a famous theorem of Fatou and Zygmund are obtained for compact Abelian groups. Given a compact [Hausdorff] Abelian group G with character group X, and an increasing sequence $\mathscr{S} = (X_n)_{n=1}^{\infty}$ of finite symmetric subsets of X, we consider a subset P of $X_\infty = \bigcup_{n=1}^{\infty} X_n$ and write $\mathfrak{F}_h(P)$ for the linear space of all Hermitian complex-valued functions on P. Write $P_n = P \cap X_n$ and for $u \in \mathfrak{F}_h(P)$, write

$$s_n u = \sum_{\chi \in P_n} u(\chi) \cdot \chi.$$

For a measurable set $W \subset G$ such that $W \subset (\operatorname{int}(W))^-$, the following property is investigated:

(*) *if* $u \in \mathfrak{F}_h(P)$ *and* $\sup_{n \geq 1} [\sup_{t \in W} s_n u(t)] < \infty$, *then* $u \in l^1(P)$.

The validity of this implication is shown to be independent of the choice of $\mathscr{S}$. Accordingly, if (*) holds, we say that the $FZ(P, W)$ property holds and we call P an $FZ(W)$-set. A number of properties of P are shown to be equivalent to property $FZ(P, W)$. In particular, certain matching properties of bounded Hermitian functions on P are shown to characterise $FZ(W)$-sets. For example, P is an $FZ(G)$-set if and only if every bounded Hermitian function on P is matched on P by the Fourier–Stieltjes transform of a nonnegative measure in $M(G)$. A large class of $FZ(G)$-sets is identified and the union of two $FZ(G)$-sets is shown to be another $FZ(G)$-set. Every $FZ(W)$-set is a Sidon set; the converse is an open question for $W = G$.

§ 1. Introduction.

1.1. History. This paper is of course related to the first two in the sequence [4], [5], but may be read independently of [4]. We will occasion-

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ally refer to [5]. In the present paper we take up a famous theorem for trigonometric series on the circle group which admit Hadamard gaps. Consider a trigonometric series

$$\sum_{k\in Z} c_k \exp(in_k x),$$

where the c_k are complex numbers and the n_k integers,

$$0 \leqslant n_1 < n_2 < n_3 < \ldots, \quad c_{-k} = \bar{c}_k,\ n_{-k} = -n_k,$$

$$\inf_{k>0} n_{k+1}/n_k = q > 1 \text{ (Hadamard's gap condition).}$$

The symmetric partial sums

(1) $$s_n(x) = \sum_{k=-n}^{n} c_k \exp(in_k x)$$

of this series are plainly real valued. Suppose that these partial sums satisfy the condition

(2) $$\sup_{n\geqslant 1} s_n(x) < \infty$$

for every x in some nonvoid open interval; the conclusion is that

$$\sum_{k\in Z} |c_k| < \infty.$$

We refer to this result as the *Fatou–Zygmund theorem*. Note that the hypothesis (2) is equivalent to the condition

(3) $$\sup_{n\geqslant 1} s_n^+(x) < \infty,$$

where $t^+ = \max(t, 0)$ for every real number t. The version (3) is more convenient than (2), and we will use it henceforth.

The Fatou–Zygmund theorem goes back to Fatou, who in [6], p. 397, announced without proof the result for $q > 2$, $\mathrm{Re}(c_k) = 0$, and the variant hypothesis that $s_n(x)$ converges for all x in some nonvoid open interval. The full theorem is due to Zygmund [16]. A proof appears in Zygmund [17], Vol. I, p. 247, Th. (6.3). The Fatou–Zygmund theorem has been extended to a much wider class of lacunary sets $\{n_k\}_{k\in Z}$ by Gapoškin [7]. An analogous but apparently not identical property has been studied for connected compact Abelian groups by Déchamps-Gondim [1]. We will discuss the contributions of these writers at appropriate places *infra*.

1.2. Mise en scène. Our aim is to extend the Fatou–Zygmund theorem, or more properly, to study the lacunarity property embodied in it, for sets of characters of compact Abelian groups. Let G be a compact infinite Abelian group with character group X. Let P be a symmetric

subset of X, and let $\mathfrak{U}$ be a certain set of complex-valued functions u on P which are Hermitian in the sense that $u(\chi^{-1}) = \overline{u(\chi)}$ for all $\chi \in P$. With every u we may associate the *formal* "trigonometric series":

(1) $$\sum_{\chi\in P} u(\chi)\chi.$$

Suppose that we are given a method of assigning to each $u \in \mathfrak{U}$ a sequence $(s_n u)_{n=1}^{\infty}$ of real-valued finite linear combinations of χ's in P that may serve as partial sums in some reasonable sense for the series (1). (For the classical case, $\mathfrak{U}$ consists of all Hermitian functions on $\{n_k\}_{k\in Z}$ and the functions $s_n u$ are the symmetric partial sums 1.1.(1). As we shall see, many other possibilities present themselves.) We ask the following question. What sort of lacunarity for the set P is expressed by the requirement that

(2) $$\sum_{\chi\in P} |u(\chi)| < \infty$$

for every $u \in \mathfrak{U}$ for which the functions $s_n^+ u$ are bounded in some preassigned sense? That is, we turn the conclusion of the Fatou–Zygmund theorem into a definition of a lacunarity property of P. Plainly the possibilities at this stage are very wide, since we have left open the definitions of $\mathfrak{U}$, of $s_n u$, and of boundedness of $s_n^+ u$. We will call the property of P expressed by this assumption a *generalised Fatou–Zygmund property*. The precise nature of this property obviously depends upon our choices of $\mathfrak{U}$, of the convergence or summability method defining $s_n u$, and of the definition of boundedness of the functions $s_n^+ u$.

The Fatou–Zygmund theorem suggests that at least some variants of the generalised Fatou–Zygmund property of P may be related to Sidonicity of P. Our reasoning here is tenuous at best: all we have to go on is the fact that sets with Hadamard gaps are Sidon sets and also have the Fatou–Zygmund property. We investigate this connection from a functional analytic point of view. We will express some generalised Fatou–Zygmund properties in terms of the possibility of matching more or less arbitrary bounded Hermitian functions on P by Fourier–Stieltjes transforms of nonnegative real-valued measures on G having restricted supports (analogous to the corresponding well-known characterisation of Sidon sets; see, for example ([8] 37.2. ii)).

1.3. Conventions. All notation and terminology not explained here are as in [3] and [8]. We will adhere throughout to the following notation. The symbol G will denote a compact Abelian Hausdorff group and X will denote its character group. Normalised Haar measure on G will be denoted by λ. For $0 < p < \infty$, $\mathfrak{L}^p(G)$ is the usual Lebesgue space of pth power integrable functions on G with respect to λ. The symbol

$\mathfrak{C}(G)$ denotes the space of all complex-valued continuous functions on G. The symbol $\mathfrak{T}(G)$ denotes the linear space of all trigonometric polynomials $\sum_{k=1}^{n} a_k \chi_k$ on G. The symbol $\mathfrak{A}(G)$ denotes the subspace of $\mathfrak{C}(G)$ consisting of all f having the form $\sum_{\chi \in X} a_\chi \chi$ where $\sum_{\chi\in X} |a_\chi| = \|f\|_{\mathfrak{A}}$ is finite.

The symbol $\boldsymbol{M}(G)$ denotes the space of all complex Radon measures on G, defined as in [8], § 14. For a subset S of G, the symbol $\boldsymbol{M}(S)$ denotes the set of all $\mu \in \boldsymbol{M}(G)$ such that $\operatorname{Supp}|\mu| \subset S$. The symbols $\boldsymbol{M}_r(S)$ and $\boldsymbol{M}_+(S)$ denote respectively the sets of real-valued and nonnegative real-valued measures in $\boldsymbol{M}(S)$.

For a complex-valued function f on any group G, $f^\sim$ denotes the function $f^\sim(x) = \overline{f(x^{-1})}$. For a set E of complex-valued functions, the symbols E_r and E_+ denote respectively the sets of real-valued and nonnegative real-valued functions in E. The symbol E_h denotes the set of all $f \in E$ such that $f = f^\sim$.

The mappings $f \to f^\wedge$ and $\mu \to \mu^\wedge$ are the Fourier and Fourier–Stieltjes transforms, defined on $\mathfrak{L}^1(G)$ and $\boldsymbol{M}(G)$, respectively.

If E is a subset of $\mathfrak{L}^1(G)$ or $\boldsymbol{M}(G)$ and if Y is a subset of X, then E_Y will denote the set of $f \in E$ such that $f^\wedge(\chi) = 0$ for $\chi \in X \setminus Y$.

§ 2. Some abstract lemmas.

We set down here some needed lemmas from functional analysis.

2.1. Definition. Let E be a real linear space. Let $\Phi(E)$ denote the set of all functions $\tau\colon E \to [0, \infty]$ such that

$$\tau(x+y) \leqslant \tau(x)+\tau(y) \quad \text{and} \quad \tau(ax) = a\tau(x)$$

for all $x, y \in E$ and $a \in [0, \infty[$. (We adopt the usual conventions concerning ∞; in particular, the product $0\cdot\infty$ is taken to be 0.) If $\tau \in \Phi(E)$ and $\tau(-x) = \tau(x)$ for all $x \in E$, τ is called *symmetric*. If E is a topological real linear space, we define $\Phi_0(E)$ as the set $\{\tau \in \Phi(E)\colon \tau$ is lower semicontinuous on $E\}$.

2.2. Remarks. We list without proof some simple facts.

(a) A function $\tau \in \Phi(E)$ belongs to $\Phi_0(E)$ if and only if the set

$$\{x \in E\colon\ \tau(x) \leqslant 1\}$$

is closed in E.

(b) If $\tau, \tau' \in \Phi_0(E)$ (or $\Phi(E)$), then $\tau+\tau'$ belongs to $\Phi_0(E)$ (or $\Phi(E)$).

(c) If Ψ is a nonvoid subset of $\Phi_0(E)$, then the function $\tau = \sup\{\varphi\colon \varphi \in \Psi\}$ belongs to $\Phi_0(E)$.

(d) If $\tau \in \Phi_0(E)$, then the function $x \to \tau(-x) = \tau^*(x)$ also belongs to $\Phi_0(E)$.

Our first lemma is simple and surprisingly useful.

2.3. Lemma. *Let E be a complete, first countable, locally convex topological linear space (not necessarily satisfying any separation axiom). Let φ and τ be elements of $\Phi_0(E)$. Suppose that*

(i) *$x \in E$ and $\tau(x) \neq \infty$ imply that $\varphi(x) \neq \infty$ and $\varphi(-x) \neq \infty$.*

Then there exist a positive real number $\varkappa$ and a continuous seminorm σ on E such that

(ii) *$\varphi(x) \leqslant \max\{\sigma(x), \varkappa\cdot\tau(x)\}$ for every $x \in E$.*

If E is a Banach space, then the seminorm σ in (ii) *can be taken as a multiple of the norm in E.*

Proof. This is, in essence, a straight category argument. Take continuous seminorms $\sigma_1, \sigma_2, \sigma_3, \ldots$ such that $\sigma_1 \leqslant \sigma_2 \leqslant \ldots$ and such that the sets $\sigma_n^{-1}([0,1])$ form a base at 0 in E. Introduce the associated semimetric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n}\sigma_n(x-y)/\big(1+\sigma_n(x-y)\big).$$

Plainly E is complete in the semimetric d, since it defines the uniformity of E.

Consider the set $B = \{x \in E\colon\ \tau(x) \leqslant 1\}$ as a topological subspace of E. It is closed in E, since $\tau \in \Phi_0(E)$; and nonvoid since $0 \in B$. The set B is therefore a nonvoid complete semimetrisable space. By (i), the restriction $\varphi|B$ assumes only finite values. The function $\varphi|B$ is also lower semicontinuous, and so for every positive integer m, the set

$$B_m = \{x \in B\colon\ \varphi(x) \leqslant m\}$$

is closed in the relative topology of B. Since $B = \bigcup_{m=1}^{\infty} B_m$, Baire's theorem entails that some B_m has nonvoid interior in the relative topology of B That is, there exist $x_0 \in B$ and a positive real number r such that

$$x \in B \quad\text{and}\quad d(x, x_0) \leqslant r \quad \text{imply that } \varphi(x) \leqslant m. \tag{1}$$

Now choose a real number a such that $0 < a < 1$ and so small that $d((1-a)x_0, x_0) \leqslant \frac{1}{2}r$. We write $(1-a)x_0 = x_1$. From (1) we see that

$$x \in B \text{ and } d(x, x_1) \leqslant \tfrac{1}{2}r \text{ imply that } \varphi(x) \leqslant m. \tag{2}$$

Since $1-a > 0$, we also have

$$\tau(x_1) = (1-a)\tau(x_0) \leqslant 1-a. \tag{3}$$

From (3) and (i) we have $\varphi(-x_1) < \infty$. Now write $m_1 = m+\varphi(-x_1)$. Choose b and n so that $0 < b \leqslant a$ and

$$y \in E \quad\text{and}\quad \sigma_n(y) \leqslant b \quad \text{imply that } d(y, 0) \leqslant \tfrac{1}{2}r. \tag{4}$$

If E is a Banach space, each σ_n can trivially be taken as a multiple of the norm in E. By (2), (4), and (3) it may be seen that

$$\varphi(x_1+y) \leqslant m$$

whenever $y \in E$, $\sigma_n(y) \leqslant b$, and $\tau(y) \leqslant a$. Thus we have proved:

(5) $\quad y \in E, \quad \sigma_n(y) \leqslant b, \quad$ and $\quad \tau(y) \leqslant b \quad$ imply that $\varphi(y) \leqslant m_1$.

Now write $\varkappa = m_1 b^{-1}$ and $\sigma = \varkappa\sigma_n$. Then (5) shows that

(6) $\quad x \in E, \quad \sigma(x) \leqslant 1, \quad$ and $\quad \varkappa\tau(x) \leqslant 1 \quad$ imply $\varphi(x) \leqslant 1$.

The relation (6) is obviously equivalent to (ii). ■

We continue with two lemmas somewhat like lemmas given by Kahane and Salem ([11], p. 141) and Kahane ([10], p. 106), but different enough to warrant in our opinion separate treatment.

2.4. Iteration Lemma. *Let E be a topological linear space, and B a bounded, convex, sequentially complete subset of E such that $0 \in B$. Let F be a normed linear space and T a linear map of E into F whose graph is closed in $E \times F$. Let F_1 denote the closed unit ball in F. Suppose that there is a real number a such that $0 < a < 1$ and such that for every $y \in F_1$ there exists an $x \in B$ such that*

(i) $$\|y - Tx\| \leqslant a.$$

Then the inclusion

(ii) $$F_1 \subset (1-a)^{-1} T(B)$$

holds.

Proof. The proof is by "iteration". Given $y \in F_1$, choose $x_0 \in B$ such that

$$\|y - Tx_0\| \leqslant a.$$

We proceed by induction. Suppose that $x_0, x_1, \ldots, x_n$ in B have been chosen so that

$$\Big\| y - T\Big(\sum_{j=0}^{n} a^j x_j\Big)\Big\| \leqslant a^{n+1}.$$

Then the element $y_n = a^{-(n+1)}\big(y - T\big(\sum_{j=0}^{n} a^j x_j\big)\big)$ belongs to F_1, and by hypothesis we can choose $x_{n+1} \in B$ so that $\|y_n - Tx_{n+1}\| \leqslant a$. Write $w_n = \sum_{j=0}^{n} a^j x_j$. Then we have

(1) $$\|y - Tw_n\| \leqslant a^{n+1}$$

for $n \in \{0, 1, 2, 3, \ldots\}$. Since B is convex and contains 0, we have

$$w_n \in (1-a)^{-1} B.$$

For $0 \leqslant m < n$, the same properties of B imply that

(2) $$w_n - w_m \in (1-a)^{-1}(a^{m+1} - a^{n+1}) B \subset (1-a)^{-1} a^{m+1} B.$$

Since B is bounded, (2) implies that the sequence $((1-a)w_n)_{n=1}^{\infty}$ is a Cauchy sequence in B. Let w' be its limit, which belongs to B. Then $w = (1-a)^{-1}w'$ belongs to $(1-a)^{-1}B$. By (1), we see that $\lim_{n\to\infty} Tw_n = y$. Since T is a closed mapping, we infer that $Tw = y$. Thus (ii) holds. ■

2.5. Lemma. *Let E, F, F_1, and T be as in Lemma 2.4, with the added hypothesis that F be a Banach space. Let $(A_n)_{n=1}^{\infty}$ be a sequence of subsets of E satisfying the following conditions.*

(i) *For all m and n, the set $A_m + A_n$ is contained in a bounded, convex, sequentially complete subset $B_{m,n}$ of E that contains 0.*

(ii) *The equality $T\big(\bigcup_{n=1}^{\infty} A_n\big) = F$ obtains.*

Then there exist positive integers q, n_0, and n_1 such that

(iii) $F_1 \subset T(qB_{n_0,n_1})$.

Proof. Since $F = \bigcup_{n=1}^{\infty} T(A_n) = \bigcup_{n=1}^{\infty} (T(A_n))^-$, Baire's theorem implies the existence of a positive integer n_0 for which $(T(A_{n_0}))^-$ has nonvoid interior in F. That is, there exist a positive integer r and an element $y_0 \in F$ such that

(1) $\quad y \in F \quad$ and $\quad \|y - y_0\| \leqslant r^{-1} \quad$ imply that $y \in (T(A_{n_0}))^-$.

There is also a positive integer n_1 such that

(2) $$-y_0 \in T(A_{n_1}).$$

Thus for $y \in F$ such that $\|y\| \leqslant r^{-1}$, we have

(3) $$\begin{aligned} y = (y_0 + y) - y_0 &\in (T(A_{n_0}))^- + T(A_{n_1}) \subset (T(A_{n_0}))^- + (T(A_{n_1}))^- \\ &\subset (T(A_{n_0} + A_{n_1}))^- \subset (T(B_{n_0,n_1}))^-; \end{aligned}$$

in the last line of (3), B_{n_0,n_1} is as in (i). From (3) we see that

(4) $$F_1 \subset (T(rB_{n_0,n_1}))^-.$$

The set rB_{n_0,n_1} is bounded, convex, and sequentially complete, and contains 0. From (4) we see that (i) of Lemma 2.4 holds with $B = rB_{n_0,n_1}$ for all positive real numbers a, and so (ii) of Lemma 2.4 holds for all $a \in]0, 1[$. Therefore (iii) holds for any integer $q > r$. ■

§ 3. Generalised Fatou-Zygmund properties. In this section we establish the notation and terminology for the remainder of the paper.

3.1. Standing conventions. We shall select a sequence $(h_n)_{n=1}^{\infty}$ of elements of $\mathfrak{A}_r(G)$, the rôle of which is to generate convergence or summability methods for formal trigonometric series on G. Further specification is left until 3.4 and 3.7.

In any case we shall write

$$X_\infty = \{\chi \in X\colon \lim_{n\to\infty} \hat{h_n}(\chi) = 1\};$$

since every h_n is real valued, X_∞ is a symmetric subset of X. Each set $\{\chi \in X\colon \hat{h_n}(\chi) \neq 0\}$ is countable and so X_∞ is also a countable subset of X.

Our Fatou–Zygmund properties are studied for certain subsets P of X. Except where the contrary is explicitly indicated, we suppose that

(i) P is a symmetric subset of X_∞.

The symbol $\mathfrak{F}(P)$ denotes the space of all complex-valued functions defined on P, $\mathfrak{B}(P)$ the space of all bounded functions in $\mathfrak{F}(P)$, and $c_0(P)$ the subspace of $\mathfrak{B}(P)$ consisting of all functions that are arbitrarily small in absolute value outside of appropriately chosen finite subsets of P. Let Top denote the topology of pointwise convergence in the real linear space $\mathfrak{F}_h(P)$; see 1.3 for the meaning of the suffix "h". It is obvious that $(\mathfrak{F}_h(P), \mathrm{Top})$ is a Fréchet space.

3.2. The function spaces $\mathfrak{U}$. We will examine Fatou–Zygmund properties for P based on certain subspaces $\mathfrak{U}$ of $\mathfrak{F}_h(P)$. We will suppose that $\mathfrak{U}$ is a (real) linear subspace of $\mathfrak{F}_h(P)$ with the following properties:

(i) $\mathfrak{U} \supset l_h^1(P)$;

(ii) $\mathfrak{U}$ is a Fréchet space under some topology;

(iii) the topology of $\mathfrak{U}$ is equal to or stronger than $\mathrm{Top}|\mathfrak{U}$;

(iv) there is a basis $(\mathfrak{N}_n)_{n=1}^{\infty}$ at 0 in $\mathfrak{U}$ such that $\mathfrak{N}_1 \supset \mathfrak{N}_2 \supset \ldots \supset \mathfrak{N}_n \supset \supset \ldots$, each $\mathfrak{N}_n$ is convex and balanced, and if $u, v \in \mathfrak{U}$, $|u| \leqslant |v|$ and $v \in \mathfrak{N}_n$, then $u \in \mathfrak{N}_n$.

Since $\mathfrak{U}$ is a Fréchet space, its topology can be described by a sequence $(\sigma_n)_{n=1}^{\infty}$ of seminorms where σ_n is the Minkowski gauge of $\mathfrak{N}_n$:

$$\sigma_n(u) = \inf\left\{a\colon\ a > 0,\ \frac{1}{a}u \in \mathfrak{N}_n\right\}.$$

From this definition it is clear that (iv) is equivalent to:

(v) the topology of $\mathfrak{U}$ is defined by an increasing sequence $(\sigma_n)_{n=1}^{\infty}$ of seminorms that are monotone in the sense that $\sigma_n(u) \leqslant \sigma_n(v)$ whenever $u, v \in \mathfrak{U}$ and $|u| \leqslant |v|$.

Finally we suppose that

(vi) for every positive integer n and every function $u \in \mathfrak{U}$, we have

$$\sum_{\chi\in P} |\hat{h_n}(\chi)u(\chi)| < \infty.$$

In view of (vi) we may define

$$s_n u = \sum_{\chi\in P} \hat{h_n}(\chi)u(\chi)\chi \tag{1}$$

for every n and every $u \in \mathfrak{U}$. Each $s_n u$ is trivially an element of $\mathfrak{A}_{P,r}(G)$. In our study of Fatou–Zygmund properties, we examine not the sums $s_n u$ but their nonnegative parts $s_n^+ u = (s_n u)^+ = \max(s_n u, 0)$. These functions are in $\mathfrak{C}_+(G)$ but not in general in $\mathfrak{A}(G)$.

3.3. Lemma. *Let W be any λ-measurable subset of G, let n be any positive integer, and let p be in* $[1, \infty]$. *Then the mapping*

$$u \to \|\xi_W s_n u\|_p \tag{i}$$

is a continuous seminorm on $\mathfrak{U}$ and the mapping

$$u \to \|\xi_W s_n^+ u\|_p \tag{ii}$$

is a continuous gauge on $\mathfrak{U}$. Hence both of these mappings belong to $\Phi_0(\mathfrak{U})$.

Proof. By 3.2.(vi), the mapping

$$u \to \sum_{\chi\in P} |\hat{h_n}(\chi)u(\chi)| = \|s_n u\|_{\mathfrak{A}}$$

is a seminorm on $\mathfrak{U}$. By 3.2. (iii), this seminorm is lower semicontinuous on $\mathfrak{U}$. By 3.2. (ii) and [3] 6.2.3. and 7.2.1, this seminorm is necessarily continuous. Thus $u \to s_n u$ is a continuous mapping of $\mathfrak{U}$ into $\mathfrak{A}(G)$. The lemma now follows from the inequalities

$$\big|\|\xi_W s_n u\|_p - \|\xi_W s_n v\|_p\big| \leqslant \|\xi_W s_n(u-v)\|_p \leqslant \|s_n(u-v)\|_p \leqslant \|s_n(u-v)\|_{\mathfrak{A}}$$

and

$$\big|\|\xi_W s_n^+ u\|_p - \|\xi_W s_n^+ v\|_p\big| \leqslant \max\{\|\xi_W s_n^+(u-v)\|_p,\ \|\xi_W s_n^+(v-u)\|_p\}$$
$$\leqslant \|s_n(u-v)\|_p \leqslant \|s_n(u-v)\|_{\mathfrak{A}}. \quad \blacksquare$$

3.4. Case A: ordinary partial sums. Our major attention will be given to a direct generalisation of the Fatou–Zygmund theorem in the following setting. Let $\mathscr{S} = (X_n)_{n=1}^{\infty}$ be an increasing sequence of finite symmetric subsets of X. Given $\mathscr{S}$, we define convergence factors h_n as in 3.1 by

$$h_n = \sum_{\chi\in X_n} \chi. \tag{i}$$

We obviously have $X_\infty = \bigcup_{n=1}^{\infty} X_n$ in this case. By analogy with the standard Dirichlet kernel for $G = T$, $X = Z$, and $X_n = \{m \in Z: |m| \leqslant n\}$, we will denote these particular functions by D_n.

Now consider our symmetric subset P of X. We write P_n for the set $X_n \cap P$, $n = 1, 2, \ldots$ For $u \in \mathfrak{U}$ and positive integers m and n, we have

(1) $$s_n u = \sum_{\chi \in P_n} u(\chi)\chi$$

and

(2) $$D_n * (s_m u) = s_{\min(m,n)} u.$$

3.5. Remark. We think of P-spectral trigonometric series

(1) $$\sum_{\chi \in P} u(\chi)\chi$$

as special instances of general trigonometric series

(2) $$\sum_{\chi \in X} c(\chi)\chi,$$

namely those for which $c(\chi) = 0$ for $\chi \in X \setminus P$. Likewise, we seek to arrange matters so that the partial sums $s_n u$ arise from applying to (1) a procedure for forming partial sums which is natural for general series (2). Thus, although it would be possible to arrange that

$$s_n u = \sum_{\chi \in P_n} u(\chi)\chi$$

for *any* increasing sequence $(P_n)_{n=1}^{\infty}$ of finite symmetric subsets of P with union equal to P, we have elected in Case A to arrange that P_n is in fact chosen to be $X_n \cap P$, where $\mathscr{S} = (X_n)_{n=1}^{\infty}$ is a sequence independent of P yielding sensible partial sums for any trigonometric series (2) for which c vanishes off X_∞. Normally, one will try to make X_∞ as "fat" as possible. (We can make $X_\infty = X$ if and only if G is first countable, i.e., metrisable. In the contrary case, sequences would have to be replaced by nets, which bring complications of their own.)

3.6. Definition. Let $\mathscr{J}$ denote the family of all nonvoid λ-measurable subsets W of G such that $W \subset (\mathrm{int}(W))^-$.

Note that for $g \in \mathfrak{C}(G)$ and $W \in \mathscr{J}$, we have

$$\|\xi_W g\|_\infty = \|\xi_W g\|_u = \|\xi_{W^-} g\|_u.$$

3.7. Case B: summability factors. In this case the h_n are subjected to different conditions, as follows:

(i) each h_n is in $\mathfrak{A}_+(G)$ and $M = \sup_{n \geqslant 1} \|h_n\|_1 < \infty$;

(ii) there is a sequence $(W_r)_{r=1}^{\infty}$ of (not necessarily open) symmetric neighbourhoods of e in G such that $W_r \in \mathscr{J}$ and $W_{r+1} \subset W_r$ for every positive integer r, and to every such r there corresponds a positive integer $n_0(r)$ such that

$$W_r^- \cdot (\mathrm{Supp}\, h_n) \subset W_{r-1}$$

for every $n \geqslant n_0(r)$ (W_0 is understood to be G).

3.8. Remarks. (a) In case 3.7 we admit the possibility that all W_r are equal; they may for example all be equal to G.

(b) If the torsion subgroup of X is finite, then we cannot satisfy 3.7. (i) with $h_n = D_n$ except in the trivial case that X_∞ is finite. This was proved by Hewitt and Zuckerman [9]. Note also that D_n can never be nonnegative for all n. For if $D_n \geqslant 0$, we have

$$\|D_n\|_1 = \hat{D_n}(1),$$

which is 1 or 0 according as the character 1 is in X_n or is not in X_n, and by Hewitt and Zuckerman, *loc. cit.*, we have $\lim_{n\to\infty} \|D_n\|_1 = \infty$.

(c) The case of connected G deserves special mention. For an arbitrary G, let f be a trigonometric polynomial on G that vanishes on a nonvoid open set U. Then f vanishes on each connected component of G that intersects U. (Let C denote the connected component of G containing e, and suppose that $x \in U$. Let φ be any continuous homomorphism of R into G. Then $f_x \circ \varphi$ is a trigonometric polynomial on R vanishing in an interval about 0. It follows that $f_x \circ \varphi$ vanishes identically on R, and so f_x vanishes identically on $\varphi(R)$. The union of all subgroups $\varphi(R)$ is dense in C (see for example [8], (25.20)). Thus f_x vanishes throughout C and so f vanishes throughout xC.) If G itself is connected, the kernels D_n must therefore have support equal to G, and 3.7. (ii) holds if and only if $W_r = G$ for all r.

(d) Remarks (b) and (c) explain why Cases A and B demand separate treatments.

(e) As was adumbrated in 3.1, the h_n are to play the rôle of summability kernels of the type frequently used in connection with trigonometric series. A given sequence of summability kernels $(h_n)_{n=1}^{\infty}$ may or may not satisfy the conditions laid down in 3.7 for a given choice of the W_r. Although 3.7. (i) and positivity of h_n are natural enough, the inclusions 3.7. (ii) fail for many familiar kernels if $(W_r)_{r=1}^{\infty}$ collapses to e.

3.9. What we shall come to term Fatou–Zygmund properties are special cases of the following type of property. Let P and $(h_n)_{n=1}^{\infty}$ be as in 3.1, let $\mathfrak{U}$ and $s_n u$ be as in 3.2, let W be a λ-measurable subset of G, let p be in $[1, \infty]$, and let $\varphi \in \Phi(\mathfrak{U})$. Consider the statements

(i) $$u \in \mathfrak{U} \quad \text{and} \quad \sup_{n \geqslant 1} \|\xi_W \cdot s_n^+ u\|_p < \infty,$$

and

(ii) $$u \in \mathfrak{U} \quad \text{and} \quad \varphi(u) < \infty.$$

The implication (i) implies (ii) will be taken as expressing a type of generalised Fatou–Zygmund property of the system $\{P, \mathfrak{U}, (h_n)_{n=1}^{\infty}, W, p, \varphi\}$.

The special cases singled out for closer study are detailed in Sections 5 and 6.

§ 4. Necessary and sufficient inequalities. This section contains abstract formulations of Fatou–Zygmund properties. We begin with a necessary condition, which we formulate in more generality than is immediately needed.

4.1. In the following theorem, notation is as in 3.1 and 3.2; W denotes a λ-measurable subset of G and p lies in $[1, \infty]$. The theorem is based on Lemma 2.3.

4.2. THEOREM. *Let φ be a symmetric element of $\Phi(\mathfrak{U})$ that is lower semicontinuous for* Top$|\mathfrak{U}$. *Suppose that the inequality*

(i) $$\varphi(u) < \infty$$

holds for every $u \in \mathfrak{U}$ satisfying the condition

(ii) $$\sup_{n\geqslant 1}\{\|\xi_W s_n^+ u\|_p\} < \infty.$$

Then to every positive integer n_0 there correspond a continuous seminorm σ on $\mathfrak{U}$ and a positive real number $\varkappa$ (both possibly depending upon n_0) such that

(iii) $$\varphi(f^\wedge) \leqslant \max\{\sigma(f^\wedge), \varkappa\cdot\sup_{n\geqslant n_0}\{\|\xi_W(h_n * f)^+\|_p\}\}$$

for every $f \in \mathfrak{T}_{P,r}(G)$.

Proof. Suppose that (i) holds for every $u \in \mathfrak{U}$ satisfying (ii). For a given positive integer n_0, we define τ on the linear space $\mathfrak{U}$ by

$$\tau(u) = \sup_{n\geqslant n_0}\{\|\xi_W s_n^+ u\|_p\}.$$

Plainly τ is in $\Phi(\mathfrak{U})$, and from 3.3 and 2.2. (c) we see that τ is actually in $\Phi_0(\mathfrak{U})$. Since φ is lower semicontinuous for Top$|\mathfrak{U}$, it is lower semicontinuous on $\mathfrak{U}$ (see 3.2. (iii)), i.e., is in $\Phi_0(\mathfrak{U})$. We now apply Lemma 2.3. If $\tau(u) < \infty$ for a given $u \in \mathfrak{U}$, then (ii) holds and so (i) holds. Our present hypotheses thus imply the hypothesis of Lemma 2.3, and the conclusion 2.3. (ii) becomes

(1) $$\varphi(u) \leqslant \max\{\sigma(u), \varkappa\cdot\sup_{n\geqslant n_0}\{\|\xi_W s_n^+ u\|_p\}\}.$$

Now let f be any function in $\mathfrak{T}_{P,r}(G)$. The restriction $f^\wedge|P$ of $f^\wedge$ to P belongs to the function space $\mathfrak{U}$, by 3.2. (i). Using the identity $s_n f^\wedge = h_n * f$, we obtain (iii) at once from (1). ∎

We now establish the converse of Theorem 4.2 for the special functions $h_n = D_n$ of Case A.

4.3. THEOREM. *Notation and hypotheses are as in 4.1 and 4.2 with the restriction that $h_n = D_n$ for all n, as in 3.2. Suppose that there exist a continuous seminorm σ on $\mathfrak{U}$ and a positive real constant $\varkappa$ such that 4.2. (iii) holds with $n_0 = 1$ for all $f \in \mathfrak{T}_{P,r}(G)$. Then if $u \in \mathfrak{U}$ and 4.2. (ii) holds, 4.2. (i) holds as well.*

Proof. Let u be any element of $\mathfrak{U}$ for which the left side of 4.2. (ii) is finite: write this number as L. The function $s_m u$ belongs to $\mathfrak{T}_{P,r}(G)$. By 3.4. (1) and 3.4. (2), we have $(s_m u)^\wedge = \xi_{P_m} u$ and $h_n * (s_m u) = s_{\min(m,n)} u$ for all positive integers n. Applying 4.2. (iii) with $n_0 = 1$, we find that

(1) $$\varphi(\xi_{P_m} u) \leqslant \max\{\sigma(\xi_{P_m} u), \varkappa\cdot\sup_{1\leqslant j\leqslant m}\{\|\xi_W(s_j u)^+\|_p\}\} \leqslant \max\{\sigma(\xi_{P_m} u), \varkappa L\}.$$

We may suppose that σ is one of the defining seminorms σ_n of the topology of $\mathfrak{U}$, and so 3.2. (v) and (1) combine to show that

$$\varphi(\xi_{P_m} u) \leqslant \max\{\sigma(u), \varkappa L\}.$$

Since φ is lower semicontinuous for Top$|\mathfrak{U}$, we have

$$\varphi(u) \leqslant \liminf_{m\to\infty}\varphi(\xi_{P_m} u) \leqslant \max\{\sigma(u), \varkappa L\} < \infty,$$

i.e., 4.2. (ii) implies 4.2. (i) under the hypothesis 4.2. (iii). ∎

4.4. Note. One might hope to prove Theorems 4.2 and 4.3 for asymmetric φ, replacing 4.2. (i) by

(i) $$\varphi(u) < \infty \quad \text{and} \quad \varphi(-u) < \infty.$$

Since there need be no connection between f^+ and $(-f)^+$, we see no way to prove that $\varphi(-u) < \infty$ for asymmetric φ under the hypothesis 4.2. (iii).

4.5. We now take up abstract versions of Fatou–Zygmund properties in Case B (see 3.7). In Theorems 4.6 and 4.7, $(h_n)_{n=1}^{\infty}$ and $(W_r)_{r=1}^{\infty}$ are as in 3.7, p is an element of $[1, \infty]$, $\mathfrak{U}$ is as in 3.2, and φ is a symmetric element of $\Phi(\mathfrak{U})$ that is lower semicontinuous for Top$|\mathfrak{U}$.

4.6. THEOREM. *Suppose that*

(i) $$\varphi(u) < \infty$$

for all $u \in \mathfrak{U}$ for which

(ii) $$\inf_{r\geqslant 1}\{\sup_{n\geqslant 1}\{\|\xi_{W_r} s_n^+ u\|_p\}\} < \infty.$$

Then for every positive integer r, there exist a positive constant $\varkappa_r$ and a continuous seminorm σ_r on $\mathfrak{U}$ such that

(iii) $$\varphi(f^\wedge) \leqslant \max\{\sigma_r(f^\wedge), \varkappa_r\|\xi_{W_r} f^+\|_p\}$$

for all $f \in \mathfrak{T}_{P,r}(G)$.

Proof. For all $r \geqslant 1$, we define $n_0 = n_0(r)$ as in 3.7. (ii). Our hypotheses obviously imply that $\varphi(u) < \infty$ if

$$\sup_{n\geqslant 1}\{\|\xi_{W_r} s_n^+ u\|_p\} < \infty,$$

and so by Theorem 4.2, there exist a positive real constant $\varkappa_r'$ and a continuous seminorm σ_r' on $\mathfrak{U}$ such that

$$\varphi(f^\wedge) \leqslant \max\{\sigma_r'(f^\wedge),\ \varkappa_r'\cdot \sup_{n\geqslant n_0}\{\|\xi_{W_r}(h_n * f)^+\|_p\}\} \tag{1}$$

for every $f \in \mathfrak{T}_{P,r}(G)$. To obtain (iii) from (1), we need to majorise the supremum in (1) by a constant multiple of $\|\xi_{W_r} f^+\|_p$. To accomplish this, we must use the special hypotheses of Case B.

Since h_n is nonnegative, its convolution $h_n * w$ with any function $w \in \mathfrak{L}^1_+(G)$ is nonnegative. From this a simple argument (which we omit) shows that

$$(h_n * w)^+ \leqslant h_n * (w^+)$$

for all $w \in \mathfrak{L}^1_r(G)$. Thus for $f \in \mathfrak{T}_{P,r}(G)$ we obtain

$$\xi_{W_r}(h_n * f)^+ \leqslant \xi_{W_r}\big(h_n * (f^+)\big). \tag{2}$$

Now let g be any function in $\mathfrak{L}^{p'}_+(G)$ (we agree as usual that $1' = \infty$ and that $\infty' = 1$). For typographical convenience, write θ for the function $\xi_{W_r} g$ and θ^* for the function $x \to \theta(x^{-1})$. It is clear that $\mathrm{Supp}\,\theta \subset W_r^-$. Let n be any integer greater than or equal to n_0. Now using (2), elementary properties of convolution, and 3.7. (ii), we write

$$\int_G \xi_{W_r}(h_n * f)^+ g\,d\lambda = \int_G (h_n * f)^+ \theta\,d\lambda \leqslant \int_G \big(h_n * (f^+)\big)\theta\,d\lambda \tag{3}$$

$$= \big(h_n * (f^+)\big) * \theta^*(e) = (f^+) * (h_n * \theta^*)(e) = \int_{W_{r-1}} f^+ (h_n * \theta^*)^*\,d\lambda.$$

Applying Hölder's inequality to the last part of (3), then the norm inequality $\|h_n * \theta^*\|_{p'} \leqslant \|h_n\|_1 \|\theta^*\|_{p'}$ (see for example [8], (20.14)), and finally 3.7. (i), we obtain

$$\int_{W_{r-1}} f^+ (h_n * \theta^*)^*\,d\lambda \leqslant \|\xi_{W_{r-1}} f^+\|_p \cdot \|h_n * \theta^*\|_{p'} \leqslant \|\xi_{W_{r-1}} f^+\|_p M \|\theta\|_{p'} \tag{4}$$

$$\leqslant \|\xi_{W_{r-1}} f^+\|_p M \|g\|_{p'}.$$

Combining (3) and (4), we find

$$\int_G \xi_{W_r}(h_n * f)^+ g\,d\lambda \leqslant M \|\xi_{W_{r-1}} f^+\|_p \|g\|_{p'}. \tag{5}$$

By a well-known property of $\mathfrak{L}^p$ norms (see for example [8], (12.13)), (5) implies that

$$\|\xi_{W_r}(h_n * f)^+\|_p \leqslant M \|\xi_{W_{r-1}} f^+\|_p.$$

This inequality combines with (1) to yield (iii) with $\sigma_r = \sigma_{r+1}'$ and $\varkappa_r = M\varkappa_{r+1}'$. ■

We now state and prove the converse of Theorem 4.6.

4.7. Theorem. *All notation and hypotheses are as in 4.5. Suppose that for each positive integer r, there exist a positive constant $\varkappa_r$ and a continuous seminorm σ_r on $\mathfrak{U}$ such that 4.6. (iii) holds for all $f \in \mathfrak{T}_{P,r}(G)$. Then the condition 4.6. (ii) implies 4.6. (i), for all $u \in \mathfrak{U}$.*

Proof. As noted in 3.2. (v), our continuous seminorm σ_r can be taken to be monotone. Let g be any function in $\mathfrak{A}_{P,r}(G)$. By 3.2. (i), the Fourier transform $g^\wedge|P$ belongs to $\mathfrak{U}$. Let $(P_j)_{j=1}^\infty$ be an increasing sequence of finite symmetric sets with union P. For every positive integer j, let f_j be the polynomial such that

$$f_j^\wedge = \xi_{P_j} g^\wedge.$$

It is obvious that $f_j \in \mathfrak{T}_{P,r}(G)$, $|f_j^\wedge| \leqslant |g^\wedge|$, $\lim_{j\to\infty} f_j^\wedge(\chi) = g^\wedge(\chi)$ for all $\chi \in X$, and $\lim_{j\to\infty}\|f_j - g\|_u = 0$. Since φ is lower semicontinuous for $\mathrm{Top}|\mathfrak{U}$, we infer that

$$\varphi(g^\wedge) \leqslant \liminf_{j\to\infty} \varphi(f_j^\wedge). \tag{1}$$

By 4.6. (iii), we have

$$\varphi(f_j^\wedge) \leqslant \max\{\sigma_r(f_j^\wedge),\ \varkappa_r\|\xi_{W_r} f_j^+\|_p\} \leqslant \max\{\sigma_r(g^\wedge),\ \varkappa_r\|\xi_{W_r} f_j^+\|_p\}. \tag{2}$$

It is plain that

$$\big|\|\xi_{W_r} f_j^+\|_p - \|\xi_{W_r} g^+\|_p\big| \leqslant \|f_j^+ - g^+\|_p. \tag{3}$$

Since $\|f_j - g\|_u \to 0$, we also have $\|f_j^+ - g^+\|_u \to 0$, and this with (3) shows that

$$\lim_{j\to\infty}\|\xi_{W_r} f_j^+\|_p = \|\xi_{W_r} g^+\|_p.$$

Applying this to (2) and using (1), we find

$$\varphi(g^\wedge) \leqslant \max\{\sigma_r(g^\wedge),\ \varkappa_r\|\xi_{W_r} g^+\|_p\}, \tag{4}$$

which is 4.6. (iii) for the function g.

Now let u be any element of $\mathfrak{U}$ for which 4.6. (ii) holds. Fix any positive integer r for which

$$L = \sup_{n\geqslant 1}\|\xi_{W_r} s_n^+ u\|_p < \infty.$$

As noted in 3.2, the functions $s_n u$ are in $\mathfrak{A}_{P,r}(G)$, and so we may apply (4) to write

$$\varphi(h_n^\wedge u) \leqslant \max\{\sigma_r(h_n^\wedge u),\ \varkappa_r L\}. \tag{5}$$

Since

$$|h_n^\wedge| \leqslant \|h_n\|_1 \leqslant M,$$

we have $|h_n^\wedge u| \leqslant M|u|$ and by the monotonicity of σ_r and (5) we find

(6) $$\varphi(h_n^\wedge u) \leqslant \max\{M\cdot\sigma_r(u), \varkappa_r L\} < \infty.$$

By 3.1. (i), we have

$$\lim_{n\to\infty} h_n^\wedge(\chi)u(\chi) = u(\chi) \quad \text{for all} \quad \chi \in P.$$

Again the lower semicontinuity of φ for Top$|\mathfrak{U}$ implies that

$$\varphi(u) \leqslant \liminf_{n\to\infty} \varphi(h_n^\wedge u),$$

and so (6) guarantees that $\varphi(u) < \infty$. ■

§ 5. Examples of case A. The theorems of § 4 are an abstract formulation of a large variety of theorems of Fatou–Zygmund type, which we obtain by special choices of $(h_n)_{n=1}^\infty$, $\mathfrak{U}$, φ, p, W, and $(W_r)_{r=1}^\infty$. We consider here and in Section 6 some particular cases.

5.1. Let $\mathscr{S} = (X_n)_{n=1}^\infty$ and $(D_n)_{n=1}^\infty$ be as in 3.4, and let P be a symmetric subset of $X_\infty = \bigcup_{n=1}^\infty X_n$. We take $\mathfrak{U}$ to be the entire space $\mathfrak{F}_h(P)$, with the topology Top of pointwise convergence, φ to be the function

$$\varphi(u) = \sum_{\chi\in P} |u(\chi)| = \|u\|_1,$$

and $\mathscr{J}$ to be as in 3.6; we also set $p = \infty$.

We note first that the continuous seminorms σ on $\mathfrak{U}$ are exactly those majorised by some seminorm

$$u \to \text{Const}\cdot\max_{\chi\in\Sigma} |u(\chi)|,$$

where Σ is a finite subset of P. The proof is simple and is omitted.

5.2. Definition. Given a set W in $\mathscr{J}$, we denote by $FZ(P, \mathscr{S}, W)$ the following statement. If $u \in \mathfrak{F}_h(P)$ and

(i) $$\sup_{n\geqslant 1} \|\xi_W s_n^+ u\|_u < \infty,$$

then

(ii) $$u \in l^1(P).$$

If $FZ(P, \mathscr{S}, W)$ holds, we say that P *has the* $FZ(\mathscr{S}, W)$ *property* and that P is an $FZ(\mathscr{S}, W)$*-set*. Plainly $FZ(P, \mathscr{S}, W)$ expresses a special Fatou–Zygmund property of the type described in 3.9.

5.3. Definition. If P has the $FZ(\mathscr{S}, W)$ property for every W in $\mathscr{J}$, then we say that P *has the full* $FZ(\mathscr{S})$ *property* and that P is a *full* $FZ(\mathscr{S})$*-set*. In other words, P is a full $FZ(\mathscr{S})$-set if every u that satisfies 5.2. (i) for *some* W in $\mathscr{J}$ belongs to $l^1(P)$.

We will show in 8.8 that the $FZ(\mathscr{S}, W)$ properties and the full $FZ(\mathscr{S})$ property do not depend on the choice of $\mathscr{S}$, and so after 8.8 we will refer simply to the $FZ(W)$ property, to $FZ(W)$-sets, etc. See 8.9.

5.4. The original Fatou–Zygmund theorem deals with the circle group T and its character group Z with $X_n = \{m \in Z\colon |m| \leqslant n\}$ and $\mathscr{S} = (X_n)_{n=1}^\infty$. The Fatou–Zygmund theorem asserts that any symmetric Hadamard set in Z possesses the full $FZ(\mathscr{S})$ property.

Gapoškin [7] has extended the Fatou–Zygmund theorem to a large class of sets, again in the group Z. These are the sets discovered by Stečkin [15]; see 10.2 and 10.4.

For the special situation described in 5.1, we can state Theorems 4.2 and 4.3 as follows.

5.5. Condition. Notation is as in 5.1; n_0 is a positive integer and there exist a finite subset Σ of P and a positive real number $\varkappa$ (both perhaps depending upon n_0) such that the inequality

(i) $$\|f^\wedge\|_1 \leqslant \varkappa\max\{\max_{\chi\in\Sigma} |f^\wedge(\chi)|, \sup_{n\geqslant n_0} \|\xi_W(D_n * f)^+\|_u\}$$

obtains for all $f \in \mathfrak{T}_{P,r}(G)$.

5.6. Theorem. *The set P has the $FZ(\mathscr{S}, W)$ property if and only if Condition 5.5 holds for all positive integers n_0. If 5.5 holds for some n_0, then P has the $FZ(\mathscr{S}, W)$ property.*

5.7. Corollary. *If P has the $FZ(\mathscr{S}, W)$ property, then P is a Sidon set.*

Proof. From 5.6 it follows that 5.5. (i) holds for every $f \in \mathfrak{T}_{P,r}(G)$. For $f \in \mathfrak{T}_{P,r}(G)$, we have $\|f^\wedge\|_\infty \leqslant \|f\|_u$ and $D_n * f = f$ for sufficiently large n, and so

(1) $$\|f^\wedge\|_1 \leqslant \varkappa\sup_n \|D_n * f\|_u.$$

Let us write $P_n = P \cap X_n$ and $s_{P_n} f = \sum_{\chi\in P_n} f^\wedge(\chi)\chi$. Thus (1) signifies that

(2) $$\|f^\wedge\|_1 \leqslant \varkappa\sup_n \|s_{P_n} f\|_u$$

for every $f \in \mathfrak{T}_{P,r}(G)$. If $f \in \mathfrak{T}_P(G)$ we may, since P is symmetric, apply (2) to each of $\mathrm{Re} f$ and $\mathrm{Im} f$ and so conclude that

$$\|f^\wedge\|_1 \leqslant 2\varkappa\sup_n \|s_{P_n} f\|_u$$

for every $f \in \mathfrak{T}_P(G)$. Applying 3.9 of [5] (with $w = 0$, $p = 2$ and every σ_j equal to the unit mass at e), we see that P is a Sidon set. ■

§ 6. Examples of case B.

6.1. We make specialisations a little different from those in § 5. As in § 5, φ is still defined by $\varphi(u) = \sum_{\chi \in P} |u(\chi)|$ and again we take $p = \infty$. The set P is again an arbitrary symmetric subset of X_∞, and the sequences $(W_r)_{r=1}^\infty$ and $(h_n)_{n=1}^\infty$ are as in 3.7. The functions s_n are again as in 3.2. The space $\mathfrak{U}$ this time is taken to be the space $\mathfrak{B}_h(P)$, topologised with the usual uniform or l^∞ norm, denoted by $\|u\|_\infty$.

The continuous seminorms on $\mathfrak{U}$ this time are exactly those seminorms on $\mathfrak{U}$ that are majorised by a constant times $\|\cdot\|_\infty$.

We think of the implication "4.6. (ii) implies 4.6. (i)" in this case as a *generalised* Fatou–Zygmund property. If W is a set in $\mathscr{J}$, $GFZ\big(P, (h_n), W\big)$ denotes the following statement: 5.2. (i) implies 5.2. (ii) for all functions u in $\mathfrak{B}_h(P)$. This is another specialisation of the ideas in 3.9.

The symbol $GFZ\big(P, (h_n)\big)$ denotes the statement: if 5.2. (ii) holds for W equal to some W_r, then 5.2. (i) holds. We say that $\{P, (h_n)\}$ *has the GFZ-property* if $GFZ\big(P, (h_n)\big)$ holds.

The *GFZ*-property has no analogue, so far as we know, in classical Fourier analysis.

6.2. CONDITION. Notation is as in 6.1. For every positive integer r, there exists a positive real number $\varkappa_r$ such that for all $f \in \mathfrak{T}_{P,r}(G)$, we have

(i) $$\|f^\wedge\|_1 \leqslant \varkappa_r \max\{\|f\|_\infty, \|\xi_{W_r} f^+\|_u\}.$$

Theorems 4.6 and 4.7 can be stated as follows.

6.3. THEOREM. *The assertion $GFZ\big(P, (h_n)\big)$ holds if and only if Condition* 6.2 *holds.*

We note also the following version of Theorem 4.2.

6.4. THEOREM. *Suppose that $GFZ\big(P, (h_n), W\big)$ holds for some W in $\mathscr{J}$. Then for every positive integer n_0, there is a positive real number $\varkappa$ such that the inequality*

(i) $$\|f^\wedge\|_1 \leqslant \varkappa \max\{\|f^\wedge\|_\infty, \sup_{n \geqslant n_0} \|\xi_W (h_n * f)^+\|_u\}$$

holds for all $f \in \mathfrak{T}_{P,r}(G)$.

6.5. COROLLARY. *If $GFZ\big(P, (h_n), W\big)$ holds, then P is a Sidon set.*

Proof. The hypothesis implies that 6.4 (i) holds for every $f \in \mathfrak{T}_{P,r}(G)$. In view of 3.7. (i), it follows at once that

$$\|f^\wedge\|_1 \leqslant \varkappa \sup_n \|h_n * f\|_u \leqslant M\varkappa \|f\|_u$$

for every $f \in \mathfrak{T}_{P,r}(G)$. So, as in the proof of 5.7, we have

$$\|f^\wedge\|_1 \leqslant 2M\varkappa \|f\|_u$$

for every $f \in \mathfrak{T}_P(G)$; this implies ([8], 37.2. vii) that P is a Sidon set. ∎

§ 7. Matching properties and the *FZ*-property. Sidonicity of a subset P of X can be expressed as a matching property: P is a Sidon set if and only if every bounded complex-valued function on P is matched on P by the Fourier–Stieltjes transform $\mu^\wedge$ of some (complex) measure μ in $\boldsymbol{M}(G)$. A similar characterisation exists for *FZ*-sets, with the refinement that the measures μ are in $\boldsymbol{M}_+(G)$.

For the reader's convenience, we repeat here a definition from [5], § 3.

7.1. DEFINITION. Let S be a complex normed linear space with norm $s \to \|s\|_S$ and I any infinite index set. Let $f\colon \iota \to f(\iota)$ be an element of S^I. Suppose that there is an element $s_\infty \in S$ such that for every $\varepsilon > 0$ in S, there is a finite subset J of I with the property that

$$\|f(\iota) - s_\infty\|_S < \varepsilon$$

for all $\iota \in I \setminus J$. (Plainly s_∞ is unique if it exists at all.) We then write

$$\lim_I f = s_\infty,$$

s_∞ denoting the constant function $\iota \to s_\infty$ in S^I. Let $c(I, S)$ denote the set of all $f \in S^I$ for which $\lim_I f$ exists, and $c_0(I, S)$ the set of all $f \in c(I, S)$ for which $\lim_I f = 0$. Let $l^1(I, S)$ denote the set of all functions f in S^I for which

$$|||f|||_1 = \sum_{\iota \in I} \|f(\iota)\|_S < \infty.$$

7.2. Remarks. Plainly $f(I)$ is a bounded set in S for all $f \in c(I, S)$, and so we may define

$$|||f|||_\infty = \sup_{\iota \in I} \|f(\iota)\|_S,$$

as in [5], § 3. It is trivial to check that $c(I, S)$ is a normed linear space under coordinatewise linear operations and the norm $f \to |||f|||_\infty$. It is easy to check that $c(I, S)$ is a Banach space if S is a Banach space. Analogous remarks apply to $l^1(I, S)$.

We now describe the conjugate space of $c(I, S)$.

7.3. LEMMA. *Let S be as in 7.1 and S' the space of all bounded linear functionals on S. Let L be a bounded linear functional on $c(I, S)$. Then there exist an element λ of $l^1(I, S')$ and an element λ_∞ of S' such that for all $f \in c(I, S)$, we have*

(i) $$L(f) = \lambda_\infty(\lim_I f) + \sum_{\iota \in I} \lambda(\iota)\big(f(\iota)\big).$$

Conversely, every function on $c(I, S)$ of the form (i) *is a bounded linear functional. The norm of the functional L is*

(ii) $$\|L\| = \|\lambda_\infty\| + |||\lambda|||_1.$$

Proof. The proof is an easy extension of the proof of Lemma 3.3 of [5], to which the reader is referred. ■

We can now state and prove our main matching theorem.

7.4. THEOREM. *Notation and hypotheses are as in* 3.1 *and* 3.4. *Let* W *be a set in* $\mathscr{J}$. *The following are equivalent.*

(a) *The set* P *has the* $FZ(\mathscr{S}, W)$ *property.*

(b) *Every function* β *in* $\mathfrak{B}_h(P)$ *admits an expression*

$$\beta(\chi) = a(\chi) + \nu_\infty^\frown(\chi) + \sum_{n=1}^{\infty} D_n^\frown(\chi)\nu_n^\frown(\chi) \tag{i}$$

for all $\chi \in P$, *where* a *is in* $\mathfrak{B}_h(P)$ *and has finite support, the measures* ν_n *and* ν_∞ *are in* $\boldsymbol{M}_+\big((W^-)^{-1}\big)$, *and*

$$\sum_{n=1}^{\infty} \|\nu_n\| < \infty. \tag{ii}$$

Furthermore, if (b) *holds, there exist a finite subset* Σ *of* P *and a positive real number* $\varkappa$ (*both depending on* P *and* W) *such that a representation* (i) *can be found for which*

$$\operatorname{Supp} a \subset \Sigma \tag{iii}$$

and

$$\|a\|_\infty + \|\nu_\infty\| + \sum_{n=1}^{\infty} \|\nu_n\| \leqslant \varkappa \|\beta\|_\infty. \tag{iv}$$

Proof. The proof, while not intrinsically difficult, is rather long. Suppose first that $FZ(P, \mathscr{S}, W)$ holds. We cite Theorem 5.6 and so may apply the inequality 5.5. (i) with $n_0 = 1$, and with the Σ of the present theorem being any finite symmetric subset of P containing the set Σ defined in 5.5. Thus there is a positive constant $\varkappa''$ such that

$$\|f^\frown\|_1 \leqslant \varkappa'' \max\{\max_{\chi\in\Sigma} |f^\frown(\chi)|, \sup_{n\geqslant 1} \|\xi_W(D_n * f)^+\|_u\} \tag{1}$$

for all $f \in \mathfrak{T}_{P,r}(G)$. To obtain a representation (i) for our function $\beta \in \mathfrak{B}_h(P)$, we first define a linear functional l on $\mathfrak{T}_{P,r}(G)$ by the rule

$$l(f) = \sum_{\chi\in P} \beta(\chi) f^\frown(\chi). \tag{2}$$

It is easy to see that l is real valued and real linear on the real linear space $\mathfrak{T}_{P,r}(G)$. Applying (1), we find

$$|l(f)| \leqslant \|\beta\|_\infty \|f^\frown\|_1 \leqslant \varkappa' \max\{\max_{\chi\in\Sigma} |f^\frown(\chi)|, \sup_{n\geqslant 1}\{\|\xi_W(D_n * f)^+\|_u\}\} \tag{3}$$

for all $f \in \mathfrak{T}_{P,r}(G)$, where $\varkappa' = \varkappa'' \|\beta\|_\infty$.

Next we introduce the real linear space

$$Y = \mathfrak{F}_h(P) \times c\big(I, \mathfrak{C}_r(W^-)\big),$$

where the index set I is $\{1, 2, 3, \ldots\}$. For $\big(u, (g_n)_{n=1}^\infty\big)$ in Y, define the gauge p by

$$p\big(u, (g_n)_{n=1}^\infty\big) = \max\{\max_{\chi\in\Sigma} |u(\chi)|, \sup_{n\geqslant 1}\{\|g_n^+\|_u\}\}. \tag{4}$$

Let us map $\mathfrak{T}_{P,r}(G)$ into Y by the mapping γ, which we define as

$$\gamma(f) = \big(f^\frown|P, ((D_n * f)|W^-)_{n=1}^\infty\big) \in Y.$$

The inequality (3) in our new notation asserts that

$$|l(f)| \leqslant \varkappa' p\big(\gamma(f)\big) \tag{5}$$

for all $f \in \mathfrak{T}_{P,r}(G)$.

We now claim that there is a real-valued real-linear functional l_0 on $\gamma\big(\mathfrak{T}_{P,r}(G)\big)$ such that

$$l(f) = l_0\big(\gamma(f)\big) \tag{6}$$

for all $f \in \mathfrak{T}_{P,r}(G)$. In fact, if $\gamma(f_1) = \gamma(f_2)$ for $f_1, f_2 \in \mathfrak{T}_{P,r}(G)$, (5) shows that

$$|l(f_1) - l(f_2)| = |l(f_1 - f_2)| \leqslant \varkappa' p\big(\gamma(f_1 - f_2)\big)$$
$$= \varkappa' p\big(\gamma(f_1) - \gamma(f_2)\big) = \varkappa' p(0) = 0.$$

Accordingly l_0 is well defined on $\gamma\big(\mathfrak{T}_{P,r}(G)\big)$. Plainly l_0 is real valued and linear. The inequality (5) and the definition (6) show that

$$l_0(y) \leqslant \varkappa' p(y) \tag{7}$$

for all $y \in \gamma\big(\mathfrak{T}_{P,r}(G)\big) \subset Y$. By the Hahn–Banach theorem, we may extend l_0 to a linear functional on Y (which we will still write as l_0) for which (7) holds for all $y \in Y$.

Restricting l_0 to $\mathfrak{F}_h(P) \times \{0\} \subsetneqq Y$, we see from (7) and (4) that l_0 on this subspace is a real linear functional satisfying the inequality

$$\big|l_0\big(u, (0_n)_{n=1}^\infty\big)\big| \leqslant \varkappa' \max_{\chi\in\Sigma} |u(\chi)|$$

for all $u \in \mathfrak{F}_h(P)$. Since Σ is finite and symmetric, there accordingly exists a function a in $\mathfrak{F}_h(P)$ such that (iii) holds,

$$\|a\|_\infty \leqslant \|a\|_1 \leqslant \varkappa', \tag{8}$$

and

$$l_0\big(u, (0_n)_{n=1}^\infty\big) = \sum_{\chi\in\Sigma} a(\chi) u(\chi) \tag{9}$$

for all $u \in \mathfrak{F}_h(P)$.

Similarly, observe that the mapping

$$(g_n)_{n=1}^{\infty} \to l_0\big(0, (g_n)_{n=1}^{\infty}\big)$$

is a bounded linear functional on $c\big(I, \mathfrak{C}_r(W^-)\big)$. Since the conjugate space of $\mathfrak{C}_r(W^-)$ is $\boldsymbol{M}_r(W^-)$, we infer from Lemma 7.3 that there are measures μ_∞ and $\mu_1, \mu_2, \mu_3, \ldots, \mu_n, \ldots$ in $\boldsymbol{M}_r(W^-)$ such that

$$(10) \qquad l_0\big(0, (g_n)_{n=1}^{\infty}\big) = \int_G g_\infty d\mu_\infty + \sum_{n=1}^{\infty} \int_G g_n d\mu_n$$

for all $(g_n)_{n=1}^{\infty} \epsilon\, c\big(I, \mathfrak{C}_r(W^-)\big)$. (Here g_∞ is the uniform limit on W^- of $(g_n)_{n=1}^{\infty}$.) Lemma 7.3 also combines with (7) to guarantee that

$$(11) \qquad \|\mu_\infty\| + \sum_{n=1}^{\infty} \|\mu_n\| = \text{norm of } l_0 \text{ on } c\big(I, \mathfrak{C}_r(W^-)\big) \leqslant \varkappa'.$$

Combining (10), (7), and (4), we obtain

$$(12) \qquad \int_G g_\infty d\mu_\infty + \sum_{n=1}^{\infty} \int_G g_n d\mu_n \leqslant \varkappa' \sup_{n \geqslant 1} \{\|g_n^+\|_u\}.$$

Given $g \epsilon \mathfrak{C}_r(W^-)$ and a positive integer m, first let $g_n = \delta_{mn} g$ ($n \epsilon \{1, 2, 3, \ldots\}$). Putting this $(g_n)_{n=1}^{\infty}$ into (12), we find

$$\int_G g d\mu_m \leqslant \varkappa' \|g^+\|_u,$$

which implies that $\mu_m \epsilon \boldsymbol{M}_+(W^-)$. Next let $g_n = 0$ if $n \leqslant m$ and $g_n = g$ if $m > n$. Putting this $(g_n)_{n=1}^{\infty}$ into (12), we find

$$(13) \qquad \int_G g d\mu_\infty \leqslant \varkappa' \|g^+\|_u - \sum_{n=m+1}^{\infty} \int_G g d\mu_n \leqslant \varkappa' \|g^+\|_u + \|g\|_u \sum_{n=m+1}^{\infty} \|\mu_n\|.$$

In view of (11), the last sum in (13) is arbitrarily small for m sufficiently large. This implies that $\mu_\infty \epsilon \boldsymbol{M}_+(W^-)$.

Combining (9) and (10), we obtain

$$l_0\big(u, (g_n)_{n=1}^{\infty}\big) = \sum_{\chi \epsilon \Sigma} a(\chi) u(\chi) + \int_G g_\infty d\mu_\infty + \sum_{n=1}^{\infty} \int_G g_n d\mu_n,$$

for all $\big(u, (g_n)_{n=1}^{\infty}\big) \epsilon Y$. This equality, (2), (6), and the definition of γ show that for all $f \epsilon \mathfrak{T}_{P,r}(G)$ we have the identities

(14)

$$\sum_{\chi \epsilon P} \beta(\chi) f^\wedge(\chi) = l(f) = l_0\big(\gamma(f)\big) = \sum_{\chi \epsilon P} a(\chi) f^\wedge(\chi) + \int_G f d\mu_\infty + \sum_{n=1}^{\infty} \int_G (D_n * f) d\mu_n.$$

Define the measures ν_∞ and ν_n by $\nu_\infty = \tilde{\mu}_\infty$ and $\nu_n = \tilde{\mu}_n$. Then (14) shows that

(15)

$$\sum_{\chi \epsilon P} \beta(\chi) f^\wedge(\chi) = \sum_{\chi \epsilon P} a(\chi) f^\wedge(\chi) + \sum_{\chi \epsilon P} \nu_\infty^\wedge(\chi) f^\wedge(\chi) + \sum_{\chi \epsilon P} \Big[\sum_{n=1}^{\infty} D_n^\wedge(\chi) \nu_n^\wedge(\chi)\Big] f^\wedge(\chi)$$

for $f \epsilon \mathfrak{T}_{P,r}(G)$. For any f in $\mathfrak{T}_P(G)$, (15) applies to both $\mathrm{Re} f$ and $\mathrm{Im} f$ and so (15) also holds for f. Putting $f = \psi$ in (15) for each ψ in P, we see that (i) holds. The inclusion (iii) has been established above. Finally, writing $\varkappa = 2\varkappa''$, we at once infer (iv) from (8) and (11). This completes the proof that (a) implies (i)–(iv).

We turn now to a proof of the converse. Supposing that the matching property (i) and (ii) of (b) obtains, we wish to show that $FZ(P, \mathscr{S}, W)$ holds. As above, let $I = \{1, 2, 3, \ldots\}$, and form the real Banach spaces $l^1\big(I, \boldsymbol{M}_r\big((W^-)^{-1}\big)\big)$ and $l_h^1(P)$ and the product space

$$E = l_h^1(P) \times \boldsymbol{M}_r\big((W^-)^{-1}\big) \times l^1\big(I, \boldsymbol{M}_r\big((W^-)^{-1}\big)\big).$$

Clearly E is a Banach space under the norm

$$(16) \qquad \big\|\big(a, \nu_\infty, (\nu_n)_{n=1}^{\infty}\big)\big\| = \|a\|_1 + \|\nu_\infty\| + \sum_{n=1}^{\infty} \|\nu_n\|.$$

Define a mapping T of E into $\mathfrak{B}_h(P)$ by

$$(17) \qquad T\big(a, \nu_\infty, (\nu_n)_{n=1}^{\infty}\big) = a + \Big(\nu_\infty^\wedge + \sum_{n=1}^{\infty} D_n^\wedge \nu_n^\wedge\Big)\Big|P.$$

Plainly T is a bounded linear transformation whose norm does not exceed 1.

For every positive integer t, let A_t be the subset of E defined by

$$(18) \qquad \mathrm{Supp}\, a \subset P_t, \quad \nu_\infty, \quad \nu_n \epsilon \boldsymbol{M}_+\big((W^-)^{-1}\big), \qquad \|a\|_1 + \|\nu_\infty\| + \sum_{n=1}^{\infty} \|\nu_n\| \leqslant t.$$

Plainly each A_t is closed in E (and so is sequentially complete), and is bounded and convex and contains 0. Furthermore, the inclusions $A_t + A_u \subset A_{t+u}$, $qA_t \subset A_{qt}$ are easy to check, for all positive integers t, u, and q. The hypothesis (b) simply asserts that

$$T\Big(\bigcup_{t=1}^{\infty} A_t\Big) = \mathfrak{B}_h(P).$$

Hence we may apply Lemma 2.5, with $B_{n_0, n_1} = A_{n_0 + n_1}$, and infer that there is a positive integer k such that

$$(19) \qquad \text{the unit ball in } \mathfrak{B}_h(P) \text{ is contained in } T(A_k).$$

Now we choose any nonzero $f\in\mathfrak{T}_{P,r}(G)$ and define $\beta\in\mathfrak{B}_h(P)$ as $\overline{(\operatorname{sgn} f^{\wedge})}\,|P$. Since $\|\beta\|_\infty = 1$, we apply (19), (17), and (18) to write β as

(20) $$\beta = \alpha+\Big(\nu_\infty^{\wedge}+\sum_{n=1}^{\infty} D_n^{\wedge}\nu_n^{\wedge}\Big)\Big|P,$$

where $\operatorname{Supp}\alpha\subset P_k$ and

$$\|\alpha\|_1+\|\nu_\infty\|+\sum_{n=1}^{\infty}\|\nu_n\|\leqslant k.$$

Recall too that ν_∞ and all ν_n are in $\boldsymbol{M}_+\big((W^-)^{-1}\big)$. Now multiply (20) through by $f^{\wedge}$ and sum over P (since f is a trigonometric polynomial, the sum is actually finite). We obtain

(21) $$\|f^{\wedge}\|_1 = \sum_{\chi\in P} f^{\wedge}(\chi)\beta(\chi)$$
$$= \sum_{\chi\in P_k} f^{\wedge}(\chi)\alpha(\chi)+\sum_{\chi\in P} f^{\wedge}(\chi)\nu_\infty^{\wedge}(\chi)+\sum_{n=1}^{\infty}\Big(\sum_{\chi\in P} f^{\wedge}(\chi)D_n^{\wedge}(\chi)\nu_n^{\wedge}(\chi)\Big).$$

For the first sum on the right side of (21), we have

(22) $$\Big|\sum_{\chi\in P_k} f^{\wedge}(\chi)\alpha(\chi)\Big|\leqslant \max_{\chi\in P_k}\{|f^{\wedge}(\chi)|\}\,\|\alpha\|_1.$$

To estimate the second and third sums, we note that

(23) $$\sum_{\chi\in P} f^{\wedge}(\chi)\nu_\infty^{\wedge}(\chi) = \sum_{\chi\in P}(f*\nu_\infty)^{\wedge}(\chi) = f*\nu_\infty(e)$$
$$= \int_{(W^-)^{-1}} f(y^{-1})\,d\nu_\infty(y) = \int_{W^-} f\,d\tilde{\nu}_\infty$$
$$= \int_{W^-}\max\{f,0\}\,d\tilde{\nu}_\infty+\int_{W^-}\min\{f,0\}\,d\tilde{\nu}_\infty$$
$$\leqslant \int_{W^-} f^+\,d\tilde{\nu}_\infty\leqslant\|\nu_\infty\|\cdot\|\xi_W f^+\|_u.$$

Similar estimates apply to each summand in the third sum, and so we combine (21), (22), and (23) to write

(24) $$\|f^{\wedge}\|_1\leqslant\|\alpha\|_1\max_{\chi\in P_k}\{|f^{\wedge}(\chi)|\}+\|\nu_\infty\|\cdot\|\xi_W f^+\|_u+\sum_{n=1}^{\infty}\|\nu_n\|\cdot\|\xi_W(D_n*f)^+\|_u.$$

For all sufficiently large n, we have $D_n*f = f$, and so (24) and (18) yield

(25) $$\|f^{\wedge}\|_1\leqslant k\max\Big\{\max_{\chi\in P_k}\{|f^{\wedge}(\chi)|\},\ \sup_{n\geqslant 1}\|\xi_W(D_n*f)^+\|_u\Big\}.$$

The inequality (25) is exactly 5.5. (i) with $\varkappa = k$, $\Sigma = P_k$, and $n_0 = 1$, and so Theorem 5.6 implies that (a) of the present theorem obtains. ■

We end this section with the analogue of Theorem 7.4 applying to the *GFZ*-property.

7.5. THEOREM. *Notation and hypotheses are as in* 6.1. *The following statements are equivalent.*

(a) *The set P has the GFZ-property.*

(b) *For every $r\in\{0,1,2,\ldots\}$, every $\beta\in\mathfrak{B}_h(P)$ admits an expression*

(i) $$\beta(\chi) = \alpha_r(\chi)+\nu_r^{\wedge}(\chi)$$

for all $\chi\in P$, where $\alpha_r\in l_h^1(P)$ and $\nu_r\in\boldsymbol{M}_+\big((W_r^-)^{-1}\big)$.

Furthermore, if (b) *holds, there exists for every $r\in\{0,1,2,\ldots\}$ a positive real number $\varkappa_r$ such that every $\beta\in\mathfrak{B}_h(P)$ admits an expression* (i) *in which $\alpha_r\in l_h^1(P)$, $\nu_r\in\boldsymbol{M}_+\big((W_r^-)^{-1}\big)$, and*

(ii) $$\|\alpha_r\|_1+\|\nu_r\|\leqslant\varkappa_r\|\beta\|_\infty.$$

Proof. This is based upon Theorem 6.3 in exactly the same way as the proof of Theorem 7.4 is based upon Theorem 5.6. In the present case the details are a good deal simpler and are omitted. ■

Throughout the rest of the paper we will concentrate on *FZ*-properties [i.e. Case A] and the exploitation of Theorem 7.4.

§ 8. More matching properties. Throughout this section, V is an arbitrary but fixed neighbourhood of e in G, W is a set in $\mathscr{I}$, and $\mathscr{S} = (X_n)_{n=1}^{\infty}$, P, P_n, and $(D_n)_{n=1}^{\infty}$ are as in 3.1 and 3.4. We begin with a technical fact.

8.1. THEOREM. *Suppose that $FZ(P,\mathscr{S},W)$ holds. Then there exist a positive real number $\varkappa$ and a finite subset Σ of P (both depending upon P, $\mathscr{S}$, and W) with the following property. To every $f\in\mathfrak{L}_{P,r}^1(G)$ there corresponds a sequence $(\eta_m)_{m=1}^{\infty}$ of nonnegative real numbers (depending upon f and also upon $\mathscr{S}$ and W) such that*

(i) $$\lim_{m\to\infty}\eta_m = 0;$$

(ii) $$(1-\eta_m)\|f^{\wedge}\|_1\leqslant\varkappa\Big\{\operatorname{card}(\Sigma\cup P_m)\cdot\max_{\chi\in\Sigma\cup P_m}\{|f^{\wedge}(\chi)|\}+\|\xi_{W^-\cdot V}f^+\|_\infty\Big\}.$$

Proof. Let $\beta = \overline{\operatorname{sgn} f^{\wedge}}\,|P$. Applying Theorem 7.4, we write β in the form 7.4. (i), with $\operatorname{Supp}\alpha\subset\Sigma$, where Σ is as in Theorem 7.4. For each positive integer m, let

(1) $$\alpha_m = \alpha+\sum_{n=1}^{m} D_n^{\wedge}\nu_n^{\wedge}.$$

It is clear that $\operatorname{Supp}\alpha_m\subset\Sigma\cup P_m$ and that $\alpha_m\in\mathfrak{B}_h(P)$. We define η_m by

(2) $$\eta_m = \sum_{n=m+1}^{\infty}\|\nu_n\|.$$

By 7.4. (ii) we see that $\lim_{m\to\infty}\eta_m = 0$, i.e., (i) holds. It is also clear from 7.4. (i) that for all $\chi \in P$, the inequality

(3) $$|\beta(\chi) - a_m(\chi) - \hat{\nu}_\infty(\chi)| \leqslant \eta_m$$

holds.

In proving (ii), we may restrict ourselves to m such that $\eta_m < 1$. Let us write a_m for the trigonometric polynomial such that $\hat{a}_m = a_m$: plainly a_m is in $\mathfrak{T}_{P,r}(G)$.

Now consider a sequence $(K_j)_{j=1}^{\infty}$ of continuous, nonnegative, positive-definite functions on G such that:

(4) $$\operatorname{Supp} K_j \subset V^{-1}; \quad \int_G K_j\,d\lambda = 1; \quad \lim_{j\to\infty}\|\hat{K}_j\hat{f}\|_1 = \|\hat{f}\|_1.$$

(For the existence of such a sequence, see [8], Vol. II, Theorem 33.11.) For each positive integer j, form the function

$$g_j = K_j * a_m * f + K_j * \nu_\infty * f.$$

Plainly g_j is in $\mathfrak{C}_r(G)$, and on the set P, we have

(5) $$\hat{g}_j = \hat{K}_j\hat{f}(a_m + \hat{\nu}_\infty) = \hat{K}_j|\hat{f}|\Big(1 - \Big(\sum_{n=m+1}^{\infty}\hat{D}_n\hat{\nu}_n\Big)\operatorname{sgn}\hat{f}\Big).$$

It is clear that

(6) $$\Big|\sum_{n=m+1}^{\infty}\hat{D}_n(\chi)\hat{\nu}_n(\chi)\operatorname{sgn}\hat{f}(\chi)\Big| \leqslant \sum_{n=m+1}^{\infty}\|\nu_n\| = \eta_m$$

for all $\chi \in P$ and so $\operatorname{Re}\hat{g}_j$ is nonnegative on P. On $X \setminus P$, $\hat{g}_j$ vanishes. Now define the function h_j by $h_j = \frac{1}{2}(g_j + g_j^*)$. It is obvious that $\hat{h}_j = \operatorname{Re}\hat{g}_j$, i.e., h_j is a continuous real-valued function on G with nonnegative Fourier transform. By [8], Vol. II, Theorem 31.42, $\hat{h}_j$ is in $l^1(X)$ and by ibid., 31.44. c, $(\hat{h}_j)^{\vee}$ is equal to h_j everywhere on G. Combining (5) and (6), we find that

$$(1-\eta_m)\hat{K}_j|\hat{f}| \leqslant \hat{h}_j,$$

and so

(7) $$(1-\eta_m)\|\hat{K}_j\hat{f}\|_1 \leqslant \|\hat{h}_j\|_1 = \sum_{\chi\in X}\hat{h}_j(\chi) = (\hat{h}_j)^{\vee}(e)$$
$$= h_j(e) = g_j(e) = K_j * a_m * f(e) + K_j * \nu_\infty * f(e).$$

Since $K_j * a_m * f$ is a trigonometric polynomial, we have

(8) $$K_j * a_m * f(e) = \sum_{\chi\in X}\hat{K}_j(\chi)a_m(\chi)\hat{f}(\chi) = \sum_{\chi\in\Sigma\cup P_m}\hat{K}_j(\chi)a_m(\chi)\hat{f}(\chi)$$
$$\leqslant \sum_{\chi\in\Sigma\cup P_m}|a_m(\chi)|\cdot|\hat{f}(\chi)| \leqslant \|a_m\|_1\max_{\chi\in\Sigma\cup P_m}\{|\hat{f}(\chi)|\}.$$

From (1) we see that

$$\|a_m\|_1 \leqslant \Big(\|a\|_\infty + \Big\|\sum_{n=1}^{m}\hat{D}_n\hat{\nu}_n\Big\|_\infty\Big)\cdot\operatorname{card}(\Sigma\cup P_m),$$

and from 7.4. (iv) that

$$\|a\|_\infty + \Big\|\sum_{n=1}^{m}\hat{D}_n\hat{\nu}_n\Big\|_\infty \leqslant \|a\|_\infty + \sum_{n=1}^{\infty}\|\nu_n\| \leqslant \varkappa.$$

Combining these estimates with (8), we obtain

(9) $$K_j * a_m * f(e) \leqslant \varkappa\cdot\operatorname{card}(\Sigma\cup P_m)\cdot\max_{\chi\in\Sigma\cup P_m}\{|\hat{f}(\chi)|\}.$$

In estimating $K_j * \nu_\infty * f(e)$, we may and will suppose that f has been redefined on a set of λ-measure 0 so as to satisfy the inequality

$$\sup_{x\in W^-\cdot V}\{f^+(x)\} = \|\xi_{W^-\cdot V}f^+\|_\infty.$$

Since $K_j \geqslant 0$ and $\nu_\infty \geqslant 0$, we may cite (4) to write

$$K_j * \nu_\infty * f(e) \leqslant K_j * \nu_\infty * f^+(e) = \int_G K_j(x^{-1})(\nu_\infty * f^+)(x)\,d\lambda(x) \leqslant \|\xi_V\nu_\infty * f^+\|_\infty.$$

Also, for λ-almost all $x \in V$, we have

$$\nu_\infty * f^+(x) = \int_{(W^-)^{-1}} f^+(y^{-1}x)\,d\nu_\infty(y) \leqslant \|\nu_\infty\|\sup_{x\in W^-\cdot V}\{f^+(x)\} = \|\nu_\infty\|\,\|\xi_{W^-\cdot V}f^+\|_\infty.$$

Since $\|\nu_\infty\| \leqslant \varkappa$ by 7.4. (iv), we see that

(10) $$K_j * \nu_\infty * f(e) \leqslant \varkappa\|\xi_{W^-\cdot V}f^+\|_\infty.$$

Combining (7), (9), and (10), and using the fact that $\lim_{j\to\infty}\|\hat{K}_j\hat{f}\|_1 = \|\hat{f}\|_1$, we obtain (ii). ∎

We now draw some easy inferences from Theorem 8.1.

8.2. Corollary. *Suppose that $FZ(P, \mathscr{S}, W)$ holds. Let f be any function in $\mathfrak{L}^1_{P,r}(G)$ such that $\|\xi_{W^-\cdot V}f^+\|_\infty < \infty$. Then $\hat{f}$ is in $l^1(X)$.*

Proof. This follows at once from (ii) of 8.1. ∎

8.3. Corollary. *Suppose that $FZ(P, \mathscr{S}, W)$ holds. Then there is a positive real number $\varkappa$ (which depends upon P, $\mathscr{S}$, W, and V) such that for all $f \in \mathfrak{L}^1_{P,r}(G)$, the inequality*

(i) $$\|\hat{f}\|_1 \leqslant \varkappa\cdot\max\{\|f\|_1, \|\xi_{W^-\cdot V}f^+\|_\infty\}$$

holds.

Proof. We wish to apply Lemma 2.3. For $f \in \mathfrak{L}^1_{P,r}(G)$, let $\varphi(f) = \|\hat{f}\|_1$ and $\tau(f) = \|\xi_{W^-\cdot V}f^+\|_\infty$. Routine arguments, which we omit, show that both φ and τ belong to $\Phi_0(\mathfrak{L}^1_{P,r}(G))$. By Corollary 8.2, we see that if $\tau(f) < \infty$, then $\varphi(f)$ and $\varphi(-f)$ are also finite. Thus Lemma 2.3 is applicable with $\mathfrak{L}^1_{P,r}(G) = E$, and we need only note that for the seminorm σ of 2.3 we may take a multiple of the $\mathfrak{L}^1$-norm in $\mathfrak{L}^1_{P,r}(G)$. ∎

We now obtain a new matching property.

8.4. THEOREM. *Suppose that $FZ(P, \mathscr{S}, W)$ holds and that V is a compact neighbourhood of e in G. Then there is a positive real number $\varkappa$ (which depends upon P, $\mathscr{S}$, W, and V) with the following property. For every function β in $\mathfrak{B}_h(P)$, there exist a function $g \in \mathfrak{L}^\infty_r(G)$ and a measure $\nu \in \boldsymbol{M}_+((W^-\cdot V)^{-1})$ such that:*

(i) $$\beta = g\hat{\ } + \nu\hat{\ } \quad on\ P;$$

(ii) $$\|g\|_\infty + \|\nu\| \leqslant \varkappa\|\beta\|_\infty.$$

Proof. This theorem follows from Corollary 8.3 much as the first implication in Theorem 7.4 follows from Theorem 5.6. We outline the proof. For $f \in \mathfrak{T}_{P,r}(G)$, define $l(f)$ by

(1) $$l(f) = \sum_{\chi\in P} f\hat{\ }(\chi)\beta(\chi).$$

From 8.3. (i), we infer that

(2) $$|l(f)| \leqslant \varkappa\|\beta\|_\infty\cdot\max\{\|f\|_1, \|\xi_{W^-\cdot V}f^+\|_u\}.$$

Now consider the linear space $E = \mathfrak{L}^1_{P,r}(G)\times\mathfrak{C}_r(W^-\cdot V)$ with the norm

$$(\varphi, \psi) \to \max\{\|\varphi\|_1, \|\psi\|_u\} = \tau(\varphi, \psi)$$

and the gauge

$$(\varphi, \psi) \to \max\{\|\varphi\|_1, \|\psi^+\|_u\} = \sigma(\varphi, \psi).$$

Imbed $\mathfrak{T}_{P,r}(G)$ into E by the injective map $f \to (f, f|W^-\cdot V)$. From (2) and the Hahn–Banach theorem we see that there is a linear functional on E, which we continue to call l, such that

$$l(f, f|W^-\cdot V) = l(f) \quad \text{for all} \quad f \in \mathfrak{T}_{P,r}(G)$$

and

(3) $$l(\varphi, \psi) \leqslant \varkappa\cdot\|\beta\|_\infty\sigma(\varphi, \psi) \leqslant \varkappa\cdot\|\beta\|_\infty\tau(\varphi, \psi).$$

The space E is a Banach space under the norm τ, and (3) informs us that l is a bounded linear functional on E. It follows that

(4) $$l(\varphi, \psi) = \int_G \varphi(x)h(x)\,dx + \int_{W^-\cdot V}\psi(x)\,d\mu(x)$$

for some $h \in \mathfrak{L}^\infty_r(G)$ and $\mu \in \boldsymbol{M}_r(W^-\cdot V)$ satisfying

(5) $$\|h\|_\infty + \|\mu\| \leqslant \varkappa\cdot\|\beta\|_\infty.$$

The first of the inequalities in (3) shows that $\mu \in \boldsymbol{M}_+(W^-\cdot V)$. For $f \in \mathfrak{T}_{P,r}(G)$, (4) becomes

(6) $$l(f) = \int_G f(x)h(x)\,dx + \int_{W^-\cdot V} f(x)\,d\mu(x).$$

For $\chi \in P$ and a complex number c, (6) yields

(7) $$l(c\chi + \bar{c}\chi^{-1}) = ch\hat{\ }(\chi^{-1}) + \bar{c}h\hat{\ }(\chi) + c\mu\hat{\ }(\chi^{-1}) + \bar{c}\mu\hat{\ }(\chi).$$

Combine (1) and (7) and set $c = \frac{1}{2}$ and $c = -\frac{1}{2}i$ in turn. This yields

(8) $$\beta(\chi) = h\hat{\ }(\chi^{-1}) + \mu\hat{\ }(\chi^{-1}).$$

Defining $g = h\tilde{\ }$ and $\nu = \mu\tilde{\ }$, we obtain (i) from (8) and (ii) from (5). ∎

8.5. Note. If P has the full $FZ(\mathscr{S})$ property 5.3, then the set $W^-\cdot V$ in 8.4 may be replaced by an arbitrary compact neighbourhood of e.

We continue with analogues for FZ-properties of certain approximation properties known to be equivalent to Sidonicity.

8.6. DEFINITION. Let $\mathfrak{D}(P)$ be the set of all complex-valued functions β on P such that $|\beta(\chi)| = 1$ for all $\chi \in P$.

8.7. THEOREM. *The following statements are equivalent.*

(a) *The set P possesses the $FZ(\mathscr{S}, W)$ property.*

(b) *There exists a positive real number $\varkappa$ such that to every β in $\mathfrak{B}_h(P)$ there corresponds a $\mu \in \boldsymbol{M}_+((W^-)^{-1})$ satisfying*

(i) $$\|\mu\| \leqslant \varkappa\|\beta\|_\infty$$

and

(ii) $$\lim_{m\to\infty}[\sup_{\chi\in P\setminus P_m}\{|\mu\hat{\ }(\chi) - \beta(\chi)|\}] = 0.$$

(c) *For every β in $\mathfrak{D}_h(P)$, there exists a $\mu \in \boldsymbol{M}_+((W^-)^{-1})$ satisfying*

(iii) $$\limsup_{m\to\infty}[\sup_{\chi\in P\setminus P_m}\{|\mu\hat{\ }(\chi) - \beta(\chi)|\}] < 1.$$

Proof. To prove that (a) implies (b), we apply Theorem 7.4. Write β as in 7.4. (i) and take $\mu = \nu_\infty$. Then we have

$$\beta(\chi) - \mu\hat{\ }(\chi) = \alpha(\chi) + \sum_{n=1}^{\infty} D_n\hat{\ }(\chi)\nu_n\hat{\ }(\chi).$$

For m so large that $\operatorname{Supp}\alpha \subset P_m$, we find for all $\chi \in P\setminus P_m$ that

(1) $$|\beta(\chi) - \mu\hat{\ }(\chi)| \leqslant \sum_{n=m+1}^{\infty}|\nu_n\hat{\ }(\chi)| \leqslant \sum_{n=m+1}^{\infty}\|\nu_n\|.$$

Since the last term in (1) has limit 0 as $m \to \infty$, (ii) holds. The existence of $\varkappa$ and the inequality (i) follow from 7.4. (iv).

It is trivial that (b) implies (c). Suppose now that (c) holds. Consider any function u in $\mathfrak{F}_h(P)$ and let $\beta = \operatorname{sgn}\bar{u}$. Given a μ satisfying (iii) for this β, we can find a finite symmetric subset Σ of P and a real number d such that $0 < d < 1$ for which the inequality

$$|\beta(\chi) - \mu^{\wedge}(\chi)| < 1 - d$$

holds for all $\chi \in P\setminus\Sigma$. For a given positive integer n, write $\sum\limits_n{}'$ for a sum over $P_n\setminus\Sigma$, $\sum\limits_n{}''$ for a sum over $P_n \cap \Sigma$, and $\sum\limits_n$ for a sum over P_n. Then we have

$$\sum_n{}' |u(\chi)| = \sum_n{}' \beta(\chi)u(\chi) \leqslant \sum_n{}' \mu^{\wedge}(\chi)u(\chi) + (1-d)\sum_n{}' |u(\chi)|,$$

which implies that

(2) $$d\sum_n{}' |u(\chi)| \leqslant \sum_n{}' \mu^{\wedge}(\chi)u(\chi).$$

We also have

(3) $$\sum_n{}' \mu^{\wedge}(\chi)u(\chi) = \int\limits_{(W^-)^{-1}} \Big[\sum_n u(\chi)\overline{\chi(t)}\Big]d\mu(t) - \sum_n{}'' \mu^{\wedge}(\chi)u(\chi).$$

We can estimate the integral in (3) as follows:

(4) $$\int\limits_{(W^-)^{-1}} \Big[\sum_n u(\chi)\overline{\chi(t)}\Big]d\mu(t) = \int\limits_{W^-} \Big[\sum_n u(\chi)\chi(t)\Big]d\mu^{\sim}(t) = \int\limits_{W^-} s_n u(t)\,d\mu^{\sim}(t) \leqslant \|\mu\|\cdot\|\xi_{W^-} s_n^+ u\|_u.$$

Combining (2), (3) and (4), we find

(5) $$d\sum_n{}' |u(\chi)| \leqslant \|\mu\|\cdot\|\xi_{W^-} s_n^+ u\|_u + \sum_n{}'' |\mu^{\wedge}(\chi)u(\chi)|.$$

Now suppose that $\sup\limits_{n\geqslant 1}\{\|\xi_{W^-} s_n^+ u\|_u\} < \infty$. Letting $n \to \infty$ in (5), we see that $u \in l^1(P)$; that is, P possesses the $FZ(\mathscr{S}, W)$ property. This proves that (c) implies (a). ∎

8.8. Corollary. *If $\mathscr{S} = (X_n)_{n=1}^{\infty}$ and $\mathscr{S}^+ = (X_n^+)_{n=1}^{\infty}$ are sequences as in 3.4, and if P is a symmetric subset of X satisfying $P \subset \bigcup\limits_{n=1}^{\infty} X_n$ and $P \subset \bigcup\limits_{n=1}^{\infty} X_n^+$, then P possesses property $FZ(\mathscr{S}, W)$ if and only if it possesses property $FZ(\mathscr{S}^+, W)$. Likewise P possesses the full $FZ(\mathscr{S})$ property if and only if it possesses the full $FZ(\mathscr{S}^+)$ property.*

Proof. The left side of 8.7. (ii) is actually equal to

$$\inf_{\Sigma}\sup_{\chi\in P\setminus\Sigma}\{|\mu^{\wedge}(\chi) - \beta(\chi)|\},$$

where Σ runs over all finite subsets of P. Hence the validity of statement (b) in 8.7 depends only on P and not on any particular sequence $\mathscr{S}$. ∎

8.9. Definition. Let P be a countable symmetric subset of X. We write $FZ(P, W)$ provided $FZ(P, \mathscr{S}, W)$ holds for some (and hence for every) $\mathscr{S}$ the union of whose elements contains P; we say that *P has the $FZ(W)$ property* or that *P is an $FZ(W)$-set*. Similarly, *P has the full FZ property* and P is called a *full FZ-set* if it has the full $FZ(\mathscr{S})$ property for some (and hence every) $\mathscr{S}$ whose union contains P.

We now obtain a matching property strongly reminiscent of F. Riesz's matching property for Sidonicity.

8.10. Theorem. *Let P be a countable symmetric subset of X. The property $FZ(P, G)$ obtains if and only if every function in $\mathfrak{B}_h(P)$ can be represented in the form*

(i) $$\beta(\chi) = \mu^{\wedge}(\chi) \quad \textit{for all} \quad \chi \in P\setminus\{1\},$$

μ being a measure in $\boldsymbol{M}_+(G)$.

Proof. Suppose that (i) holds for all β as described. Then we can write β in the form 7.4. (i) with $\mu = \nu_\infty$, all $\nu_n = 0$, and $\alpha = \big(\beta(1) - \mu^{\wedge}(1)\big)\xi_{\{1\}}$ if $1 \in P$ and $\alpha = 0$ if $1 \notin P$. By Theorem 7.4, $FZ(P, G)$ holds.

To prove the necessity of our condition, we cite Theorem 8.4 with $V = G$. Given $\beta \in \mathfrak{B}_h(P)$, we find $g \in \mathfrak{L}_r^{\infty}(G)$ and $\mu_0 \in \boldsymbol{M}_+(G)$ such that

(1) $$\beta = (g^{\wedge} + \mu_0^{\wedge})|P.$$

Now write $\mu = (\|g\|_\infty + g)\lambda + \mu_0$. Plainly μ belongs to $\boldsymbol{M}_+(G)$, and from (1), $\mu^{\wedge}$ matches β on $P\setminus\{1\}$, inasmuch as $\lambda^{\wedge} = \xi_{\{1\}}$. ∎

8.11. Remarks. (a) The foregoing theorem clarifies the relation of the Fatou–Zygmund property to Sidonicity. A subset P of X is of course a Sidon set if and only if every bounded complex-valued function on P is matchable [on P] by a Fourier–Stieltjes transform $\mu^{\wedge}$, where μ is some *complex* measure in $\boldsymbol{M}(G)$ (see for example [8], Vol. II, Theorem 37.2). A countable symmetric subset P of X is an $FZ(G)$-set if and only if every bounded Hermitian function on P is matched except perhaps at 1 by the Fourier–Stieltjes transform of a *nonnegative* measure.

(b) Consider a countably infinite symmetric subset P of X that is an $FZ(G)$-set. A bounded function β on P such that $\beta(\chi^{-1}) = \overline{\beta(\chi)}$ can be redefined at 1 (if $1 \in P$) so that β admits a positive-definite extension over all of X. This follows from 8.10 since $\mu^{\wedge}$ is positive-definite for $\mu \in \boldsymbol{M}_+(G)$. Since $t\xi_{\{1\}}$ is positive-definite for $t \geqslant 0$ and sums of positive-definite functions are positive-definite, we see that $\beta + t\xi_{\{1\}}$ admits a positive-definite extension over X for all $t \geqslant \mu^{\wedge}(1) - \beta(1)$.

(c) Now let Φ be a finite subset of X, and consider the finite symmetric set $\Phi\Phi^{-1} = \Psi$. Plainly Ψ is an $FZ(G)$-set and so every symmetric function on Ψ can be redefined at 1 so as to admit a positive-definite extension over X. For previous results in this direction, see [14].

We next establish an analogue of 8.10 for functions in $c_0(P)$, suggested by a well known fact for Sidon sets. We precede this by a technical lemma.

8.12. LEMMA. *Let Φ be a finite subset of $X \setminus \{1\}$ and let ε be a positive real number. There exists a function $f \in \mathfrak{T}_+(G)$ such that $f^\wedge(\chi) = 1$ for all $\chi \in \Phi$ and $\|f\|_1 \leqslant 1+\varepsilon$.*

Proof. With no loss of generality, we may suppose that Φ is symmetric. By [8], Vol. II, Theorem 28.57, there is a sequence $(K_n)_{n=1}^{\infty}$ of functions in $\mathfrak{T}_+(G)$ such that $\|K_n\|_1 = 1$ and

$$\lim_{n\to\infty} K_n^\wedge(\chi) = 1 \quad \text{for all} \quad \chi \in \Phi.$$

Define

$$f_n = K_n + \sum_{\chi\in\Phi} (1-K_n^\wedge(\chi))\chi + \sum_{\chi\in\Phi} |1-K_n^\wedge(\chi)|.$$

Plainly f_n is in $\mathfrak{T}_+(G)$, and since $1 \notin \Phi$, we have $f_n^\wedge(\chi) = 1$ for all $\chi \in \Phi$. Also we have

$$\|f_n\|_1 = f_n^\wedge(1) = K_n^\wedge(1) + \sum_{\chi\in\Phi} |1-K_n^\wedge(\chi)| = 1 + \sum_{\chi\in\Phi} |1-K_n^\wedge(\chi)|,$$

and this is less than $1+\varepsilon$ for n sufficiently large. ∎

8.13. THEOREM. *The property $FZ(P, G)$ holds if and only if every function β in $c_{0h}(P)$ can be represented in the form*

(i) $$\beta(\chi) = f^\wedge(\chi) \quad \textit{for all } \chi \in P \setminus \{1\},$$

where f is a function in $\mathfrak{L}^1_+(G)$.

Proof. A glance at 8.10. (i) and (i) shows that we lose no generality in supposing that $1 \notin P$.

Suppose that (i) holds. We will apply Lemma 2.5 with $E = \mathfrak{L}^1_r(G)$, $F = c_{0h}(P)$ (both with the usual norms) and T the mapping $f \to f^\wedge|P$. For A_n $(n \in \{1, 2, 3, \ldots\})$ we take $\{f \in \mathfrak{L}^1_+(G)\colon \|f\|_1 \leqslant n\}$. The mapping T is linear and continuous, hence closed. Plainly $A_n + A_m$ is contained in A_{n+m}, and A_{n+m} is closed, convex, contains 0, and is sequentially complete. Also the equality $qA_n = A_{nq}$ holds for all positive integers q and n. The matching property (i) is just the assertion $T\big(\bigcup_{n=1}^{\infty} A_n\big) = c_{0h}(P)$. Hence the conclusion of Lemma 2.5 holds. Reworded slightly, this asserts that there is a positive real number $\varkappa$ such that for all $\beta \in c_{0h}(P)$, the function f in (i) may be chosen so that

(1) $$\|f\|_1 \leqslant \varkappa \|\beta\|_\infty.$$

Now given a function γ in $\mathfrak{B}_h(P)$, we can trivially find a sequence $(\beta_j)_{j=1}^{\infty}$ such that

(2) $$\begin{cases} \lim\limits_{j\to\infty} \beta_j(\chi) = \gamma(\chi) & \text{for all } \chi \in P, \\ \beta_j \in c_{0h}(P), & \|\beta_j\|_\infty \leqslant \|\gamma\|_\infty. \end{cases}$$

For each j, choose $f_j \in \mathfrak{L}^1_+(G)$ for which

(3) $$f_j^\wedge|P = \beta_j$$

and $\|f_j\|_1 \leqslant \varkappa\|\beta_j\|_\infty \leqslant \varkappa\|\gamma\|_\infty$.

The sequence $(f_j)_{j=1}^{\infty}$ admits a convergent subnet in the weak-* topology of $\{\nu \in \boldsymbol{M}(G)\colon \|\nu\| \leqslant \varkappa\|\gamma\|_\infty\}$. Let μ be the limit of this subnet. Plainly μ is in $\boldsymbol{M}_+(G)$, and from (2) and (3) it is clear that $\hat{\mu}|P = \gamma$.

Let us prove the converse. Suppose that $FZ(P, G)$ holds, and cite the matching theorem 8.10. Another application of Lemma 2.5 shows the existence of a positive real number $\varkappa$ such that the measure μ in 8.10. (i) can be chosen so that

(4) $$\|\mu\| \leqslant \varkappa\|\beta\|_\infty.$$

Consider any $\beta \in c_{0h}(P)$. For every positive integer j, define

$$P_j = \{\chi \in P\colon 2^{-j}\|\beta\|_\infty < |\beta(\chi)| \leqslant 2^{-j+1}\|\beta\|_\infty\}.$$

Plainly P_j is a finite symmetric subset of P. Define β_j as $\beta\xi_{P_j}$. It is clear that

(5) $$\|\beta_j\|_\infty \leqslant 2^{-j+1}\|\beta\|_\infty, \qquad \sum_{j=1}^{\infty} \beta_j = \beta.$$

Using (4) and (5), we find measures $\mu_j \in \boldsymbol{M}_+(G)$ such that

(6) $$\mu_j^\wedge|P = \beta_j, \qquad \|\mu_j\| \leqslant 2^{-j+1}\varkappa\|\beta\|_\infty.$$

Now use Lemma 8.12 to choose polynomials p_j in $\mathfrak{T}_+(G)$ such that

(7) $$\|p_j\|_1 < \tfrac{3}{2} \quad \text{and} \quad p_j^\wedge = 1 \text{ on } P_j.$$

Define the function f as $\sum_{j=1}^{\infty} p_j * \mu_j$. It is easy to check from (6) and (7) that $f \in \mathfrak{L}^1_+(G)$ (and incidentally that $\|f\|_1 \leqslant 3\varkappa\|\beta\|_\infty$). From (6) and (7) it is also clear that

$$f^\wedge|P = \sum_{j=1}^{\infty} p_j^\wedge \mu_j^\wedge|P = \sum_{j=1}^{\infty} \beta_j = \beta. \quad ∎$$

8.14. Remarks. We do not know whether or not every symmetric countably infinite Sidon set P is an $FZ(G)$-set. Nevertheless, we can set down some conditions equivalent to $FZ(P, G)$ that *look* considerably stronger than Sidonicity. For a real-valued function f on G, we write $\max(f)$ and $\min(f)$ for $\max\{f(x): x \in G\}$ and $\min\{f(x): x \in G\}$, respectively.

8.15. THEOREM. *Let P be a countably infinite symmetric subset of X not containing* 1. *The following conditions are equivalent:*

(i) *P satisfies the property $FZ(P, G)$;*

(ii) *every function β in $\mathfrak{B}_h(P)$ is matchable on P by $\hat{\mu}$ for some $\mu \in M_+(G)$;*

(iii) *there is a positive constant $\varkappa$ such that $\|\hat{f}\|_1 \leqslant \varkappa \cdot \max(f)$ for all $f \in \mathfrak{T}_{P,r}(G)$;*

(iv) *there is a positive constant $\varkappa$ such that $\|\hat{f}\|_1 \leqslant \varkappa \cdot \operatorname{ess\,sup}(f)$ for all $f \in \mathfrak{L}^1_{P,r}(G)$;*

(v) *$f \in \mathfrak{L}^1_{P,r}(G)$ and $\operatorname{ess\,sup}(f) < \infty$ imply $\hat{f} \in l^1(X)$;*

(vi) *$\mu \in M_{P,r}(G)$ and $\mu \leqslant c\lambda$ for some real number c (which depends on μ) imply $\hat{\mu} \in l^1(X)$;*

(vii) *there is a positive constant $\varkappa$ such that μ in $M_{P,r}(G)$ and $\mu \leqslant c\lambda$ (c depending on μ) imply $\|\hat{\mu}\|_1 \leqslant \varkappa c$.*

The constants in (iii), (iv) *and* (vii) *may be taken equal.*

Proof. The equivalence of (i) and (ii) is Theorem 8.10. It is simple to establish directly that (iii) implies (i), and we will now do this. Suppose that (iii) holds, and that u is any function in $\mathfrak{F}_h(P)$. If the functions $s_n = \sum_{\chi \in P_n} u(\chi)\chi$ are bounded above by say α on G, then from (iii) we have

$$\|\hat{s}_n\|_1 = \sum_{\chi \in P_n} |u(\chi)| \leqslant \varkappa \cdot \max(s_n) \leqslant \varkappa\alpha,$$

and so taking the limit as $n \to \infty$, we find that $u \in l^1(P)$, i.e., property $FZ(P, G)$ holds.

We now prove that (ii) implies (iii). We first apply Lemma 2.5 with $E = M(G)$, $F = \mathfrak{B}_h(P)$, $T(\mu) = \hat{\mu}|P$, and $A_n = \{\mu \in M_+(G): \|\mu\| \leqslant n\}$. Lemma 2.5 shows that there exists a positive integer m such that $\{\beta \in \mathfrak{B}_h(P): \|\beta\|_\infty \leqslant 1\} \subset (A_m)^\wedge$. In other terms, there is a positive constant $\varkappa$ such that for every $\beta \in \mathfrak{B}_h(P)$, there is a $\mu \in M_+(G)$ for which $\|\mu\| \leqslant \varkappa\|\beta\|_\infty$ and for which $\hat{\mu}|P = \beta$. Now let f be a polynomial in $\mathfrak{T}_{P,r}(G)$, define β as $\overline{\operatorname{sgn}\hat{f}}$, and choose $\mu \in M_+(G)$ such that $\|\mu\| \leqslant \varkappa$ and $\hat{\mu}|P = \beta$. Then we have

$$\|\hat{f}\|_1 = \sum_{\chi \in P} \hat{f}(\chi)\beta(\chi) = \sum_{\chi \in P} \hat{f}(\chi)\hat{\mu}(\chi) = \sum_{\chi \in P} (f * \mu)^\wedge(\chi)\chi(e)$$
$$= f * \mu(e) = \int_G f(y^{-1})\,d\mu(y) \leqslant \max(f)\,\|\mu\| \leqslant \max(f) \cdot \varkappa.$$

This is exactly (iii). We have established the equivalence of (i), (ii), and (iii).

For μ in $M_{P,r}(G)$, we have $0 = \hat{\mu}(1) = \int_G 1\,d\mu$, and so any c as in (vi) or (vii) must be nonnegative. Condition (vii) trivially implies (iv) with the same value of $\varkappa$, and (iv) trivially implies (iii) with the same value of $\varkappa$. Suppose now that (iii) holds; we prove (vii). Let μ in $M_{P,r}(G)$ have the property that $\mu \leqslant c\lambda$. By [8], Vol. II, 28.57, there is a sequence $(K_j)_{j=1}^\infty$ of functions in $\mathfrak{T}_+(G)$ such that $\int_G K_j\,d\lambda = 1$ and $\lim_{j\to\infty} \hat{K}_j(\chi) = 1$ for all $\chi \in P$. Consider the polynomials $f_j = \mu * K_j$. For each of them and for all $x \in G$, we have

$$f_j(x) = \int_G K_j(y^{-1}x)\,d\mu(y) \leqslant c\int_G K_j(y^{-1}x)\,d\lambda(y) = c\int_G K_j\,d\lambda = c.$$

By (iii) and the evident fact that $f_j \in \mathfrak{T}_{P,r}(G)$, we have

$$\|\hat{f}_j\|_1 \leqslant \varkappa c.$$

By Fatou's lemma for series, we have

$$\|\hat{\mu}\|_1 = \sum_{\chi \in P} \lim_{j\to\infty} |\hat{K}_j(\chi)\hat{\mu}(\chi)| \leqslant \liminf_{j\to\infty} \Big[\sum_{\chi \in P} |\hat{K}_j(\chi)\hat{\mu}(\chi)|\Big]$$
$$= \liminf_{j\to\infty} \|\hat{f}_j\|_1 \leqslant \varkappa c.$$

Thus (vii) holds with the same value of $\varkappa$.

We have thus proved the equivalence of (i), (ii), (iii), (iv) and (vii). Since (vii) obviously implies (vi) and (vi) obviously implies (v), it suffices for us to prove that (v) implies (iii). First we show that

(1) $$\|f\|_1 \leqslant 2\max(f) \quad \text{for } f \in \mathfrak{T}_{P,r}(G).$$

We have $f = f^+ - f^-$ where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$. Since 1 is not in P, we have

$$0 = \hat{f}(1) = \int_G f\,d\lambda = \int_G f^+\,d\lambda - \int_G f^-\,d\lambda = \|f^+\|_1 - \|f^-\|_1,$$

and hence

$$\|f\|_1 = \|f^+\|_1 + \|f^-\|_1 = 2\|f^+\|_1 \leqslant 2\|f^+\|_u = 2\max(f).$$

This establishes (1). Now we apply Lemma 2.3 with

$$E = \mathfrak{L}^1_{P,r}(G), \quad \varphi(f) = \|\hat{f}\|_1 \quad \text{and} \quad \tau(f) = \operatorname{ess\,sup}(f).$$

Hypothesis 2.3. (i) holds in view of (v). We conclude that there is a constant $\varkappa$ such that

(2) $$\|\hat{f}\|_1 \leqslant \varkappa \max\{\|f\|_1, \operatorname{ess\,sup}(f)\}$$

for all $f \in \mathfrak{L}^1_{P,r}(G)$. Inequalities (1) and (2) yield

$$\|f^\wedge\|_1 \leqslant 2\varkappa \max(f) \quad \text{for } f \in \mathfrak{T}_{P,r}(G),$$

and so (iii) holds. ■

We list yet another property equivalent to $FZ(P, G)$.

8.16. THEOREM. *Let P be a countable symmetric subset of X not containing 1. In order that P be an $FZ(G)$-set it is necessary and sufficient that*

(a) *P be a Sidon set and*

(b) *there exists a real number $\varkappa > 1$ such that either*

(i) $\varkappa^{-1} \leqslant \max(f) \leqslant \varkappa$ *for every $f \in \mathfrak{T}_{P,r}(G)$ satisfying* $\min(f) = -1$,

or (what is equivalent)

(ii) $-\varkappa \leqslant \min(f) \leqslant -\varkappa^{-1}$ *for every $f \in \mathfrak{T}_{P,r}(G)$ satisfying* $\max(f) = 1$.

Proof. The equivalence of (i) and (ii) is trivial.

Suppose that (ii) holds, and let f be any nonzero polynomial in $\mathfrak{T}_{P,r}(G)$. Since $\int_G f d\lambda = 0$, we have $\beta = \max(f) > 0$. For the function f/β, we have by (ii) that $\min(f/\beta) \geqslant -\varkappa$, hence $\min(f) \geqslant -\varkappa\beta$; and so

$$\|f\|_u = \max[\max(f), -\min(f)] \leqslant \varkappa\beta = \varkappa \max(f).$$

If (a) also holds, we select a constant $\varkappa'$ such that $\|f^\wedge\|_1 \leqslant \varkappa' \|f\|_u$ and find

$$\|f^\wedge\|_1 \leqslant \varkappa\varkappa' \max(f)$$

for all $f \in \mathfrak{T}_{P,r}(G)$. That is, the property $FZ(P, G)$ holds, by Theorem 8.15.

Conversely, suppose that P is an $FZ(G)$-set. By 5.7, P is a Sidon set. By 8.15, there is a positive constant $\varkappa$ such that

$$\|f^\wedge\|_1 \leqslant \varkappa \max(f)$$

for all $f \in \mathfrak{T}_{P,r}(G)$. It is trivial that $\|f\|_u \leqslant \|f^\wedge\|_1$, and so we have

(1) $$\max[\max(f), -\min(f)] \leqslant \varkappa \max(f)$$

for all $f \in \mathfrak{T}_{P,r}(G)$. Similarly we have

(2) $$\max[\max(-f), -\min(-f)] \leqslant \varkappa \max(-f),$$

and combining (1) and (2), we find

(3) $$\max[\max(f), -\min(f)] \leqslant \varkappa \min[\max(f), -\min(f)].$$

Again note that if $f \neq 0$, then both $a = -\min(f)$ and $b = \max(f)$ are positive numbers. We rewrite (3) as

$$\tfrac{1}{2}(a+b+|a-b|) \leqslant \tfrac{1}{2}\varkappa(a+b-|a-b|),$$

or equivalently

(4) $$|a-b| \leqslant \theta(a+b),$$

where

(5) $$\theta = \frac{\varkappa-1}{\varkappa+1}, \quad \text{so that } 0 < \theta < 1.$$

The inequality (4) translates readily into

(6) $$a \leqslant \varkappa b \quad \text{if } a \geqslant b,$$

(7) $$b \leqslant \varkappa a \quad \text{if } a \leqslant b.$$

Thus if a has a fixed value, say 1, then (6) becomes

(6′) $$b \geqslant \frac{1}{\varkappa} \quad \text{if } b \leqslant 1$$

and (7) becomes

(7′) $$b \leqslant \varkappa \quad \text{if } b \geqslant 1.$$

That is, (i) holds. ■

We next set down some matching properties enjoyed by certain Sidon sets, which so far as we know are new.

8.17. THEOREM. *Let P be a symmetric Sidon set containing no elements χ such that $\chi^2 = 1$. Let $\varkappa$ be as in* [8], Vol. II, (37.2.v). *That is, we have $\|f^\wedge\|_1 \leqslant \varkappa \|f\|_\infty$ for all $f \in \mathfrak{L}^\infty_P(G)$. Let β be any function in $\mathfrak{B}_r(P)$ such that $\beta(\chi) = -\beta(\chi^{-1})$ for all $\chi \in P$. Then there is a measure $\mu \in \boldsymbol{M}_+(G)$ such that:*

(i) $$\|\mu\| \leqslant \varkappa \|\beta\|_\infty,$$

(ii) $$\operatorname{Im} \mu^\wedge(\chi) = \beta(\chi) \quad \textit{for all } \chi \in P.$$

Proof. Let $\mathfrak{S}_P(G)$ be the real linear space of real-valued functions on G spanned by all functions $i(\chi - \chi^{-1})$ for $\chi \in P$. Plainly if $f \in \mathfrak{S}_P(G)$, we have

(1) $$f(x^{-1}) = -f(x) \quad \text{for all } x \in G,$$

and we say that f is an *odd function*. Associated with our function β is a real linear functional M_0 on $\mathfrak{S}_P(G)$, defined by

(2) $$M_0(f) = \frac{1}{i} \sum_{\chi \in P}{}' \beta(\chi) f^\wedge(\chi).$$

If $f = i \sum_{k=1}^{n} c_k(\chi_k - \chi_k^{-1})$ with real c_k, then we have

(3) $$M_0(f) = \sum_{k=1}^{n} c_k\big(\beta(\chi_k) - \beta(\chi_k^{-1})\big) = \sum_{k=1}^{n} c_k 2\beta(\chi_k).$$

Since P is a Sidon set, we have

(4) $$|M_0(f)| \leqslant \|f^\wedge\|_1 \|\beta\|_\infty \leqslant \varkappa \|\beta\|_\infty \|f\|_u.$$

Thus the number

$$\gamma = \|M_0\| = \sup\{M_0(f): f \in \mathfrak{S}_P(G), \|f\|_u \leqslant 1\}$$

is a nonnegative real number, equal to zero if and only if β is the zero function.

The function 1 is not in $\mathfrak{S}_P(G)$. We extend M_0 to a functional M_1 on $\{f+\alpha 1: f \in \mathfrak{S}_P(G), \alpha \in R\}$ by the rule

(5) $$M_1(f+\alpha 1) = M_0(f) + \alpha\cdot\gamma.$$

We compute the norm of M_1. Suppose that $\|f+\alpha 1\|_u \leqslant 1$, with $f \in \mathfrak{S}_P(G)$ and $\alpha \in R$. Then we have

$$-1-\alpha \leqslant f \leqslant 1-\alpha.$$

Since f is an odd function, we must have $-1 \leqslant \alpha \leqslant 1$, and also

$$-\min\{1-\alpha, 1+\alpha\} \leqslant f \leqslant \min\{1-\alpha, 1+\alpha\}.$$

That is, we have $\|f\|_u \leqslant 1-|\alpha|$, and so

$$|M_1(f+\alpha 1)| \leqslant |M_0(f)| + |\alpha|\cdot\gamma \leqslant \gamma(1-|\alpha|) + |\alpha|\cdot\gamma = \gamma.$$

This proves that $\|M_1\| = \gamma = \|M_0\|$.

Now use the Hahn–Banach theorem to extend M_1 to a real linear functional M on $\mathfrak{C}_r(G)$ such that $\|M\| = \|M_1\| = \gamma$. Since $M(1) = \|M\|$, M is a nonnegative linear functional (see e.g. [8], Vol. II, (34.48.b)). Thus there is a measure μ in $M_+(G)$ for which

$$\int_G \varphi d\mu = M(\varphi) \quad \text{for all } \varphi \in \mathfrak{C}_r(G).$$

Taking $\varphi = i(\chi-\chi^{-1})$ with $\chi \in P$, and using (3), we find

(6) $$M_0(\varphi) = \beta(\chi) - \beta(\chi^{-1}) = 2\beta(\chi)$$

and also

(7) $$M_0(\varphi) = \int_G \varphi d\mu = i[\mu^\wedge(\chi^{-1}) - \mu^\wedge(\chi)] = 2\,\mathrm{Im}\,\mu^\wedge(\chi).$$

Equalities (6) and (7) imply (ii). The inequality (i) follows from (4) and the definition of γ. ∎

8.18. Corollary. *Let P_0 be a Sidon set such that $\chi \in P_0$ implies $\chi^{-1} \notin P_0$. Let β_0 be any bounded real-valued function on P_0. There is a measure $\mu \in M_+(G)$ such that $\mathrm{Im}\,\mu^\wedge(\chi) = \beta_0(\chi)$ for all $\chi \in P_0$.*

Proof. By Drury's theorem [2], the set $P = P_0 \cup P_0^{-1}$ is a Sidon set. Define β as β_0 on P_0 and as $-\beta_0$ on P_0^{-1}, and apply 8.17. ∎

To conclude this section, we give an analogue of 8.10 for the full FZ property; see 8.9 and 5.3.

8.19 Theorem. *Let P be a countable symmetric subset of X. Then P has the full FZ property if and only if for every $\beta \in \mathfrak{B}_h(P)$ and every compact symmetric neighbourhood U of e, there exist $\mu \in M_+(U)$ and $g \in \mathfrak{L}_r^\infty(G)$ such that*

(i) $$\beta = (\mu^\wedge + g^\wedge)|P.$$

Proof. Supposing that (i) holds for all $\beta \in \mathfrak{B}_h(P)$, consider a fixed but arbitrary $\beta \in \mathfrak{D}_h(P)$ and choose $\mu \in M_+(U)$ and $g \in \mathfrak{L}_r^\infty(G)$ for which (i) holds. By the Riemann–Lebesgue lemma, $g^\wedge|P$ belongs to $c_0(P)$. Hence for m large enough and $\chi \in P\setminus P_m$, we have $|g^\wedge(\chi)| < \frac{1}{2}$, so that

$$|\beta(\chi) - \mu^\wedge(\chi)| < \tfrac{1}{2}$$

for all $\chi \in P\setminus P_m$. Thus 8.7. (iii) holds and so by Theorem 8.7, P is an $FZ(U)$-set.

Now suppose that W is any set in $\mathscr{I}$, and let u be a function in $\mathfrak{F}_h(P)$ such that $s_n u(x) = \sum_{\chi \in P_n} u(\chi)\chi(x)$ is bounded above for all n and for all $x \in W$. Let x_0 be in $\mathrm{int}(W)$ and U a compact symmetric neighbourhood of e such that $U \in \mathscr{I}$ and $x_0 U \subset W$. Then for all $t \in U$, we have

$$\sum_{\chi \in P_n} u(\chi)\chi(x_0)\chi(t) = s_n u(x_0 t).$$

Since P is an $FZ(U)$-set, the function $\chi \to \chi(x_0)u(\chi)$ is in $l^1(P)$ and so therefore is u itself. That is, P is an $FZ(W)$-set for all $W \in \mathscr{I}$, which is to say, P has the full FZ property.

The converse is simple. If P has the full FZ property, we merely cite 8.5 to see that the matching property (i) holds. ∎

§ 9. Drury's theorem for $FZ(G)$-sets. Our aim here is to prove an analogue of Drury's theorem [2].

9.1. Theorem. *Let P_1 and P_2 be countably infinite (symmetric) $FZ(G)$-sets in X. Then the union $P_1 \cup P_2$ is also an $FZ(G)$-set.*

The proof is broken up into several parts. We follow Drury's construction, with some simplifications permitted by our current situation. We will suppose throughout that 1 is in none of our sets P, P_1, or P_2.

9.2. Definition. Let c be a real number $\geqslant 1$, and let P be a symmetric countable subset of X not containing 1. Suppose that for every function $\beta \in \mathfrak{D}_h(P)$ (see 8.6), there is a measure $\mu \in M_+(G)$ such that $\mu^\wedge|P = \beta$ and such that $\|\mu\| = \mu^\wedge(1) \leqslant c$. Then P is called *an FZ-c set.*

9.3. Theorem. *Every $FZ(G)$-set P not containing 1 is an FZ-c set for some $c \geqslant 1$.*

Proof. We use Lemma 2.5, with $E = \boldsymbol{M}_r(G)$, $F = \mathfrak{B}_h(P)$, and $T(\mu)$ defined as $\hat{\mu}|P$ for all $\mu \in \boldsymbol{M}_r(G)$. Theorem 8.10 implies that $T(\boldsymbol{M}_+(G)) = \mathfrak{B}_h(P)$. Defining A_n as the set $\{\mu \in \boldsymbol{M}_+(G)\colon \|\mu\| \leqslant n\}$ for $n \in \{1, 2, 3, \ldots\}$, we see that 2.5. (i) and 2.5. (ii) hold, with $B_{m,n} = A_{m+n}$. By Lemma 2.5, there is a positive integer m such that for all $\beta \in \mathfrak{B}_h(P)$ with $\|\beta\|_\infty \leqslant 1$, there is a $\mu \in \boldsymbol{M}_+(G)$ such that $\|\mu\| \leqslant m$ and $\hat{\mu}|P = \beta$. Thus we may take $c = m$. ■

9.4. THEOREM. *Let P be an FZ-c set such that either* (a) $\chi^2 = 1$ *for all* $\chi \in P$ *or* (b) $\chi^2 \neq 1$ *for all* $\chi \in P$. *Let ε be a real number in* $]0, 1]$. *There is a measure $\mu \in \boldsymbol{M}_+(G)$ such that $\hat{\mu}(\chi) = 1$ for all $\chi \in P$, $|\hat{\mu}(\chi)| \leqslant \varepsilon$ for all $\chi \in X \setminus (P \cup \{1\})$, and $\hat{\mu}(1) \leqslant 2c^4\varepsilon^{-1}$ in Case* (a) *and $\hat{\mu}(1) \leqslant 4c^4\varepsilon^{-1}$ in Case* (b).

The proof of 9.4 is not really very complicated, but involves several small computations. We first make a reduction.

9.5. *If the conclusion of Theorem 9.4 holds for all finite FZ-c sets, then it holds for all FZ-c sets.*

Proof. Given a finite symmetric subset Φ of P, let μ_Φ be such that $\hat{\mu}_\Phi(\chi) = 1$ for all $\chi \in \Phi$, $|\hat{\mu}_\Phi(\chi)| \leqslant \varepsilon$ for $\chi \in X \setminus (\Phi \cup \{1\})$ and $\hat{\mu}_\Phi(1) \leqslant Ac^4\varepsilon^{-1}$. Here $A = 2$ in Case (a) and $A = 4$ in Case (b). Under inclusion the family $\{\Phi\}$ is a directed set. The set of measures $\{\mu_\Phi\}$ lies in the weak-* compact set $\{\nu \in \boldsymbol{M}_+(G)\colon \|\nu\| \leqslant Ac^4\varepsilon^{-1}\}$ and so the *net* $\{\mu_\Phi\}$ admits a weak-* convergent subnet with limit μ. It is clear that $\hat{\mu}(\chi)$ is a cluster point of the complex-valued net $\{\hat{\mu}_\Phi(\chi)\}$ for all $\chi \in X$, and so the conclusion of Theorem 9.4 holds for P. ■

9.6. Notation. For a fixed but arbitrary positive integer m, let Ω denote the multiplicative group $\prod_{k=1}^{m} \{1, -1\}_{(k)}$. For $\omega \in \Omega$ and $k \in \{1, 2, \ldots, m\}$, let ω_k denote the kth coordinate of ω. Let Γ denote the character group of Ω. Note that all $\gamma \in \Gamma$ are real valued and that $\omega = \omega^{-1}$ for all $\omega \in \Omega$.

A subset P of X is called *asymmetric* if $\chi \in P$, $\chi^{-1} \in P$ imply $\chi = \chi^{-1}$.

9.7. LEMMA. *Let $\{\chi_1, \ldots, \chi_m\}$ be an asymmetric set in X such that $P = \{\chi_1, \ldots, \chi_m\} \cup \{\chi_1^{-1}, \ldots, \chi_m^{-1}\}$ is an FZ-c set. For each $\omega \in \Omega$, there is a measure $\nu_\omega \in \boldsymbol{M}_+(G)$ such that:*

(i) $$\hat{\nu}_\omega(\chi_k) = \omega_k \quad \text{for } k \in \{1, 2, \ldots, m\};$$

(ii) $$\|\nu_\omega\| = \hat{\nu}_\omega(1) \leqslant c^2.$$

For $\chi \in X$, let f_χ be the function on Ω such that $f_\chi(\omega) = \hat{\nu}_\omega(\chi)$. Then we have

(iii) $$\sum_{\gamma \in \Gamma} |\hat{f}_\chi(\gamma)| \leqslant c^2 \quad \text{for all } \chi \in X.$$

Proof. For $\omega \in \Omega$, let μ_ω be a measure in $\boldsymbol{M}_+(G)$ such that

(1) $$\hat{\mu}_\omega(\chi_k) = \omega_k \quad \text{for } k \in \{1, 2, \ldots, m\}$$

and

(2) $$\|\mu_\omega\| \leqslant c.$$

For each $\omega \in \Omega$, define ν_ω as

$$\nu_\omega = 2^{-m} \sum_{\alpha \in \Omega} \mu_{\omega\alpha^{-1}} * \mu_\alpha.$$

Plainly ν_ω is in $\boldsymbol{M}_+(G)$, and (ii) is immediate. To check (i), we write

$$\hat{\nu}_\omega(\chi_k) = 2^{-m} \sum_{\alpha \in \Omega} \hat{\mu}_{\omega\alpha^{-1}}(\chi_k)\hat{\mu}_\alpha(\chi_k) = 2^{-m} \sum_{\alpha \in \Omega} (\omega\alpha^{-1})_k \alpha_k = 2^{-m} \sum_{\alpha \in \Omega} \omega_k = \omega_k.$$

We now prove (iii). We have

$$f_\chi(\omega) = \hat{\nu}_\omega(\chi) = 2^{-m} \sum_{\alpha \in \Omega} \hat{\mu}_{\omega\alpha^{-1}}(\chi)\hat{\mu}_\alpha(\chi),$$

and so for $\gamma \in \Gamma$,

$$\hat{f}_\chi(\gamma) = 2^{-m} \sum_{\omega \in \Omega} f_\chi(\omega)\gamma(\omega) = 2^{-m} \sum_{\omega \in \Omega} \Big[2^{-m} \sum_{\alpha \in \Omega} \hat{\mu}_{\omega\alpha^{-1}}(\chi)\hat{\mu}_\alpha(\chi)\Big]\gamma(\omega)$$
$$= 2^{-m} \sum_{\omega \in \Omega} 2^{-m} \sum_{\alpha \in \Omega} \hat{\mu}_\omega(\chi)\hat{\mu}_\alpha(\chi)\gamma(\omega\alpha) = \Big[2^{-m} \sum_{\omega \in \Omega} \hat{\mu}_\omega(\chi)\gamma(\omega)\Big]^2.$$

This equality implies that

$$|\hat{f}_\chi(\gamma)| = \Big[2^{-m} \sum_{\omega \in \Omega} \hat{\mu}_\omega(\chi)\gamma(\omega)\Big]\Big[2^{-m} \sum_{\alpha \in \Omega} \overline{\hat{\mu}_\alpha(\chi)}\gamma(\alpha)\Big]$$
$$= 2^{-m} \sum_{\omega \in \Omega} 2^{-m} \sum_{\alpha \in \Omega} \hat{\mu}_\omega(\chi)\overline{\hat{\mu}_\alpha(\chi)}\gamma(\omega\alpha)$$

and so

(3) $$\sum_{\gamma \in \Gamma} |\hat{f}_\chi(\gamma)| = 2^{-m} \sum_{\omega \in \Omega} 2^{-m} \sum_{\alpha \in \Omega} \hat{\mu}_\omega(\chi)\overline{\hat{\mu}_\alpha(\chi)} \sum_{\gamma \in \Gamma} \gamma(\omega\alpha).$$

The final sum in (3) is equal to 2^m if $\omega = \alpha$ and is zero otherwise (see e.g. [8], Vol. I, (23.19)). Thus from (3) we have

$$\sum_{\gamma \in \Gamma} |\hat{f}_\chi(\gamma)| = 2^{-m} \sum_{\omega \in \Omega} |\hat{\mu}_\omega(\chi)|^2 \leqslant 2^{-m} \sum_{\omega \in \Omega} \|\mu_\omega\|^2 \leqslant c^2.$$

This is (iii). ■

9.8. Proof of Theorem 9.4. As noted in 9.5, we may suppose that our set P is finite, as in 9.7. Define σ by

(1) $$\sigma = \tfrac{1}{4}\varepsilon c^{-2}.$$

We construct the measure μ actually as a trigonometric polynomial times Haar measure on G. To do this, define for every $\omega\in\Omega$ the Riesz polynomial

(2) $$p_\omega = \prod_{k=1}^{m}\bigl(1+\omega_k\sigma(\chi_k+\chi_k^{-1})\bigr).$$

It is trivial that $p_\omega \geqslant 0$. Finally we define p by

(3) $$p = 2^{-m}\sum_{\omega\in\Omega} p_\omega * \nu_\omega,$$

where the measures ν_ω are as in Lemma 9.7. Plainly p is nonnegative and real. We estimate the norm $\|p\|_1$ using 9.7. (ii):

(4) $$\|p\|_1 = 2^{-m}\sum_{\omega\in\Omega}\int_G\int_G p_\omega(y^{-1}x)\,d\nu_\omega(y)\,dx = 2^{-m}\sum_{\omega\in\Omega}\int_G p_\omega(t)\,dt\,\nu_\omega(G)$$
$$\leqslant c^2\int_G\Bigl[2^{-m}\sum_{\omega\in\Omega}p_\omega(t)\Bigr]dt.$$

Write $g_k(t) = \sigma\bigl(\chi_k(t)+\chi_k^{-1}(t)\bigr)$. From (2), we see that the integrand in (4) can be rewritten as

$$\prod_{k=1}^{m}\tfrac{1}{2}[1+g_k(t)+1-g_k(t)] = 1.$$

Thus (4) implies that

(5) $$\|p\|_1 \leqslant c^2.$$

We next compute $p^\wedge(\chi_k)$, first using 9.7. (i) to write

(6) $$p^\wedge(\chi_k) = 2^{-m}\sum_{\omega\in\Omega}p_\omega^\wedge(\chi_k)\nu_\omega^\wedge(\chi_k) = 2^{-m}\sum_{\omega\in\Omega}\omega_k p_\omega^\wedge(\chi_k).$$

We multiply out the product (2), obtaining

(7) $$p_\omega(t) = \sum_S\prod_{j\in S}\bigl[\omega_j\sigma\bigl(\chi_j(t)+\chi_j^{-1}(t)\bigr)\bigr],$$

the sum in (7) being over all of the 2^m subsets S of $\{1, 2, \ldots, m\}$, and the void product being taken as 1. Multiplying out the products in (7), we have

(8) $$p_\omega(t) = \sum_S\sum_T\prod_{j\in T}\bigl(\omega_j\sigma\chi_j(t)\bigr)\prod_{j\in S\setminus T}\bigl(\omega_j\sigma\overline{\chi_j(t)}\bigr),$$

the inner sum in (8) being on all subsets T of S. The Fourier transform $p_\omega^\wedge(\chi_k)$ is the coefficient of χ_k in the polynomial (8). That is, we have

(9) $$p_\omega^\wedge(\chi_k) = \sum_{(T,S)}(\sigma)^{\mathrm{card}(S)}\prod_{j\in S}\omega_j,$$

the sum in (9) being over all pairs (T, S) such that $T\subset S\subset\{1, 2, \ldots, m\}$ and

(10) $$\prod_{j\in T}\chi_j\prod_{j\in S\setminus T}\chi_j^{-1} = \chi_k.$$

We know no one of the numbers $p_\omega^\wedge(\chi_k)$, but we can nonetheless evaluate $p^\wedge(\chi_k)$. By (6) and (9), we have

(11) $$p^\wedge(\chi_k) = \sum_{(T,S)}(\sigma)^{\mathrm{card}(S)}2^{-m}\sum_{\omega\in\Omega}\omega_k\prod_{j\in S}\omega_j.$$

Suppose that a given S contains an element $j_0\neq k$. Then the mapping $\omega\to\omega_k\prod_{j\in S}\omega_j$ is a character of Ω that is not identically 1 and so we have

$$2^{-m}\sum_{\omega\in\Omega}\omega_k\prod_{j\in S}\omega_j = 0.$$

The only pairs (T, S) that can make nonzero contributions to the sum (11) are therefore $(\varnothing, \varnothing)$, $(\varnothing, \{k\})$, and $(\{k\}, \{k\})$. The corresponding characters on the left side of (10) are 1, χ_k^{-1}, and χ_k, respectively. Since $1\notin\{\chi_1, \ldots, \chi_m\}$, the pair $(\varnothing, \varnothing)$ cannot yield χ_k. If χ_k has order different from 2, then $\chi_k^{-1}\neq\chi_k$, the pair $(\varnothing, \{k\})$ contributes nothing, and only the pair $(\{k\}, \{k\})$ contributes to (11). If χ_k has order 2, then both $(\varnothing, \{k\})$ and $(\{k\}, \{k\})$ contribute to (11). Thus we have:

(12) $$p^\wedge(\chi_k) = \sigma 2^{-m}\sum_{\omega\in\Omega}\omega_k^2 = \sigma\quad\text{if } \chi_k^2\neq 1;$$

(12$_2$) $$p^\wedge(\chi_k) = \sigma+\sigma = 2\sigma\quad\text{if } \chi_k^2 = 1.$$

We now estimate $|p^\wedge(\chi)|$ for $\chi\notin P\cup\{1\}$, beginning with the obvious equality

(13) $$p^\wedge(\chi) = 2^{-m}\sum_{\omega\in\Omega}p_\omega^\wedge(\chi)\nu_\omega^\wedge(\chi).$$

We use the polynomial f_χ on Ω that was introduced in 9.7. We write f_χ in its own Fourier expansion:

$$\nu_\omega^\wedge(\chi) = f_\chi(\omega) = \sum_{\gamma\in\Gamma}f_\chi^\wedge(\gamma)\gamma(\omega).$$

Substituting this value in (13) and citing 9.7. (iii), we find

(14) $$|p^\wedge(\chi)| = \Bigl|\sum_{\gamma\in\Gamma}2^{-m}\sum_{\omega\in\Omega}p_\omega^\wedge(\chi)f_\chi^\wedge(\gamma)\gamma(\omega)\Bigr|$$
$$\leqslant\sum_{\gamma\in\Gamma}|f_\chi^\wedge(\gamma)|\cdot\Bigl|2^{-m}\sum_{\omega\in\Omega}p_\omega^\wedge(\chi)\gamma(\omega)\Bigr|$$
$$\leqslant\Bigl[\max_{\gamma\in\Gamma}\Bigl|2^{-m}\sum_{\omega\in\Omega}p_\omega^\wedge(\chi)\gamma(\omega)\Bigr|\Bigr]\cdot c^2.$$

We now estimate the quantity $[\ldots]$ in (14). Every $\gamma \in \Gamma$ has the form $\gamma(\omega) = \prod_{j\in S} \omega_j$, where S is a subset of $\{1, 2, \ldots, m\}$, or equivalently, $\gamma(\omega) = \prod_{k=1}^{m} \omega_k^{e_k}$, where $\{e_k\}_{k=1}^m$ is a sequence consisting of 0's and 1's. Again write $g_k = \sigma(\chi_k + \chi_k^{-1})$. Then by (2) we have

$$(15)\qquad 2^{-m}\sum_{\omega\in\Omega} p_\omega\hat{}\,(\chi)\gamma(\omega) = 2^{-m}\sum_{\omega\in\Omega}\int_G \prod_{k=1}^{m}\big(1+\omega_k g_k(t)\big)\overline{\chi(t)}\,dt\cdot\prod_{l=1}^{m}\omega_l^{e_l}$$

$$= \int_G \Big[2^{-m}\sum_{\omega\in\Omega}\Big(\prod_{k=1}^{m}\big(\omega_k^{e_k}+\omega_k^{1+e_k}g_k(t)\big)\Big]\overline{\chi(t)}\,dt.$$

By Fubini's theorem for $\Omega = \{-1, 1\}^m$, the expression $[\ldots]$ in the last line of (15) is equal to

$$\prod_{k=1}^{m}\tfrac{1}{2}\big[1+g_k(t)+\big((-1)^{e_k}+(-1)^{1+e_k}g_k(t)\big)\big] = \prod_{k\in S} g_k(t) = \prod_{k\in S}\sigma\big(\chi_k(t)+\chi_k^{-1}(t)\big).$$

Hence the last line of (15) is equal to

$$(16)\qquad \int_G \Big[\prod_{k\in S}\sigma\big(\chi_k(t)+\chi_k^{-1}(t)\big)\Big]\overline{\chi(t)}\,dt.$$

For $S = \emptyset$, (16) is zero because $\chi \neq 1$. For $S = \{l\}$ for some $l \in \{1, 2, \ldots, m\}$, (16) again vanishes because $\chi \neq \chi_l$ and $\chi \neq \chi_l^{-1}$. If $\operatorname{card}(S) \geqslant 2$, then the integrand in (16) has absolute value $\leqslant 4\sigma^2$, and so we see that for all $\gamma \in \Gamma$, the inequality

$$\Big|2^{-m}\sum_{\omega\in\Omega} p_\omega\hat{}\,(\chi)\gamma(\omega)\Big| \leqslant 4\sigma^2$$

obtains. Going back to (14), we infer that

$$(17)\qquad |p\hat{}\,(\chi)| \leqslant 4c^2\sigma^2$$

for all $\chi \notin P\cup\{1\}$.

At this point we distinguish between Cases (a) and (b) in the proof of 9.4. If $\chi^2 = 1$ for all $\chi \in P$, then we define μ as the absolutely continuous nonnegative measure $\dfrac{1}{2\sigma}p\lambda$. It is clear from (1) and (5) that

$$(18)\qquad \|\mu\| = \mu\hat{}\,(1) \leqslant 2c^4\varepsilon^{-1},$$

from (12_2) that

$$(19)\qquad \mu\hat{}\,(\chi) = 1 \quad \text{for } \chi \in P,$$

and from (17) that

$$(20)\qquad |\mu\hat{}\,(\chi)| \leqslant \tfrac{1}{2}\varepsilon \quad \text{for } \chi \in X\setminus(P\cup\{1\}).$$

In Case (b), we define μ as $\dfrac{1}{\sigma}p\lambda$ and obtain the desired results, using (12) instead of (12_2) and getting ε instead of $\frac{1}{2}\varepsilon$ in (20). ∎

9.9. Proof of 9.1. Let P and P^* be any countable (symmetric) $FZ(G)$-sets in X not containing 1. Write Σ for P and T for $P^*\setminus P$. Now write Σ_1 for $\{\chi\in\Sigma\colon \chi^2\neq 1\}$, and Σ_2 for $\{\chi\in\Sigma\colon \chi^2 = 1\}$; define T_1 and T_2 analogously. Plainly Σ_1, Σ_2, T_1 and T_2 are pairwise disjoint $FZ(G)$-sets, and $\Sigma_1\cup\Sigma_2\cup T_1\cup T_2 = P\cup P^*$. Now let β be any function in $\mathfrak{B}_h(P\cup P^*)$. By Theorem 8.10, there are measures $\mu_1, \mu_2, \nu_1, \nu_2$ in $M_+(G)$ such that

$$(1)\qquad \mu_j\hat{}\,|\Sigma_j = \beta|\Sigma_j,\quad \nu_j\hat{}\,|T_j = \beta|T_j \quad (j\in\{1,2\}).$$

Select ε in $]0, 1]$ so that

$$(2)\qquad \max\{\varepsilon\|\mu_1\hat{}\|_u, \varepsilon\|\mu_2\hat{}\|_u, \varepsilon\|\nu_1\hat{}\|_u, \varepsilon\|\nu_2\hat{}\|_u\} \leqslant \tfrac{1}{4}.$$

By 9.3, there is a real number $c \geqslant 1$ such that Σ_1, Σ_2, T_1, and T_2 are FZ-c sets. By 9.4, there are measures $\varrho_1, \varrho_2, \sigma_1$, and σ_2 in $M_+(G)$ such that

$$(3)\qquad \begin{cases} \varrho_j\hat{}\,|\Sigma_j = 1, & |\varrho_j\hat{}\,|\,\big|\big(X\setminus(\Sigma_j\cup\{1\})\big) \leqslant \varepsilon,\\ \sigma_j\hat{}\,|T_j = 1, & |\sigma_j\hat{}\,|\,\big|\big(X\setminus(T_j\cup\{1\})\big) \leqslant \varepsilon.\end{cases}$$

Let μ be the measure $\mu_1*\varrho_1+\mu_2*\varrho_2+\nu_1*\sigma_1+\nu_2*\sigma_2$. Plainly μ is in $M_+(G)$, and for $\chi\in\Sigma_1$, for example, we have from (1), (2), and (3) that

$$|\beta(\chi)-\mu\hat{}\,(\chi)| = |\mu_2\hat{}\,(\chi)\varrho_2\hat{}\,(\chi)+\nu_1\hat{}\,(\chi)\sigma_1\hat{}\,(\chi)+\nu_2\hat{}\,(\chi)\sigma_2\hat{}\,(\chi)|$$

$$\leqslant \varepsilon(\|\mu_2\hat{}\|_u+\|\nu_1\hat{}\|_u+\|\nu_2\hat{}\|_u) \leqslant \tfrac{3}{4}.$$

Similar estimates apply to Σ_2, T_1, and T_2, and so we have

$$|\beta(\chi)-\mu\hat{}\,(\chi)| \leqslant \tfrac{3}{4} \quad \text{for all } \chi\in P\cup P^*.$$

By 8.7, $P\cup P^*$ is an $FZ(G)$-set. ∎

§ 10. Some sufficient conditions for property $FZ(P, G)$. It is a curious fact all that symmetric Sidon sets known to the writers are also $FZ(G)$-sets. The following definitions are based on Stečkin's work [15] as extended to general compact Abelian groups by Rudin ([13], 5.7.5, pp. 124–126). Some analogous notions appear in [4], § 4.

10.1. Definition. Let X be an Abelian group, Δ any subset of X, and t a positive integer. The symbol $\mathscr{S}(\Delta, t)$ denotes the set of all functions a carrying Δ into $\{1, 0, -1\}$ such that $\sum_{\chi\in\Delta}|a(\chi)| = t$. For $\psi\in X$, the symbol $\mathscr{S}(\Delta, t, \psi)$ denotes the set of all $a\in\mathscr{S}(\Delta, t)$ such that $\prod_{\chi\in\Delta}\chi^{a(\chi)} = \psi$.

10.2. Definition. Consider the following property (R) of $\Delta\subset X$. There exists a real number $A\geqslant 1$ such that $\operatorname{card}(\mathscr{S}(\Delta, t, \psi))\leqslant A^t$ for all ψ in X and integers $t\geqslant 2$. If a subset Δ of X is a finite union, $\Delta = \bigcup_{j=1}^{s}\Delta_j$, where each Δ_j has property (R), then Δ is called a *Stečkin set*.

10.3. Remarks. (a) For $X = Z$, Stečkin sets were introduced by Stečkin [15], under the name "R_σ sets". Stečkin proved that all Stečkin subsets of Z are Sidon sets.

(b) The notion of Stečkin sets was extended to arbitrary Abelian groups by Rudin [13], 5.7.5, pp. 124–126. Rudin proved, under some mild restrictions, that Stečkin subsets of arbitrary Abelian groups are Sidon sets. Rider [12] has extended Rudin's result somewhat.

(c) The dissociate sets of Hewitt and Zuckerman (see e.g. [8], Vol. II, (37.12)), are plainly special cases of Stečkin sets.

10.4. Theorem. *If Δ is a countably infinite Stečkin set, then $\Delta \cup \Delta^{-1}$ is an $FZ(G)$-set.*

Proof. In view of 9.1, we may suppose that Δ has property (R). Splitting Δ into the subset of elements of order 2 and elements not of order 2, we may also suppose that

(1) all or none of the elements of Δ have order 2.

We may also suppose that $1 \notin \Delta$ and that

$$\chi \in \Delta \text{ and } \chi^2 \neq 1 \quad \text{imply } \chi^{-1} \notin \Delta. \tag{2}$$

Let β be an arbitrary function in $\mathfrak{D}_h(\Delta)$. According to Theorem 8.7, the present theorem will be proved if we can find a measure $\nu \in M_+(G)$ such that

$$\sup\{|\beta(\chi) - \hat{\nu}(\chi)|: \chi \in \Delta \cup \Delta^{-1}\} \leqslant d < 1. \tag{3}$$

We enumerate Δ as a sequence $(\chi_1, \chi_2, \ldots)$. If no element of Δ has order 2, we have

$$\Delta \cup \Delta^{-1} = \{\chi_1, \chi_1^{-1}, \chi_2, \chi_2^{-1}, \ldots, \chi_n, \chi_n^{-1}, \ldots\}; \tag{4}$$

all characters appearing in (4) are distinct by (2). If all elements of Δ have order 2, then clearly we have

$$\Delta \cup \Delta^{-1} = \Delta = \{\chi_1, \chi_2, \ldots, \chi_n, \ldots\}. \tag{4_2}$$

In either case, we define $\Delta_n = \{\chi_1, \chi_2, \ldots, \chi_n\}$, $n \in \{1, 2, 3, \ldots\}$.

We define certain Riesz polynomials. Let τ be the number

$$\tau = \frac{1}{A(2A+1)}, \tag{5}$$

where A is the constant in 10.2. For every positive integer n, define p_n as

$$p_n = \prod_{l=1}^{n} \left(1 + \tau\beta(\chi_l)\chi_l + \tau\beta(\chi_l^{-1})\chi_l^{-1}\right). \tag{6}$$

It is clear from (5) and the fact that $|\beta| = 1$ that $p_n \geqslant 0$. Now multiply out the product (6). We obtain

$$p_n = 1 + \sum_{l=1}^{n} [\tau\beta(\chi_l)\chi_l + \tau\beta(\chi_l^{-1})\chi_l^{-1}] + \sum_{\psi \in X} c(n, \psi)\psi, \tag{7}$$

where, for every $\psi \in X$, we have

$$c(n, \psi) = \sum_{t=2}^{n} \Big[\sum_{a \in \mathscr{S}(\Delta_n, t, \psi)} \tau^t \prod_{\chi \in \Delta_n} \beta(\chi)^{a(\chi)} \Big]. \tag{8}$$

Since Δ has property (R), the absolute value of each inner sum in (8) is majorised by

$$\tau^t \operatorname{card}(\mathscr{S}, (\Delta, t, \psi)) \leqslant (\tau A)^t,$$

and so we have

$$|c(n, \psi)| \leqslant \sum_{t=2}^{n} (\tau A)^t < \frac{\tau^2 A^2}{1 - \tau A}. \tag{9}$$

Since p_n is real and nonnegative, we have

$$\|p_n\|_1 = \int_G p_n \, d\lambda = 1 + c(n, 1). \tag{10}$$

By (10) and (9), the numbers $\|p_n\|_1$ are bounded. Thus the set of measures $\{p_n\lambda\}_{n=1}^{\infty}$ in $M_+(G)$ admits a weak-$*$ cluster point, which we call μ. Clearly μ belongs to $M_+(G)$, and

$$\|\mu\| \leqslant 1 + \frac{\tau^2 A^2}{1 - \tau A}.$$

For every ψ in X, it is clear that

$$\hat{\mu}(\psi) \text{ is a cluster point of } \{\hat{p}_n(\psi)\}_{n=1}^{\infty}. \tag{11}$$

From (7), it is clear that

$$\hat{p}_n(\chi_l^{\pm 1}) = \tau\beta(\chi_l^{\pm 1}) + c(n, \chi_l^{\pm 1}) \quad \text{for } n \geqslant l \tag{12}$$

for all l in Case (4) and that

$$\hat{p}_n(\chi_l) = 2\tau\beta(\chi_l) + c(n, \chi_l) \quad \text{for } n \geqslant l \tag{12_2}$$

for all l in Case (4_2). From (9) and (12), we obtain

$$|\hat{p}_n(\chi_l^{\pm 1}) - \tau\beta(\chi_l^{\pm 1})| < \frac{\tau^2 A^2}{1 - \tau A} \quad \text{for } n \geqslant l \tag{13}$$

in Case (4). Applying (11) and (13), we find

$$|\hat{\mu}(\chi) - \tau\beta(\chi)| \leqslant \frac{\tau^2 A^2}{1 - \tau A} \quad \text{for all } \chi \in \Delta \cup \Delta^{-1}. \tag{14}$$

Note our choice of τ in (5), and infer from (14) that

$$|\mu^{\wedge}(\chi)-\tau\beta(\chi)| \leqslant \tfrac{1}{2}\tau \text{ for all } \chi \in \Delta \cup \Delta^{-1}, \tag{15}$$

in Case (4). In like fashion, (12_2) leads to

$$|\mu^{\wedge}(\chi)-2\tau\beta(\chi)| \leqslant \tfrac{1}{2}\tau \text{ for all } \chi \in \Delta \cup \Delta^{-1}, \tag{15_2}$$

in Case (4_2). From (15) it is evident that the measure $\nu = \frac{1}{\tau}\mu$ satisfies (3) with $d = \frac{1}{2}$. From (15_2) it is evident that $\nu = \frac{1}{2\tau}\mu$ satisfies (3) with $d = \frac{1}{4}$. ∎

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A divergent multiple Fourier series of power series type

by

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We present this paper to honor a great mathematician. The first co-author wishes to thank Professor Zygmund for the personal interest he has taken and the encouragement he has given over the pre- and post-doctoral years.

Abstract A continuous complex-valued function on the torus whose (double) Fourier series diverges restrictedly rectangularly at every point has been constructed by Charles Fefferman. The present paper presents a function which has the above properties and whose Fourier series is of power series type ($a_{mn} = 0$ if $m < 0$ or $n < 0$).

Charles Fefferman [2] has given an example of a continuous function $F(x, y)$ defined on the torus T^2 with the property that the double Fourier series $\Sigma a_{mn}\exp(i(mx+ny))$ of F is everywhere restrictedly rectangularly divergent. This means that for each point (x, y) and $E > 1$,

$$S_{MN}(x, y) = \sum_{\substack{|m|\leqslant M\\ |n|\leqslant N}} a_{mn} e^{i(mx+ny)}$$

fails to tend to a limit as M and N tend to infinity with $E^{-1} \leqslant M/N \leqslant E$.

In this paper we extend Fefferman's result by proving the following.

THEOREM 1. *There is a continuous complex-valued function $H(x, y)$ on the torus whose double Fourier series is of power series type ($a_{mn} = 0$ if $m < 0$ or $n < 0$) and is restrictedly rectangularly divergent everywhere.*

On $[0, 2\pi]\times[0, 2\pi]$ set $g_0(x, y) = g_0(x, y; \lambda) = \varphi(x)\varphi(y)e^{i\lambda xy}$ where φ is a C^∞ function equal to 0 *if* $0 \leqslant t \leqslant 1/40$ or if $2\pi - \frac{1}{40} \leqslant t \leqslant 2\pi$ and to 1 *if* $\frac{1}{20} \leqslant t \leqslant 2\pi - \frac{1}{20}$ with $0 \leqslant \varphi \leqslant 1$ elsewhere on $[0, 2\pi]$. The real parameter λ is greater then 1. Then clearly g_0 is *a* C^∞ function on the torus T^2 obtained from $[0, 2\pi]\times[0, 2\pi]$ by identifications, and $\|g_0\|_\infty = \sup|g_0(x, y)| = 1$.