Lacunarity for compact groups, III

by

R. E. EDWARDS (Canberra, A.C.T.), E. HEWITT* (Seattle, Wash.),
and K. A. ROSS** (Eugene, Ore.)

Dedicated with deep homage to Professor Antoni Zygmund
on the fiftieth anniversary of his scientific work

Abstract. Analogues and generalisations of a famous theorem of Fatou and
Zygmund are obtained for compact Abelian groups. Given a compact (Hausdorff)
Abelian group $G$ with character group $X$, and an increasing sequence $S = \{X_n\}_{n=1}^\infty$
of finite symmetric subsets of $X$, we consider a subset $P$ of $X = \bigcup_{n=1}^\infty X_n$ and write
$\Psi_k(P)$ for the linear space of all Hermitian complex-valued functions on $P$. Write
$P_n = P \cap X_n$ and for $w \in \Psi_k(P)$, write
$s_n w = \sum_{x \in P_n} w(x) x.

For a measurable set $W \subset G$ such that $W \subset (\text{int}(W))^{**}$, the following property is investigated:

(*) if $u \in \Psi_k(P)$ and $\sup_{x \in W} |u(x)| < \infty$, then $u \in \mathcal{F}(P)$.

The validity of this implication is shown to be independent of the choice of $S$. Accordingly, if $(*)$ holds, we say that the $\mathcal{F}(P, W)$ property holds and we call $P$ an $\mathcal{F}(P, W)$-set. A number of properties of $P$ are shown to be equivalent to property $\mathcal{F}(P, W)$. In particular, certain matching properties of bounded Hermitian functions on $P$ are shown to characterise $\mathcal{F}(P, W)$-sets. For example, $P$ is an $\mathcal{F}(P, W)$-set if and only if every bounded Hermitian function on $P$ is matched on $P$ by the Fourier-Stieltjes transform of a nonnegative measure in $\mathcal{M}(G)$. A large class of $\mathcal{F}(P, W)$-sets is identified and the union of two $\mathcal{F}(P, W)$-sets is shown to be another $\mathcal{F}(P, W)$-set. Every $\mathcal{F}(P, W)$-set is a Borel set; the converse is an open question for $W = G$.

§ 1. Introduction.

1.1. History. This paper is of course related to the first two in the sequence [4], [5], but may be read independently of [4]. We will occasion-

* Supported by National Science Foundation Grant GP-28313.
** Supported by National Science Foundation Grant GP-28320.
ally refer to [5]. In the present paper we take up a famous theorem for trigonometric series on the circle group which admit Hadamard gaps. Consider a trigonometric series
\[ \sum_{n=0}^{\infty} c_n \exp(i\theta_n x), \]
where the \( c_n \) are complex numbers and the \( n_k \) integers,
\[ 0 \leq n_1 < n_2 < n_3 < \ldots, \quad \theta_n = \theta_{n+1} - \theta_n, \]
\[ \inf \{n_k+1/n_k = g > 1 \} \text{ (Hadamard's gap condition)}. \]
The symmetric partial sums
\[ s_n(x) = \sum_{k=0}^{n} c_k \exp(i\theta_k x) \]
of this series are plainly real valued. Suppose that these partial sums satisfy the condition
\[ \sup_{n \to \infty} s_n(x) < \infty \]
for every \( x \) in some nonvoid open interval; the conclusion is that
\[ \sum_{n=0}^{\infty} |c_n| < \infty. \]
We refer to this result as the Fatou–Zygmund theorem. Note that the hypothesis (2) is equivalent to the condition
\[ \sup_{n \to \infty} s_n^r(x) < \infty, \]
where \( r^t = \max(t,0) \) for every real number \( t \). The version (3) is more convenient than (2), and we will use it henceforth.

The Fatou–Zygmund theorem goes back to Fatou, who in [8], p. 397, announced without proof the result for \( g > 2 \), \( \Re(c_0) = 0 \), and the variant hypothesis that \( s_n(x) \) converges for all \( x \) in some nonvoid open interval. The full theorem is due to Zygmund [16]. A proof appears in Zygmund [17], Vol. I, p. 247, Th. (8.3). The Fatou–Zygmund theorem has been extended to a much wider class of lacunary sets \( \{n_k\}_{k=0}^{\infty} \) by Gapolkin [7]. An analogous but apparently not identical property has been studied for connected compact Abelian groups by Déchamps–Gondim [1]. We will discuss the contributions of these writers at appropriate places infra.

1.2. Mise en scène. Our aim is to extend the Fatou–Zygmund theorem, or more properly, to study the lacunarity property embodied in it, for sets of characters of compact Abelian groups. Let \( G \) be a compact infinite Abelian group with character group \( X \). Let \( P \) be a symmetric subset of \( X \), and let \( U \) be a certain set of complex-valued functions \( u \) on \( P \) which are Hermitian in the sense that \( u(x^{-1}) = u(x) \) for all \( x \in P \). With every \( u \) we may associate the formal "trigonometric series":
\[ \sum_{n=0}^{\infty} u(n) \exp(i\theta_n x). \]
Suppose that we are given a method of assigning to each \( u \in U \) a sequence \( (\mu_0, u)_0 \) of real-valued finite linear combinations of \( \chi_n \) in \( P \) that may serve as partial sums in some reasonable sense for the series (1). (For the classical case, \( U \) consists of all Hermitian functions on \( \{n_k\}_{k=0}^{\infty} \) and the functions \( s_n u \) are the symmetric partial sums 1.1.1.1. As we shall see, many other possibilities present themselves.) We ask the following question. What sort of lacunarity for the set \( P \) is expressed by the requirement that
\[ \sum_{n=0}^{\infty} |u(n)| < \infty. \]
for every \( u \in U \) for which the functions \( s_n^r u \) are bounded in some preassigned sense? That is, we turn the conclusion of the Fatou–Zygmund theorem into a definition of a lacunarity property of \( P \). Plainly the possibilities at this stage are very wide, since we have left open the definitions of \( U \), of \( s_n u \), and of boundedness of \( s_n^r u \). We will call the property of \( P \) expressed by this assumption a generalised Fatou–Zygmund property. The precise nature of this property obviously depends upon our choices of \( U \), of the convergence or summability method defining \( s_n u \), and of the definition of boundedness of the functions \( s_n^r u \).

The Fatou–Zygmund theorem suggests that at least some variants of the generalised Fatou–Zygmund property of \( P \) may be related to Sidonity of \( P \). Our reasoning here is tenuous at best: all we have to go on is the fact that sets with Hadamard gaps are Sidon sets and also have the Fatou–Zygmund property. We investigate this connection from a functional analytic point of view. We will express some generalised Fatou–Zygmund properties in terms of the possibility of matching more or less arbitrary bounded Hermitian functions on \( P \) by Fourier–Stieltjes transforms of nonnegative real-valued measures on \( G \) having restricted supports (analogous to the corresponding well-known characterisation of Sidon sets; see, for example [(8) 37.2.2].)

1.3. Conventions. All notation and terminology not explained here are as in [3] and [8]. We will adhere throughout to the following notation. The symbol \( G \) will denote a compact Abelian Hausdorff group and \( X \) will denote its character group. Normalised Haar measure on \( G \) will be denoted by \( \lambda \). For \( 0 < p < \infty \), \( L^p(G) \) is the usual Lebesgue space of \( p \)th power integrable functions on \( G \) with respect to \( \lambda \). The symbol
$C(G)$ denotes the space of all complex-valued continuous functions on $G$. The symbol $\mathcal{Z}(G)$ denotes the linear space of all trigonometric polynomials $\sum a_n x^n$ on $G$. The symbol $\mathfrak{A}(\mathcal{Z}(G))$ denotes the subspace of $C(G)$ consisting of all $f$ having the form $\sum a_n x^n$ where $\sum a_n = \|f\|$ is finite.

The symbol $M(G)$ denotes the space of all complex Radon measures on $G$, defined as in [8], §14. For a subset $S$ of $G$, the symbol $M(S)$ denotes the set of all $\mu \in M(G)$ such that $\text{Supp} |\mu| \subset S$. The symbols $M_n(S)$ and $M_n(S)$ denote respectively the sets of real-valued and nonnegative real-valued measures in $M(S)$.

For a complex-valued function $f$ on any group $G$, $f^*$ denotes the function $f^*(x) = \bar{f}(x^{-1})$. For a set $E$ of complex-valued functions, the symbols $E_1$ and $E_2$ denote respectively the sets of real-valued and nonnegative real-valued functions in $E$. The symbol $E_n$ denotes the set of all $f \in E$ such that $f^* = f$.

The mappings $f \to f^*$ and $\mu \to \mu^*$ are the Fourier and Fourier-Stieltjes transforms, defined on $\mathfrak{A}(G)$ and $M(G)$, respectively.

If $E$ is a subset of $\mathfrak{A}(G)$ or $M(G)$ and if $Y$ is a subset of $X$, then $E_Y$ will denote the set of $f \in E$ such that $f^*(y) = 0$ for $x \not\in X \setminus Y$.

§ 2. Some abstract lemmas. We set down here some needed lemmas from functional analysis.

2.1. Definition. Let $E$ be a real linear space. Let $\Phi(E)$ denote the set of all functions $\tau: E \to [0, \infty]$ such that
\[ \tau(x+y) \leq \tau(x) + \tau(y) \quad \text{and} \quad \tau(ax) = a \tau(x) \]
for all $x, y \in E$ and $a \in [0, \infty]$. (We adopt the usual conventions concerning $\infty$; in particular, the product $0 \cdot \infty$ is taken to be $0$.) If $\tau \in \Phi(E)$ and $\tau(-x) = \tau(x)$ for all $x \in E$, $\tau$ is called symmetric. If $E$ is a topological real linear space, we define $\Phi_0(E)$ as the set $\{\tau \in \Phi(E): \tau$ is lower semicontinuous on $E\}$.

2.2. Remarks. We list without proof some simple facts.
(a) A function $\tau \in \Phi(E)$ belongs to $\Phi_0(E)$ if and only if the set
\[ \{x \in E: \tau(x) < 1\} \]
is closed in $E$.

(b) If $\tau$, $\tau \in \Phi_0(E)$ (or $\Phi(E)$), then $\tau + \tau'$ belongs to $\Phi_0(E)$ (or $\Phi(E)$).

c. If $\mathcal{Y}$ is a nonvoid subset of $\Phi_0(E)$, then the function $\tau = \sup \{\tau: \tau \in \mathcal{Y}\}$ belongs to $\Phi_0(E)$.

(d) If $x \in \Phi_0(E)$, then the function $x \to \tau(x) = \tau(x^a)$ also belongs to $\Phi_0(E)$.

Our first lemma is simple and surprisingly useful.

2.3. Lemma. Let $E$ be a complete, first countable, locally convex topological linear space (not necessarily satisfying any separation axiom). Let $\varphi$ and $\tau$ be elements of $\Phi_0(E)$. Suppose that

(i) $x \in E$ and $\tau(x) \neq \infty$ imply that $\varphi(x) \neq \infty$ and $\varphi(-x) \neq \infty$.

Then there exist a positive real number $\kappa$ and a continuous seminorm $\sigma$ on $E$ such that

(ii) $\varphi(x) \leq \kappa \sigma(x)$ for every $x \in E$.

If $E$ is a Banach space, then the seminorm $\sigma$ in (ii) can be taken as a multiple of the norm in $E$.

Proof. This is, in essence, a straightforward argument. Take continuous seminorms $\sigma_1$, $\sigma_2$, $\sigma_3$, ..., such that $\sigma_1 \leq \sigma_2 \leq \sigma_3$ and such that the sets $\sigma_i^{-1}(0, 1]$ form a base at 0 in $E$. Introduce the associated seminorm
\[ d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \sigma_n(x-y)/(1 + \sigma_n(x-y)). \]

Plainly $E$ is complete in the semimetric $d$, since it defines the uniformity of $E$.

Consider the set $B = \{x \in E: \tau(x) \leq 1\}$ as a topological subspace of $E$. It is closed in $E$, since $\tau \in \Phi_0(E)$; and nonvoid since $\tau \in E$. The set $B$ is therefore a nonvoid complete semimetrizable space. By (b), the restriction $\varphi|B$ assumes only finite values. The function $\varphi|B$ is also lower semicontinuous, and so for every positive integer $m$, the set
\[ B_m = \{x \in B: \varphi(x) \leq m\} \]
is closed in the relative topology of $B$. Since $B = \bigcup_{m=1}^{\infty} B_m$, Baire's theorem entails that some $B_m$ has nonvoid interior in the relative topology of $B$ That is, there exist $x_0 \in B$ and a positive real number $\kappa$ such that

(i) $x \in B$ and $d(x, x_0) < \kappa$ imply that $\varphi(x) \leq m$.

Now choose a real number $a$ such that $0 < a < 1$ and so small that $d((1-a)x_0, x_0) < \frac{\kappa}{4}$. We write $(1-a)x_0 = x_1$. From (1) we see that

(ii) $x \in B$ and $d(x, x_1) < \frac{\kappa}{4}$ imply that $\varphi(x) \leq m$.

Since $1-a > 0$, we also have

(iii) $\varphi(x_1) = (1-a)\varphi(x_0) \leq 1-a$.

From (3) and (i) we have $\varphi(-x_1) < \infty$. Now write $m_1 = m + \varphi(-x_1)$. Choose $b$ and $\kappa$ so that $0 < b \leq a$ and

(iv) $y \in E$ and $\sigma_1(y) < b$ imply that $d(y, 0) < \frac{\kappa}{4}$.
If $E$ is a Banach space, each $a_n$ can trivially be taken as a multiple of the norm in $E$. By (2), (4), and (3) it may be seen that
\[ \varphi(x+y) \leq m, \]
whenever $y \leq b$, and $r(y) \leq a$. Thus we have proved:

(5) \( y \leq \mathcal{E}, \quad \alpha_n(y) \leq b, \quad \text{and} \quad r(y) \leq a. \) Then (5) shows that

(6) \( x \in \mathcal{E}, \quad \alpha(x) \leq 1, \quad \text{and} \quad r(x) \leq 1. \) The relation (6) is obviously equivalent to (ii).

We continue with two lemmas somewhat like lemmas given by Kahane and Salem ([11], p. 141) and Kahane ([10], p. 106), but different enough to warrant in our opinion separate treatment.

2.4. Iteration Lemma. Let $E$ be a topological linear space, and $B$ a bounded, convex, sequentially complete subset of $E$ such that $0 \in B$. Let $F$ be a normed linear space and $T$ a linear map of $E$ into $F$ whose graph is closed in $E \times F$. Let $F_0$ denote the closed unit ball in $F$. Suppose that there is a real number $a$ such that $0 < a < 1$ and such that for every $y \in F_0$, there exists an $x \in \mathcal{E}$ such that

(i) \[ \| y - Tx \| \leq a. \]

Then the inclusion

(ii) \[ F_1 \subset (1 - a)^{-1} T(B) \]

holds.

Proof. The proof is by "iteration". Given $y \in F_0$, choose $x_0 \in B$ such that

\[ \| y - Tx_0 \| \leq a. \]

We proceed by induction. Suppose that $x_0, x_1, \ldots, x_n$ in $B$ have been chosen so that

\[ \| y - T \left( \sum_{j=0}^{n} a_j x_j \right) \| \leq a^{n+1}. \]

Then the element $y_n = a^{n+1} \left( y - T \left( \sum_{j=0}^{n} a_j x_j \right) \right)$ belongs to $F_1$, and by hypothesis we can choose $a_{n+1} \in B$ so that $\| y_n - Ta_{n+1} \| \leq a$. Write $w_n = - a_{n+1}$. Then we have

(1) \[ \| y - Tw_n \| \leq a^{n+1}. \]

for $n \in \{ 0, 1, 2, 3, \ldots \}$. Since $B$ is convex and contains 0, we have

\[ \omega_n = (1 - a)^{-1} B. \]

For $0 \leq m < n$, the same properties of $B$ imply that

(3) \[ \omega_n - \omega_m = (1 - a)^{-1} \left( \omega^{m+1} - \omega^{m+1} \right) B = (1 - a)^{-1} a^{m+1} B. \]

Since $B$ is bounded, (2) implies that the sequence $\omega_n$ is a Cauchy sequence in $B$. Let $w$ be its limit, which belongs to $B$. Then $w = (1 - a)^{-1} w$ belongs to $(1 - a)^{-1} B$. By (1), we see that $\lim \omega_n = w$. Since $T$ is a closed mapping, we infer that $T \omega_n = w$. Thus (ii) holds.

2.5. Lemma. Let $E$, $F$, $F_1$, and $T$ be as in Lemma 2.4, with the added hypothesis that $F$ be a Banach space. Let $(A_n)_{n=1}^\infty$ be a sequence of subsets of $E$ satisfying the following conditions:

(i) \[ \text{For all } m \text{ and } n, \text{ the set } A_m + A_n \text{ is contained in a bounded, convex, sequentially complete subset } B_{m,n} \text{ of } E \text{ that contains } 0. \]

(ii) \[ \text{The equality } T \left( \bigcup_{n=1}^\infty A_n \right) = F \text{ obtains.} \]

Then there exist positive integers $m_n$ and $n_k$ such that

(iii) \[ F_1 \subset \bigcup_{k=1}^\infty (T B_{m_k,n_k}). \]

Proof. Since $F = \bigcup_{n=1}^\infty T(A_n) = \bigcup_{n=1}^\infty (T(A_n))^{-1}$, Baire's theorem implies the existence of a positive integer $m_n$ for which $(T(A_n))^{-1}$ has nonvoid interior in $F$. That is, there exist a positive integer $r$ and an element $y_\ast \in F$ such that

(1) \[ y \neq T \quad \text{and} \quad \| y - y_\ast \| < r^{-1} \quad \text{implies} \quad y \notin (T(A_n))^{-1}. \]

There is also a positive integer $n_k$ such that

(2) \[ - y_\ast \notin (T(A_n)). \]

Thus for $y \in F$ such that $\| y \| < r^{-1}$, we have

(3) \[ y - y_\ast \notin (T(A_n))^{-1} \quad \text{and} \quad (T(A_n))^{-1} = (T(A_n) + A_n)^{-1} = (T(A_n) + A_n)^{-1} = (T(B_{m_k,n_k})). \]

In the last line of (3), $B_{m_k,n_k}$ is as in (i). From (3) we see that

(4) \[ F_1 \subset \bigcup_{k=1}^\infty (T B_{m_k,n_k}). \]

The set $r B_{m_k,n_k}$ is bounded, convex, and sequentially complete, and contains 0. From (4) we see that (i) of Lemma 2.4 holds with $B = r B_{m_k,n_k}$ for all positive real numbers $a$, and so (ii) of Lemma 2.4 holds for all $a \neq 0, 1$. Therefore (iii) holds for any integer $r > 0$. \[ \]
§ 3. Generalised Fatou-Zygmund properties. In this section we establish the notation and terminology for the remainder of the paper.

3.1. Standing conventions. We shall select a sequence \((h_n)_{n=1}^\infty\) of elements of \(\mathcal{R}(G)\), the role of which is to generate convergence or summability methods for formal trigonometric series on \(G\). Further specification is left until 3.4 and 3.7.

In any case we shall write

\[ X_n = \{ x \in X : \lim_{n \to \infty} h_n(x) = 1 \} \]

since every \(h_n\) is real valued, \(X_n\) is a symmetric subset of \(X\). Each set \(\{ x \in X : h_n(x) = 0 \}\) is countable and so \(X_n\) is also a countable subset of \(X\).

Our Fatou-Zygmund properties are studied for certain subsets \(P\) of \(X\). Except where the contrary is explicitly indicated, we suppose that

(i) \(P\) is a symmetric subset of \(X\).

The symbol \(\mathcal{F}(P)\) denotes the space of all complex-valued functions defined on \(P\), \(\mathcal{F}(P)\) the space of all bounded functions in \(\mathcal{F}(P)\), and \(\mathcal{C}(P)\) the subspace of \(\mathcal{F}(P)\) consisting of all functions that are arbitrarily small in absolute value outside of appropriately chosen finite subsets of \(P\).

Let \(\text{Top}\) denote the topology of pointwise convergence in the real linear space \(\mathcal{F}(P)\); see 1.3 for the meaning of the suffix “\(\text{Top}\)”.

It is obvious that \(\mathcal{F}(P)\) is a Fréchet space.

3.2. The function spaces \(\mathcal{H}\). We will examine Fatou-Zygmund properties for \(\mathcal{H}\) based on certain subspaces \(\mathcal{H}\) of \(\mathcal{F}(P)\). We will suppose that \(\mathcal{H}\) is a (real) linear subspace of \(\mathcal{F}(P)\) with the following properties:

(i) \(\mathcal{H}\) is a \(\text{Top}\) space under some topology;

(ii) the topology of \(\mathcal{H}\) is equal to or stronger than \(\text{Top}|\mathcal{H}\); and

(iv) there is a basis \((\mathcal{H}_a)_{a \in A}\) of \(\mathcal{H}\) such that \(\mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots \supset \mathcal{H}_k \supset \ldots\), each \(\mathcal{H}_k\) is convex and balanced, and if \(u, v \in \mathcal{H}_k\), then \(u + v \in \mathcal{H}_k\).

Since \(\mathcal{H}\) is a Fréchet space, its topology can be described by a sequence \((\sigma_n)_{n=1}^\infty\) of seminorms, where \(\sigma_n\) is the Minkowski gauge of \(\mathcal{H}_n\).

\[ \sigma_n(u) = \inf \{ \alpha : \alpha > 0, \frac{1}{\alpha} \leq u \in \mathcal{H}_n \} \]

From this definition it is clear that (iv) is equivalent to:

(v) the topology of \(\mathcal{H}\) is defined by an increasing sequence \((\sigma_n)_{n=1}^\infty\) of seminorms that are monotone in the sense that \(\sigma_n(u) \leq \sigma_n(v)\) whenever \(u, v \in \mathcal{H}\) and \(|u| \leq |v|\).

Finally we suppose that

(vi) for every positive integer \(n\) and every function \(u \in \mathcal{H}\), we have

\[ \sum_{P} |h_n(x)u(x)| < \infty \]

In view of (vi) we may define

\[ s_n u = \sum_{P} h_n(x)u(x) \]

for every \(u\) and every \(n \in \mathcal{H}\). Each \(s_n u\) is trivially an element of \(\mathcal{R}\).

In our study of Fatou-Zygmund properties, we examine not the sums \(s_n u\) but their nonnegative parts \(s_n^+ u = (s_n u)^+ = \max(s_n u, 0)\). These functions are in \(\mathcal{C}_+(G)\) but not in general in \(\mathcal{C}(G)\).

3.3. Lemma. Let \(W\) be any \(\lambda\)-measurable subset of \(G\), let \(u\) be any positive integer, and let \(p\) be in \([1, \infty]\). Then the mapping

(i) \[ u \to \|s_p s_n u\|_p \]

is a continuous seminorm on \(\mathcal{H}\) and the mapping

(ii) \[ u \to \|s_p s_n u\|_\mathfrak{H} \]

is a continuous gauge on \(\mathcal{H}\). Hence both of these mappings belong to \(\sigma_\alpha(\mathfrak{H})\).

Proof. By 3.2.(vi), the mapping

\[ u \to \sum_{P} |h_n^+ u(x)| = \|s_n u\|_\mathfrak{H} \]

is a seminorm on \(\mathcal{H}\). By 3.2. (iii), this seminorm is lower semicontinuous on \(\mathcal{H}\). By 3.2. (ii) and (iii) 6.3.3. and 7.2.1, this seminorm is necessarily continuous. Thus \(u \to s_n u\) is a continuous mapping of \(\mathcal{H}\) into \(\mathcal{C}(G)\). The lemma now follows from the inequalities

\[ \|s_p s_n u\|_p - \|s_p s_n u\|_p \leq \|s_n^+ u - v\|_p \leq \|s_n^+ u - v\|_p \leq \|s_n^+ u - v\|_p \]

and

\[ \|s_p s_n u\|_p - \|s_p s_n u\|_p \leq \max(\|s_p s_n u - v\|_p, \|s_p s_n v - u\|_p) \leq \|s_n^+ u - v\|_p \leq \|s_n^+ u - v\|_p \]
We obviously have \( X_n = \bigcup_{m=1}^{\infty} X_m \) in this case. By analogy with the standard Dirichlet kernel for \( G = \mathbb{Z}, \ X = \mathbb{Z}, \) and \( X_n = \{ m \in \mathbb{Z} : |m| \leq n \}, \) we will denote these particular functions by \( D_n \).

Now consider our symmetric subsets \( P \) of \( X \). We write \( P_n \) for the set \( X_n \cap P, \ n = 1, 2, \ldots \). For \( u \in \mathbb{N} \) and positive integers \( m \) and \( n, \) we have
\[
\sum_{m \in P_n} u(m) \chi
\]
and
\[
D_n u = \sum_{m \in P_n} u(m) \chi.
\]

3.5. Remark. We think of \( P \)-spectral trigonometric series
\[
\sum_{m \in P \chi} c(x) \chi
\]
as special instances of general trigonometric series
\[
\sum_{m \in P \chi} c(x) \chi,
\]
namely those for which \( c(x) = 0 \) for \( x \in X \setminus P \). Likewise, we seek to arrange matters so that the partial sums \( s_n u \) arise from applying to (1) a procedure for forming partial sums which is natural for general series (2). Thus, although it would be possible to arrange that
\[
s_n u = \sum_{m \in P_n \chi} u(m) \chi
\]
for any increasing sequence \( (P_n)_{n=1}^{\infty} \) of finite symmetric subsets of \( P \) with union equal to \( P, \) we have elected in Case A to arrange that \( P_n \) is in fact chosen to be \( X_n \cap P, \) where \( J = (X_n)_{n=1}^{\infty} \) is a sequence independent of \( P \) yielding sensible partial sums for any trigonometric series (2) for which \( c \) vanishes off \( X_n \). Normally, one will try to make \( X_n \) as "fat" as possible.

We can make \( X_n = X \) if and only if \( G \) is first countable, i.e., metrizable. In the contrary case, sequences would have to be replaced by nets, which brings complications of their own.

3.6. Definition. Let \( \mathcal{P} \) denote the family of all nonempty \( \lambda \)-measurable subsets \( W \) of \( G \) such that \( W = \{ \text{int}(W) \} \).

Note that for \( g \in G \) and \( W \in \mathcal{P}, \) we have
\[
\|W g\|_{\infty} = \|W g\|_{1} = \|W - g\|_{1}.
\]

3.7. Case B: Summability factors. In this case the \( h_n \) are subjected to different conditions, as follows:
(i) each \( h_n \) is in \( \mathcal{K}(G) \) and \( M = \sup_{n \geq 1} \|h_n\|_1 < \infty \);
(ii) there is a sequence \( (W_n)_{n=1}^{\infty} \) of (not necessarily open) symmetric neighborhoods of \( e \) in \( G \) such that \( W_n e \cap P \) and \( W_{n+1} e \subseteq W_n e \) for every positive integer \( r \), and to every such \( r \) there corresponds a positive integer \( n_r \) such that
\[
W_{n_r} e \cap \text{Supp} h_n = W_{n_r} e \subseteq W_{n_r-1}
\]
for every \( n \geq n_r \) (\( W_0 \) is understood to be 0).

3.8. Remarks. (a) In Case 3.7 we admit the possibility that all \( W_r \) are equal; they may for example all be equal to \( G \).
(b) If the torsion subgroup of \( G \) is finite, then we cannot satisfy 3.7. (i) with \( h_n = D_n \) except in the trivial case that \( X_n \) is finite. This was proved by Hewitt and Zuckerman [9]. Note also that \( D_n \) cannot be nonnegative for all \( n \). For if \( D_n \geq 0 \), we have
\[
\|D_n\|_{1} = D_n (1),
\]
which is 1 or 0 according as the character 1 is in \( X_n \) or is not in \( X_n \), and by Hewitt and Zuckerman, loc. cit., we have \( \lim_{n \to \infty} \|D_n\|_{1} = \infty \).

(c) The case of connected \( G \) deserves special mention. For an arbitrary \( G, \) let \( f \) be a trigonometric polynomial on \( G \) that vanishes on a nonvoid open set \( U \). Then \( f \) vanishes on each connected component of \( G \) that intersects \( U. \) (Let \( C \) denote the connected component of \( G \) containing \( e \), and suppose that \( x \in U \). Let \( g \) be any continuous homomorphism of \( B \) into \( G \). Then \( f g \) is a trigonometric polynomial on \( B \) vanishing in an interval about 0. It follows that \( f g \) vanishes identically on \( B \), and so \( f \) vanishes identically on \( \gamma(B) \). The union of all subgroups \( \gamma(B) \) (\( B \) is dense in \( C \) (see for example [8], [35, 30]). Thus \( f \) vanishes throughout \( C \) and so \( f \) vanishes throughout \( \gamma(C) \). If \( G \) itself is connected, the kernels \( D_n \) must therefore have support equal to \( G \), and 3.7. (ii) holds if and only if \( W_n = G \) for all \( n \).

(d) Remarks (b) and (c) explain why Cases A and B demand separate treatments.

(e) As was adumbrated in 3.1, the \( h_n \) are to play the role of summability kernels of the type frequently used in connection with trigonometric series. A given sequence of summability kernels \( (h_n)_{n=1}^{\infty} \), may or may not satisfy the conditions laid down in 3.7 for a given choice of the \( W_n \). Although 3.7. (i) and positivity of \( h_n \) are natural enough, the inclusions 3.7. (ii) fail for many familiar kernels if \( (W_n)_{n=1}^{\infty} \) collapses to \( e \).
We now establish the converse of Theorem 4.2 for the special functions $h_n = D_n$ of Case A.

4.3. Theorem. Notation and hypotheses are as in 4.1 and 4.2 with the restriction that $h_n = D_n$ for all $n$, as in 3.2. Suppose that there exist a continuous seminorm $\sigma$ on $\mathcal{H}$ and a positive real constant $\kappa$ such that 4.2. (iii) holds with $n_0 = 1$ for all $f \in \mathcal{H}_p(G)$. Then if $u \in \mathcal{H}$ and 4.2. (ii) holds, 4.2. (i) holds as well.

Proof. Let $u$ be any element of $\mathcal{H}$ for which the left side of 4.2. (ii) is finite: write this number as $L$. The function $s_n u$ belongs to $\mathcal{H}_p(G)$. By 3.4. (1) and 3.4. (2), we have $(s_n u)^* = s_n u$ and $h_n(s_n u) = s_{n_0}(s_n u)$ for all positive integers $n$. Applying 4.2. (iii) with $n_0 = 1$, we find that

$$
\Phi(s_n u) = \max\{\sigma(s_n u), \kappa \sup_{\alpha < \beta} (s_n u, \alpha L)\} 
$$

for every $f \in \mathcal{H}_p(G)$.

Proof. Suppose that (i) holds for every $u \in \mathcal{H}$ satisfying (ii). For a given positive integer $n$, we define $\tau$ on the linear space $\mathcal{H}$ by

$$
\tau(u) = \sup_{\alpha > \beta} (s_n u, \alpha L).
$$

Plainly $\tau$ is in $\Phi(\mathcal{H})$, and from 3.3 and 2.2. (2) we see that $\tau$ is actually in $\Phi(\mathcal{H})$. Since $\Phi$ is lower semicontinuous for $\mathcal{H}_p(G)$, it is lower semicontinuous on $\mathcal{H}$ (see 3.2. (iii)), i.e., in $\Phi(\mathcal{H})$. We now apply Lemma 2.3. If $\tau(u) < \infty$ for a given $u \in \mathcal{H}$, then (ii) holds and (i) holds. Our present hypotheses thus imply the hypothesis of Lemma 2.3, and the conclusion 2.3. (ii) becomes

$$
\Phi(u) = \max\{\sigma(u), \kappa \sup_{\alpha > \beta} (s_n u, \alpha L)\}.
$$

Now let $f$ be any function in $\mathcal{H}_p(G)$. The restriction $f'$ of $f^*$ to $P$ belongs to the function space $\mathcal{H}$, by 3.2. (i). Using the identity $s_n f^* = h_n s_n f$, we obtain (iii) at once from (1).
Proof. For all \( r \geq 1 \), we define \( u_0 - u_t(r) \) as in 3.7. (ii). Our hypotheses obviously imply that \( \psi(u) < \infty \) if
\[
\sup_{n=1} \| \xi P_{n} u_t^* \|_p < \infty,
\]
and so by Theorem 4.3, there exist a positive real constant \( c_r \) and a continuous seminorm \( c_r' \) on \( U \) such that
\[
\psi(f') \leq \max \{ c_r(f') \}, \quad c_r' \sup_{n=1} \\sup_{n=1} \| \xi P_{n} u_t^* \|_p \leq \infty
\]
for every \( f \in U \). To obtain (iii) from (1), we need to majorize the supremum in (1) by a constant multiple of \( \| \xi P_{n} f \|_p \). To accomplish this, we must use the special hypotheses of Case B.

Since \( u_t \) is nonnegative, its convolution \( u_t * w \) with any function \( w \in \mathcal{S}(U) \) is nonnegative. From this a simple argument (which we omit) shows that
\[
(u_t * w)^+ \leq u_t * (w^+)
\]
for all \( w \in \mathcal{S}(U) \). Thus for \( f' \in U \) we obtain
\[
\xi P_{n} (u_t * f)^+ \leq \xi P_{n} (u_t * f')^+
\]
(2).

Now let \( g \) be any function in \( \mathcal{S}(U) \) (we agree as usual that \( 1' = \infty \) and that \( \alpha' = 1 \)). For typographical convenience, write \( \delta \) for the function \( x \to \delta(x^{-1}) \). It is clear that \( \sup \delta = \infty \in \mathcal{W}^* \).

Let \( n \) be any integer greater than or equal to \( n_0 \). Now using (3), (elementary properties of convolution), and 3.7. (ii), we write
\[
\int \xi P_{n} (u_t * f')^+ d\xi \leq \int \xi P_{n} (u_t * f')^+ d\xi \leq \int \xi P_{n} (u_t * f')^+ d\xi
\]
for all \( w \in \mathcal{S}(U) \). Thus for \( f' \in \mathbb{U} \) we obtain
\[
\xi P_{n} (u_t * f')^+ \leq \xi P_{n} (u_t * f')^+
\]
(2).

Now let \( g \) be any function in \( \mathcal{S}(U) \) (we agree as usual that \( 1' = \infty \) and that \( \alpha' = 1 \)). For typographical convenience, write \( \delta \) for the function \( x \to \delta(x^{-1}) \). It is clear that \( \sup \delta = \infty \in \mathcal{W}^* \).

Let \( n \) be any integer greater than or equal to \( n_0 \). Now using (3), (elementary properties of convolution), and 3.7. (ii), we obtain
\[
\int \xi P_{n} (u_t * f')^+ d\xi \leq \int \xi P_{n} (u_t * f')^+ d\xi \leq \int \xi P_{n} (u_t * f')^+ d\xi
\]
for all \( w \in \mathcal{S}(U) \). Thus for \( f' \in \mathbb{U} \) we obtain
\[
\xi P_{n} (u_t * f')^+ \leq \xi P_{n} (u_t * f')^+
\]
(2).

Now let \( g \) be any function in \( \mathcal{S}(U) \) (we agree as usual that \( 1' = \infty \) and that \( \alpha' = 1 \)). For typographical convenience, write \( \delta \) for the function \( x \to \delta(x^{-1}) \). It is clear that \( \sup \delta = \infty \in \mathcal{W}^* \).

Let \( n \) be any integer greater than or equal to \( n_0 \). Now using (3), (elementary properties of convolution), and 3.7. (ii), we obtain
\[
\int \xi P_{n} (u_t * f')^+ d\xi \leq \int \xi P_{n} (u_t * f')^+ d\xi \leq \int \xi P_{n} (u_t * f')^+ d\xi
\]
for all \( w \in \mathcal{S}(U) \). Thus for \( f' \in \mathbb{U} \) we obtain
\[
\xi P_{n} (u_t * f')^+ \leq \xi P_{n} (u_t * f')^+
\]
(2).
Since
\[ |h_u| \leq |h_0| \leq M, \]
we have \( |h_0| \leq M \) and by the monotonicity of \( \sigma \) and (6) we find
\[ \varphi(h_0) \leq \max \{ M \cdot \sigma(u), \sigma(L) \} < \infty. \]

By 3.1 (i), we have
\[ \lim_{n \to \infty} \lambda_n^*(x) = u(x) \quad \text{for all } x \in P. \]
Again the lower semicontinuity of \( \varphi \) for \( \mathbb{T}/H \) implies that
\[ \varphi(u) = \liminf_{n \to \infty} \varphi(h_0), \]
and so (6) guarantees that \( \varphi(u) < \infty. \]

§ 5. Examples of case A. The theorems of § 4 are an abstract formulation of a large variety of theorems of Fatou-Zygmund type, which we obtain by special choices of \( (h_0, 0, \varphi, p, W, \{ W_i \}_{i=1}^\infty) \). We consider here and in Section 6 some particular cases.

5.1. Let \( \mathcal{S} = (X_n)_{n=1}^\infty \) and \( (D_n)_{n=1}^\infty \) be as in 3.4, and let \( P \) be a symmetric subset of \( X_n = \bigcup X_n \). We take \( H \) to be the entire space \( \mathbb{B}(P) \), with the topology \( \mathbb{T}/P \) of pointwise convergence, \( \sigma \) to be the function
\[ \sigma(u) = \sum_{z \in P} |u(x)| = |u|_{1}, \]
and \( \mathcal{S} \) to be as in 3.6; we also set \( \sigma = \infty \).

We note first that the continuous seminorms \( \sigma \) on \( H \) are exactly those majorised by some seminorm
\[ u \rightarrow \text{Const} \cdot \max_{z \in P} |u(x)|, \]
where \( P \) is a finite subset of \( P \). The proof is simple and is omitted.

5.3. Definition. Given a set \( W \) in \( \mathcal{S} \), we denote by \( FZ(P, \mathcal{S}, W) \) the following statement: if \( u = \mathbb{B}(P) \) and
\[ \sup_{n \geq 1} |u_x|_{W, n} < \infty, \]
then
\[ u \in F(P). \]
If \( FZ(P, \mathcal{S}, W) \) holds, we say that \( P \) has the \( FZ(\mathcal{S}, W) \) property and that \( P \) is an \( FZ(\mathcal{S}, W) \)-set. Plainly \( FZ(P, \mathcal{S}, W) \) expresses a special Fatou-Zygmund property of the type described in 3.9.

5.3. Definition. If \( P \) has the \( FZ(\mathcal{S}, W) \) property for every \( W \) in \( \mathcal{S} \), then we say that \( P \) has the full \( FZ(\mathcal{S}, W) \) property and that \( P \) is a full \( FZ(\mathcal{S}, W) \)-set.

In other words, \( P \) is a full \( FZ(\mathcal{S}, W) \)-set if every \( u \) that satisfies 5.3 (i) for some \( W \) in \( \mathcal{S} \) belongs to \( F(P) \).

We will show in 8.8 that the \( FZ(\mathcal{S}, W) \) properties and the full \( FZ(\mathcal{S}, W) \) property do not depend on the choice of \( \mathcal{S} \), and so after 8.8 we will refer simply to the \( FZ(W) \) property, to \( FZ(W) \)-sets, etc. See 8.9.

5.4. The original Fatou-Zygmund theorem deals with the circle group \( T \) and its character group \( Z \) with \( X_n = \{ 2 \pi n : n \in \mathbb{Z} \} \) and \( \mathcal{S} = (X_n)_{n=1}^\infty \). The Fatou-Zygmund theorem asserts that any symmetric Hadamard set in \( Z \) possesses the full \( FZ(\mathcal{S}) \) property.

Gapoškin [7] has extended the Fatou-Zygmund theorem to a large class of sets, again in the group \( Z \). These are the sets discovered by Ščekin [15]; see 10.2 and 10.4.

For the special situation described in 5.1, we can state Theorems 4.2 and 4.3 as follows.

5.5. Condition. Notation is as in 5.1; \( n_0 \) is a positive integer and there exist a finite subset \( \Sigma \) of \( P \) and a positive real number \( \alpha \) (both perhaps depending upon \( n_0 \)) such that the inequality
\[ \| f \| \leq \max_{z \in \Sigma} |f(z)|, \sup_{\Sigma} |D_n(f)*f|_{n_0} \]
obtains for all \( f \in \mathcal{X}_P(\mathcal{S}) \).

5.6. Theorem. The set \( P \) has the \( FZ(\mathcal{S}, W) \) property if and only if Condition 5.5 holds for all positive integers \( n_0 \). If 5.5 holds for some \( n_0 \), then \( P \) has the \( FZ(\mathcal{S}, W) \) property.

5.7. Corollary. If \( P \) has the \( FZ(\mathcal{S}, W) \) property, then \( P \) is a Sidon set.

Proof. From 5.6 it follows that 5.5 (i) holds for every \( f \in \mathcal{X}_P(\mathcal{S}) \).

For \( f \in \mathcal{X}_P(\mathcal{S}) \), we have \( \| f \|_{n_0} \leq \| f \|_{n_0} + D_{n_0} f = \| f \|_{n_0} + D_{n_0} f \) and so
\[ \| f \|_{n_0} \leq \max_{z \in \Sigma} |f(z)|, \sup_{\Sigma} |D_n(f)*f|_{n_0} \]
for every \( f \in \mathcal{X}_P(\mathcal{S}) \). If \( f \in \mathcal{X}_P(\mathcal{S}) \) we may, since \( P \) is symmetric, apply (2) to each of \( \text{Re} f \) and \( \text{Im} f \) and so conclude that
\[ \| f \|_1 \leq 2 \sup_{\Sigma} \| \text{Re} f \|_1 \]
for every \( f \in \mathcal{X}_P(\mathcal{S}) \). Applying 3.9 of [5] (with \( w = 0 \), \( p = 2 \) and every \( c_i \) equal to the unit mass at \( e_i \)), we see that \( P \) is a Sidon set.
§ 6. Examples of case B.

6.1. We make specializations a little different from those in § 5. As in § 5, ϕ is still defined by $\mathcal{P}(\mathbb{R}) = \sum_{\mathbb{R}} |u|^2$ and again we take $p = \infty$.

The set $\mathcal{P}$ is again an arbitrary symmetric subset of $X_{rs}$, and the sequences $(\mathcal{W}, \mathcal{W}_{rs})$ and $(\ell_{rs})$ are as in 3.2. The functions $\mathfrak{e}$ are again as in 3.2. The space $\mathcal{H}$ this time is taken to be the space $\mathfrak{B}_0(\mathcal{P})$, topologised with the usual uniform or $\ell^m$ norm, denoted by $\|u\|_{\mathfrak{B}_0}$.

The continuous seminorms on $\mathcal{H}$ this time are exactly those seminorms on $\mathcal{U}$ that are majorised by a constant times $\|\cdot\|_{\mathfrak{B}_0}$.

We think of the implication “4.6. (ii) implies 4.6. (i)” in this case as a generalised Fatou–Zygmund property. If $\mathcal{W}$ is a set in $\mathcal{F}$, $\mathcal{G}Z(\mathcal{P}, \ell_{rs})$ denotes the following statement: 5.3. (i) implies 5.2. (ii) for all functions $u$ in $\mathfrak{B}_0(\mathcal{P})$. This is another specialisation of the ideas in 3.9.

The symbol $\mathcal{G}Z(\mathcal{P}, \ell_{rs})$ denotes the statement: if 5.2. (ii) holds for $\mathcal{W}$ equal to some $\mathcal{W}_{rs}$, then 5.2. (i) holds. We say that $(\mathcal{P}, \ell_{rs})$ has the $\mathcal{G}Z$-property if $\mathcal{G}Z(\mathcal{P}, \ell_{rs})$ holds.

The $\mathcal{G}Z$-property has no analogue, so far as we know, in classical Fourier analysis.

6.2. Consequence. Notation is as in 6.1. For every positive integer $r$, there exists a positive real number $\mathfrak{e}$ such that for all $f \in \mathcal{X}_r(\mathcal{G})$, we have

$\|f\|_m \leq \mathfrak{e} \max \{\|f\|_{\mathfrak{B}_0}, \|f\|_{\ell^m}\}$.

Theorems 4.6 and 4.7 can be stated as follows.

6.3. Theorem. The assertion $\mathcal{G}Z(\mathcal{P}, \ell_{rs})$ holds if and only if Condition 6.2 holds.

We note also the following version of Theorem 4.2.

6.4. Theorem. Suppose that $\mathcal{G}Z(\mathcal{P}, \ell_{rs})$ holds for some $\mathcal{W}$ in $\mathcal{F}$. Then for every positive real number $\mathfrak{e}$ there is a positive real number $\mathfrak{e}$ such that the inequality

$\|f\|_m \leq \mathfrak{e} \max \{\|f\|_{\mathfrak{B}_0}, \sup_{\mathfrak{e}} \|\hat{f}(\ell_{rs} f)^*\|\}$

holds for all $f \in \mathcal{X}_r(\mathcal{G})$.

6.5. Corollary. If $\mathcal{G}Z(\mathcal{P}, \ell_{rs})$ holds, then $\mathcal{P}$ is a Sidon set.

Proof. The hypothesis implies that 6.4 (i) holds for every $f \in \mathcal{X}_r(\mathcal{G})$. In view of 3.7. (i), it follows at once that

$\|f\|_m \leq \sup_{\mathfrak{e}} \|\hat{f}(\ell_{rs} f)^*\| \leq \mathcal{M}\|f\|_m$

for every $f \in \mathcal{X}_r(\mathcal{G})$. So, as in the proof of 5.7, we have

$\|f\|_m \leq 2\mathcal{M}\|f\|_m$

for every $f \in \mathcal{X}_r(\mathcal{G})$; this implies ([8], 37.2. vii) that $\mathcal{P}$ is a Sidon set.

§ 7. Matching properties and the $\mathcal{G}Z$-property. Sidonility of a subset $\mathcal{P}$ of $X$ can be expressed as a matching property: $\mathcal{P}$ is a Sidon set if and only if every bounded complex-valued function on $\mathcal{P}$ is matched on $\mathcal{P}$ by the Fourier–Stietjes transform $\mu$ of some (complex) measure $\mu$ in $M(G)$. A similar characterisation exists for $\mathcal{F}$-sets, with the refinement that the measures $\mu$ are in $M_1(G)$.

For the reader’s convenience, we repeat here a definition from [5], § 3.

7.1. Definition. Let $\mathcal{S}$ be a complex normed linear space with norm $\|\cdot\|_\mathcal{S}$ and $I$ any infinite index set. Let $f: I \rightarrow \mathcal{F}(\mathcal{S})$ be an element of $\mathcal{S}^I$.

Suppose that there is an element $\mathfrak{e}$ such that for every $\varepsilon > 0$ there is a finite subset $\mathcal{J}$ of $I$ with the property that

$\|f(i) - \mathfrak{e}_i\|_\mathcal{S} < \varepsilon$

for all $i \in I \setminus \mathcal{J}$. (Plainly $\mathfrak{e}_i$ is unique if it exists at all.) We then write

$\lim_{\mathcal{J}} f = \mathfrak{e}$,

$\mathfrak{e}_i$ denoting the constant function $i \mapsto \mathfrak{e}_i$ in $\mathcal{S}^I$. Let $c(I, \mathcal{S})$ denote the set of all $f \in \mathcal{S}^I$ for which $\lim f$ exists, and $c(I, \mathcal{S})$ the set of all $f \in c(I, \mathcal{S})$ for which $\lim f^* = 0$. Let $l^I(\mathcal{S})$ denote the set of all functions $f$ in $\mathcal{S}^I$ for which

$\|f\|_I = \sum_{i} \|f(i)\|_\mathcal{S} < \infty$.

7.2. Remarks. Plainly $f(I)$ is a bounded set in $\mathcal{S}$ for all $f \in l^I(\mathcal{S})$, and so we may define

$\|f\|_\infty = \sup_{\mathcal{S}} \|f(i)\|_\mathcal{S}$,

as in [5], § 3. It is trivial to check that $c(I, \mathcal{S})$ is a normed linear space under coordinatewise linear operations and the norm $f \mapsto \|f\|_\infty$. It is easy to check that $c(I, \mathcal{S})$ is a Banach space if $\mathcal{S}$ is a Banach space. Analogous remarks apply to $l^I(\mathcal{S})$.

We now describe the conjugate space of $c(I, \mathcal{S})$.

7.3. Lemma. Let $\mathcal{S}$ be as in 7.1 and $\mathcal{S}^I$ the space of all bounded linear functionals on $\mathcal{S}$. Let $L$ be a bounded linear functional on $c(I, \mathcal{S})$. Then there exist an element $\lambda$ of $l^I(\mathcal{S})$ and an element $\lambda_\mathcal{S}$ of $\mathcal{S}^I$ such that for all $f \in c(I, \mathcal{S})$, we have

$\text{(i)}$ $L(f) = \lambda_\mathcal{S}(\lim f) + \sum_{i} \lambda(i)(f(i))$.

Conversely, every function on $c(I, \mathcal{S})$ of the form (i) is a bounded linear functional. The norm of the functional $L$ is

$\|L\| = \|\lambda_\mathcal{S}\| + \|\lambda\|_I$. 
Next we introduce the real linear space

\[ Y = \mathcal{B}_b(P) \times \mathcal{C}(I, \mathcal{C}(W)) \]

where the index set \( I \) is \( \{1, 2, 3, \ldots \} \). For \( (w_1, (y_n)_{n=1}^{\infty}) \) in \( Y \), define the gauge \( p \) by

\[ p(w_1, (y_n)_{n=1}^{\infty}) = \max \{ \max |w(z)|, \sup_{n=1} \|y_n\| \} \]

Let us map \( \mathcal{P}_p(G) \) into \( Y \) by the mapping \( \gamma \), which we define as

\[ \gamma(f) = (\gamma(M/P, \|f\|_w) \sup_{n=1} \|w_n\|) \subseteq Y \]

The inequality (3) in our new notation asserts that

\[ \|f\|_{\gamma} \leq \varepsilon' p(\gamma(f)) \]

for all \( f \in \mathcal{P}_p(G) \).

We now claim that there is a real-valued real-linear functional \( I_\gamma \) on \( \gamma(\mathcal{P}_p(G)) \) such that

\[ I_\gamma(f) = I_\gamma(\gamma(f)) \]

for all \( f \in \mathcal{P}_p(G) \). In fact, if \( \gamma(f_1) = \gamma(f_2) \) for \( f_1, f_2 \in \mathcal{P}_p(G) \), (5) shows that

\[ \|f_1 - f_2\|_{\gamma} = \varepsilon' p(\gamma(f_1 - f_2)) = 0 \]

Accordingly \( I_\gamma \) is well defined on \( \gamma(\mathcal{P}_p(G)) \). Plainly \( I_\gamma \) is real valued and linear. The inequality (5) and the definition (6) show that

\[ I_\gamma(g) \leq \varepsilon' p(g) \]

for all \( g \in \gamma(\mathcal{P}_p(G)) \). By the Hahn–Banach theorem, we may extend \( I_\gamma \) to a linear functional on \( Y \) (which we will still write as \( I_\gamma \)) for which (7) holds for all \( y \in Y \).

Restricting \( I_\gamma \) to \( \mathcal{B}_b(P) \times \{0\} \subseteq Y \), we see from (7) and (4) that \( I_\gamma \) on this subspace is a real linear functional satisfying the inequality

\[ |I_\gamma(u_1, (0, y_n)_{n=1}^{\infty})| \leq \varepsilon' \max |w(z)| \]

for all \( (w_1, (0, y_n)_{n=1}^{\infty}) \in Y \). Since \( \Sigma \) is finite and symmetric, there accordingly exists a function \( \alpha \) in \( \mathcal{B}_b(P) \) such that (iii) holds,

\[ |\alpha|_{\Sigma} \leq \alpha_{\|w\|} \leq \alpha' \]

and

\[ I_\gamma(u, (0, y_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \alpha(z) u(z) \]

for all \( u \in \mathcal{B}_b(P) \).
Similarly, observe that the mapping
\[(\mu_n)_{n=1}^{\infty} \mapsto \lambda_0(0, (\mu_n)_{n=1}^{\infty})\]
is a bounded linear functional on $c(I, \mathcal{C}(W^*))$. Since the conjugate space of $\mathcal{C}(W^*)$ is $\mathcal{M}(W^*)$, we infer from Lemma 7.3 that there are measures $\mu_0$ and $\mu_1, \mu_2, \mu_3, \ldots, \mu_n, \ldots$ in $\mathcal{M}(W^*)$ such that
\[(10) \quad \lambda_0(0, (\mu_n)_{n=1}^{\infty}) = \int \int g_n d\mu_n = \sum_{n=1}^{\infty} \int g_n d\mu_n\]
for all $(g_n)_{n=1}^{\infty} \in c(I, \mathcal{C}(W^*))$. (Here $\mu_n$ is the uniform limit on $W^*$ of $(\mu_n)_{n=1}^{\infty}$.) Lemma 7.3 also combines with (7) to guarantee that
\[(11) \quad \|\mu_n\| + \sum_{n=1}^{\infty} \|\mu_n\| = \text{norm of } \lambda_0 \text{ on } c(I, \mathcal{C}(W^*)) \leq \kappa'.\]

Combining (10), (7), and (4), we obtain
\[(12) \quad \int \int g_n d\mu_n = \sum_{n=1}^{\infty} \int g_n d\mu_n \leq \kappa' \cdot \|g\|_\infty.\]

Given $g \in \mathcal{C}(W^*)$ and a positive integer $m$, first let $g_n = \delta_{mn} g$ ($n \in \{1, 2, 3, \ldots\}$). Putting this $(g_n)_{n=1}^{\infty}$ into (12), we find
\[
\int \int g d\mu_n \leq \kappa' \cdot \|g\|_\infty
\]
which implies that $\mu_n \in \mathcal{M}(W^*)$. Next let $g_n = 0$ if $n \not\in m$ and $g_n = g$ if $n \in m$. Putting this $(g_n)_{n=1}^{\infty}$ into (12), we find
\[(13) \quad \int \int g d\mu_n \leq \kappa' \cdot \|g\|_\infty + \sum_{n=1}^{\infty} \|\mu_n\|.
\]
In view of (11), the last sum in (13) is arbitrarily small for $m$ sufficiently large. This implies that $\mu_n \in \mathcal{M}(W^*)$.

Combining (9) and (10), we obtain
\[(14) \quad \lambda_0(u, (\mu_n)_{n=1}^{\infty}) = \sum_{x \in E} u(x) f(x) + \int \int g_n d\mu_n = \sum_{n=1}^{\infty} \int g_n d\mu_n,
\]
for all $(u, (\mu_n)_{n=1}^{\infty}) \in Y$. This equality, (2), (6), and the definition of $\gamma$ show that for all $f \in \mathcal{I}(G)$ we have the identities:
\[(15) \quad \sum_{x \in E} \gamma f(x) = \lambda(\gamma f) = \sum_{x \in E} u(x) f(x) + \int \int (D_n f) d\mu_n.
\]
Define the measures $\gamma_0$ and $\gamma_n$ by $\gamma_n = \mu_0\gamma$ and $\gamma_n = \mu_n\gamma$. Then (14) shows that
\[(16) \quad \sum_{x \in E} \gamma f(x) = \sum_{x \in E} u(x) f(x) + \gamma_0 f(x) + \gamma_n f(x) = \sum_{x \in E} \sum_{n=1}^{\infty} \sum_{x \in E} D_n f(x) \gamma_n(x) f(x)
\]
for $f \in \mathcal{I}(G)$. For any $f \in \mathcal{I}(G)$, (15) applies to both Re and Im and so (15) also holds for $f$. Putting $f = \gamma$ in (15) for each $\gamma$, we see that (i) holds. The inclusion (iii) has been established above. Finally, writing $\kappa = 2\kappa'$, we see that (iv) follows from (8) and (11). This completes the proof that (a) implies (i)–(iv).

We now turn to a proof of the converse. Supposing that the matching property (i) and (ii) of (b) obtains, we wish to show that $FZ(\mathcal{N}, \mathcal{M})$, $W$ holds. As above, let $I = \{1, 2, 3, \ldots\}$, and form the real Banach spaces $F(I, \mathcal{M}, [W^*])$ and $L^1(I)$ and the product space $E = L^1(I) \times L^1(I, \mathcal{M}, [W^*])$.

Clearly $E$ is a Banach space under the norm
\[(17) \quad \|a, \gamma_0, (\gamma_n)_{n=1}^{\infty}\| = \|a_1\| + \|\gamma_0\| + \sum_{n=1}^{\infty} \|\gamma_n\|.
\]

Define a mapping $T$ of $E$ into $L^1(I)$ by
\[(18) \quad T(a, \gamma_0, (\gamma_n)_{n=1}^{\infty}) = a + \gamma_0 + \sum_{n=0}^{\infty} D_n \gamma_n P.
\]

Plainly $T$ is a bounded linear transformation whose norm does not exceed 1.

For every positive integer $t$, let $A_t$ be the subset of $E$ defined by
\[(19) \quad \text{Supp } a = P_t, \quad \gamma_0, \gamma_n \in \mathcal{M}, [W^*], \quad \|a_1\| + \|\gamma_0\| + \sum_{n=1}^{\infty} \|\gamma_n\| \leq t.
\]

Plainly each $A_t$ is closed in $E$ (and so is sequentially complete), and is bounded and convex and contains 0. Furthermore, the inclusions $A_t \subseteq A_{t+1} \subseteq A_{t+2}$ are easy to check, for all positive integers $t$, $u$, and $q$. The hypothesis (b) simply asserts that
\[(20) \quad \bigcup_{t=0}^{\infty} A_t = \bigcup_{t=0}^{\infty} A_t = L^1(I).
\]

Hence we may apply Lemma 2.5, with $B_{u, u_1} = A_{u+u_1}$, and infer that there is a positive integer $k$ such that
\[(21) \quad \text{the unit ball in } L^1(I) \text{ is contained in } T(A_k).
\]
Now we choose any nonzero \( f \in \mathcal{I}_{p_0}(\theta) \) and define \( \beta \in \mathcal{B}_q(P) \) as \((\text{sgn} f') |P|\). Since \( ||\beta||_m \leq 1 \), we apply (19), (17), and (18) to write \( \beta \) as
\[
\beta = a + \left( \sum_{n=1}^{m} D_n \nu_n \right)|P|, 
\]
where \( \text{Supp } a \subset P_1 \) and
\[
|a|_1 + ||\nu||_m + \sum_{n=1}^{m} ||\nu_n|| \leq k.
\]
Recall too that \( \nu_n \) and all \( \nu_n \) are in \( M_q(\mathbb{W}^{-1} \mathbb{W}) \). Now multiply (20) through by \( f' \) and sum over \( P \) (since \( f \) is a trigonometric polynomial, the sum is actually finite). We obtain
\[
||f'||_m = \sum_{p_0} f'(x) \beta(x) = \sum_{p_0} f'(x) a(x) + \sum_{p_0} f'(x) \nu(x) + \sum_{p_0} \left( \sum_{n=1}^{m} \left( \sum_{p_0} f'(x) D_n \nu_n(x) \right) \right).
\]
For the first sum on the right side of (21), we have
\[
\left| \sum_{p_0} f'(x) a(x) \right| \leq \max_{p_0 \mathbb{Z}} |f'(x)| |a|_1.
\]
To estimate the second and third sums, we note that
\[
\sum_{p_0} f'(x) \nu(x) = \sum_{p_0} \left( f \circ \nu(x) \right)' = \int f(y^{-1}) d\nu(y) = \int f d\nu^{-1} \]
\[
- \int \max_{[y^{-1}]} f(y) d\nu^{-1} + \int \min_{[y^{-1}]} f(y) d\nu^{-1} \]
\[
\leq \int_{[y^{-1}]} f d\nu^{-1} \leq ||\nu||_m ||f||_m \nu^{-1}.
\]
Similar estimates apply to each summand in the third sum, and so we combine (21), (22), and (23) to write
\[
||f'||_m \leq ||a||_1 \max_{p_0 \mathbb{Z}} |f'(x)| + ||\nu||_m ||f||_m + \sum_{n=1}^{m} ||\nu_n|| ||f||_m (D_n f)' \nu^{-1}._m
\]
For all sufficiently large \( n \), we have \( D_n f = f \), and so (24) and (18) yield
\[
||f'||_m \leq k \max_{p_0 \mathbb{Z}} |f'(x)| + \sup_{p_0 \mathbb{Z}} ||f||_m (D_n f) |\nu_n||_m.
\]
The inequality (23) is exactly 5.5. (i) with \( k = \Sigma, \Sigma = P_1 \), and \( n_0 = 1 \), and so Theorem 5.6 implies that (a) of the present theorem obtains.

We end this section with the analogue of Theorem 7.4 applying to the \( \mathcal{S}-\mathcal{E} \)-property.

**7.5 Theorem.** Notation and hypotheses are as in 6.1. The following statements are equivalent.

(a) The set \( P \) has the \( \mathcal{S}-\mathcal{E} \)-property.

(b) For every \( r \in \{0, 1, 2, \ldots \} \), every \( \beta \in \mathcal{B}_q(P) \) admits an expression

\[
\beta(x) = a_\epsilon(x) + \nu_\epsilon(x)
\]

for all \( x \in P \), where \( a_\epsilon \in \mathcal{B}_q(P) \) and \( \nu_\epsilon \in M_q(\mathbb{W}^{-1} \mathbb{W}) \).

Furthermore, if \( (b) \) holds, there exists for every \( r \in \{0, 1, 2, \ldots \} \) a positive real number \( \lambda_r \), such that every \( \beta \in \mathcal{B}_q(P) \) admits an expression (i) in which \( a_\epsilon \in \mathcal{B}_q(P) \), \( \nu_\epsilon \in M_q(\mathbb{W}^{-1} \mathbb{W}) \).

(i) \( ||a_\epsilon||_1 + ||\nu_\epsilon||_m \leq \lambda_r ||\beta||_m \).

**Proof.** This is based upon Theorem 6.3 in exactly the same way as the proof of Theorem 7.4 is based upon Theorem 5.6. In the present case the details are a good deal simpler and are omitted.

Throughout the rest of the paper we will concentrate on \( \mathcal{E} \)-properties [i.e. Case A] and the exploitation of Theorem 7.4.

**§ 8. More matching properties.** Throughout this section, \( V \) is an arbitrary but fixed neighbourhood of \( \epsilon \in \theta \), \( W \) is a set in \( \mathcal{I} \), and \( \mathcal{S} = (\Sigma_{\mathcal{I}} \cup P_0 \mathbb{Z}) \) is as in 3.1 and 3.4. We begin with a technical fact.

**8.1. Theorem.** Suppose that \( \mathcal{E}(P, \mathcal{S}, W) \) holds. Then there exist a positive real number \( \lambda \) and a finite subset \( \Sigma \) of \( P \) (both depending upon \( P, \mathcal{S}, \) and \( W \) with the following property. To every \( \epsilon \in \mathcal{I}_{p_0}(\theta) \) there corresponds a sequence \( (\nu_\mathcal{I}_{p_0}(\theta), \lambda) \) of nonnegative real numbers (depending upon \( P \) and also upon \( \mathcal{S} \) and \( W \)) such that

(i) \( \lim_{\mathcal{I}_{p_0}(\theta)} \nu_\mathcal{I}_{p_0}(\theta) = 0 \);

(ii) \( (1 - \eta) ||f'||_m \leq \nu(\mathcal{S} \cup P_0 \mathbb{Z}) \max_{p_0 \mathbb{Z}} |f'(x)| + ||f||_m ||f||_m \).

**Proof.** Let \( \beta = \text{Supp } a \subset P \). Applying Theorem 7.4, we write \( \beta \) in the form 7.4. (i), with \( \text{Supp } a \subset \Sigma \), where \( \Sigma \) is as in Theorem 7.4. For each positive integer \( m \), let

\[
a_m = a + \sum_{n=1}^{m} D_n \nu_n.
\]

It is clear that \( \text{Supp } a_m \subset \Sigma \cup P_m \) and that \( a_m \in \mathcal{B}_q(P) \). We define \( \eta_m \) by
\[
\eta_m = \sum_{n=m+1}^{\infty} ||\nu||_m.
\]
By 7.4. (ii) we see that \( \lim_{n \to \infty} \nu_n = 0 \), i.e., (i) holds. It is also clear from
7.4. (i) that for all \( \varepsilon < 1 \), the inequality
\[
|f'(z) - a_m(z) - v_m(z)| \leq \nu_m
\]
holds.

In proving (ii), we may restrict ourselves to \( m \) such that \( \nu_m \leq 1 \). Let us write \( a_m \) for the trigonometric polynomial such that
\( a_m = a_m \); plainly \( a_m \) is in \( \mathcal{F}_{\nu_m}(\mathcal{G}) \).

Now consider a sequence \( (X_{ij})_{i,j} \) of continuous, nonnegative, positive-
definite functions on \( \mathcal{G} \) such that:
\[
\text{supp} X_j \subseteq \mathcal{G}^{-1}; \quad \int_{\mathcal{G}} X_j \, d\lambda = 1; \quad \lim_{j \to \infty} \|X_j f_j\|_1 = \|f_j\|_1.
\]
(For the existence of such a sequence, see [8], Vol. II, Theorem 33.11.) For each positive integer \( j \), form the function
\[
g_j = K_j \ast a_n \ast f + X_j \ast v_m \ast f.
\]
Plainly \( g_j \) is in \( \mathcal{C}_r(\mathcal{G}) \), and on the set \( P \), we have
\[
g_j = K_j f_j + (a_m + v_m) \cdot (1 - \sum_{n=1}^m D_n v_n) \cdot \text{sgn} f_j.
\]

It is clear that
\[
\sum_{n=1}^m D_n (v_n) (a_n + v_n) \cdot \text{sgn} f_j \leq \sum_{n=1}^m \|v_n\|_1 = \nu_m
\]
for all \( \varepsilon < 1 \) and so \( \text{sgn} f_j \) is nonnegative on \( P \). On \( X_j \cdot P \), \( g_j \) vanishes. Now define the function \( h_0 \) by \( h_j = \frac{1}{\lambda_j} (g_j + g_0) \). It is obvious that \( h_0 = \text{Reg} g_j \); i.e., \( h_0 \) is a continuous real-valued function on \( \mathcal{G} \) with nonnegative Fourier transform. By [8], Vol. II, Theorem 31.42, \( h_0 \) is in \( P(X) \) and by ibid., 31.44, (\( h_0 \cdot P \)) is equal to \( h_0 \) everywhere on \( \mathcal{G} \). Combining (3) and (4), we find that
\[
(1 - \nu_m) \|X_j f_j\|_1 = \|h_j\|_1 = \sum_{n=1}^m h_j (x) = (h_j \cdot P)(x)
\]
and so
\[
(1 - \nu_m) \|X_j f_j\|_1 \geq \sum_{n=1}^m h_j (x) = (h_j \cdot P)(x)
\]
Since \( K_j \ast a_m \ast f \) is a trigonometric polynomial, we have
\[
K_j \ast a_m \ast f = \sum_{x \in P} K_j (x) (a_m (x) f (x) = \sum_{x \in P} K_j (x) a_m (x) f (x)
\]
\[
\leq \sum_{x \in P} (a_m (x) \cdot |f_j (x)| \leq \|a_m\|_1 \cdot \max_{x \in P} |f_j (x)|),
\]
From (1) we see that
\[
\|a_m\|_1 \leq \left( \|a_m\|_1 + \sum_{n=1}^m D_n v_n \right) \cdot \text{card} (\Sigma \cup P_n),
\]
and from 7.4. (iv) that
\[
\|a_m\|_1 + \sum_{n=1}^m D_n v_n \leq \|a_m\|_1 + \sum_{n=1}^m r_n \leq \nu_m.
\]
Combining these estimates with (9), we obtain
\[
K_j \ast a_m \ast f (x) \leq \nu_m \cdot \text{card} (\Sigma \cup P_n) \cdot \max |f_j (x)|.
\]
In estimating \( K_j \ast a_m \ast f \), we may and will suppose that \( f \) has been re-defined on a set of \( \lambda \)-measure 0 so as to satisfy the inequality
\[
\sup_{x \in \mathcal{G} \cdot \nu_m} |f_j (x)| = \|f_j \|_1.
\]
Since \( K_j \geq 0 \) and \( \nu_m \geq 0 \), we may cite (4) to write
\[
K_j \ast a_m \ast f (x) = \int_{\mathcal{G}} K_j \cdot a_m \ast f (x) \, d\lambda (x) \leq \|f_j \|_1 \cdot \|a_m \|_1.
\]
Also, for \( \lambda \)-almost all \( x \in \mathcal{G} \cdot \nu_m \), we have
\[
\|f_j \|_1 = \int_{\mathcal{G} \cdot \nu_m} \phi (y) \, d\nu_m (y) \leq \|\nu_m\| \cdot \sup_{x \in \mathcal{G} \cdot \nu_m} |f_j (x)| = \|\nu_m\| \cdot \|f_j \|_1.
\]
Since \( \|\nu_m\| \leq \nu_m \) by 7.4. (iv), we see that
\[

\]
both $\alpha$ and $\beta$ belong to $\Phi_{\alpha}(\mathcal{L}^p(G))$. By Corollary 8.2, we see that if $\|f\|_\alpha < \infty$, then $\phi(f)$ and $\phi(-f)$ are also finite. Thus Lemma 2.3 is applicable with $\mathcal{L}^p(G) = E$, and we need only note that for the seminorm $\sigma$ of 2.3 we may take a multiple of the $\ell^p$-norm in $\mathcal{L}^p(G)$. \[ \\]

We now obtain a new matching property.

8.4. Theorem. Suppose that $\mathcal{E}(P, \mathcal{S}, W)$ holds and that $V$ is a compact neighbourhood of $\alpha$. Then there is a positive real number $\epsilon$ (which depends upon $P$, $\mathcal{S}$, and $V$) with the following property. For every function $\beta$ in $B_1(\mathcal{S})$, there exist a function $g \in \mathcal{L}^\infty(G)$ and a measure $\nu \in M_\alpha((W^- \cdot V)^{-1})$ such that:

(i) $\beta = g \cdot \nu^*$ on $\mathcal{S}$;

(ii) $\|g\|_\infty + \|\nu\|_\infty \leq \|\beta\|_\infty$.

Proof. This theorem follows from Corollary 8.3 much as the first implication in Theorem 7.4 follows from Theorem 5.6. We outline the proof. For $f \in \mathcal{L}^p(\mathcal{S})$, define $l(f)$ by

$$l(f) = \sum_{\alpha \in \mathcal{S}} f(\alpha) \beta(\alpha).$$

From 8.3. (i), we infer that

$$|l(f)| \leq \|f\|_p \cdot \sup \{ \|\beta\|_p, \|\nu\|_p \}. $$

Now consider the linear space $E = \mathcal{L}^p(\mathcal{S}) \times \mathcal{L}^\infty((W^- \cdot V))$ with the norm

$$\|(\varphi, \psi)\| = \max \{ \|\varphi\|_p, \|\psi\|_\infty \}$$

and the gauge

$$\varphi(\psi) = \max \{ \|\varphi\|_p, \|\psi\|_\infty \}.$$ \[ \]

Imbed $\mathcal{L}^p(\mathcal{S})$ into $E$ by the injective map $f \mapsto (f, f(W^- \cdot V))$. From (2) and the Hahn–Banach theorem we see that there is a linear functional on $E$, which we continue to call $l$, such that

$$l(f) = \int f(x) \mu(x) dx$$

and

(i) $|l(\varphi, \psi)| \leq \|\varphi\|_p \|\psi\|_\infty \leq \|\beta\|_p \|\nu\|_\infty$.

The space $E$ is a Banach space under the norm $\tau$, and (3) informs us that $l$ is a bounded linear functional on $E$. It follows that

$$l(\varphi, \psi) = \int_\mathcal{S} \varphi(\alpha) h(\alpha) d\alpha + \int_{W^- \cdot V} \psi(x) d\mu(x)$$

for some $h \in \mathcal{L}^\infty(G)$ and $\mu \in M_\alpha((W^- \cdot V)^{-1})$ satisfying

$$\|h\|_\infty + \|\mu\| \leq \|\beta\|_\infty.$$ \[ \]

The first of the inequalities in (3) shows that $\mu \in M_\alpha((W^- \cdot V)$). For $f \in \mathcal{L}^p(\mathcal{S})$, (4) becomes

$$l(f) = \int_\mathcal{S} f(\alpha) h(\alpha) d\alpha + \int_{W^- \cdot V} f(x) d\mu(x).$$

For $\chi \in \mathcal{S}$ and a complex number $c$, (6) yields

$$l(c \chi + \epsilon \mu^*) = c h^* (\chi^{-1}) + \epsilon h^* (\chi) + \epsilon \mu^* (\chi^{-1}) + \epsilon \mu^* (\chi).$$

Combine (1) and (7) and set $c = \frac{1}{2}$ and $\epsilon = -\frac{1}{2}$ in turn. This yields

$$\beta(\chi) = h^* (\chi^{-1}) + \mu^* (\chi^{-1}).$$

Defining $g = h^*$ and $\tau = \mu^*$, we obtain (i) from (8) and (ii) from (5). \[ \]

8.5. Note. If $P$ has the full $\mathcal{E}(\mathcal{S})$ property 5.3, then the set $W^- \cdot V$ in 8.4 may be replaced by an arbitrary compact neighbourhood of $\alpha$.

We continue with analogues for $\mathcal{E}$-properties of certain approximation properties known to be equivalent to Sideness

8.6. Definition. Let $\mathcal{D}(P)$ be the set of all complex-valued functions $\beta$ on $P$ such that $|\beta(\chi)| = 1$ for all $\chi \in P$.

8.7. Theorem. The following statements are equivalent.

(a) The set $P$ possesses the $\mathcal{E}(\mathcal{S})$, $W$ property.

(b) There exists a positive real number $\epsilon$ such that to every $\beta$ in $B_1(\mathcal{S})$ there corresponds a $\mu \in M_\alpha((W^- \cdot V)^{-1})$ satisfying

(i) $|\mu| \leq \|\beta\|_\infty$

and

(ii) $\lim_{m \to \infty} \sup_{\chi \in \mathcal{S}} |\mu^* (\chi) - \beta(\chi)| = 0$.

(c) For every $\beta$ in $\mathcal{D}(P)$, there exists a $\mu \in M_\alpha((W^- \cdot V)^{-1})$ satisfying

(iii) $\lim_{m \to \infty} \sup_{\chi \in \mathcal{S}} |\mu^* (\chi) - \beta(\chi)| < 1$.

Proof. To prove that (a) implies (b), we apply Theorem 7.4. Write $\beta$ as in 7.4. (i) and take $\mu = \nu - \nu_0$. Then we have

$$\beta(\chi) - \mu^* (\chi) = \alpha(\chi) + \sum_{n=1}^\infty D_n^* (\chi) \nu_n (\chi).$$

For $m$ so large that $\text{Supp} \alpha \subset P_m$, we find for all $\chi \in \mathcal{S} \setminus P_m$ that

$$|\beta(\chi) - \mu^* (\chi)| = \sum_{n=m+1}^\infty |\nu_n (\chi)| \leq \sum_{n=m+1}^\infty |\nu_n (\chi)|.$$ \[ \]

Since the last term in (1) has limit $0$ as $m \to \infty$, (ii) holds. The existence of $\alpha$ and the inequality (i) follow from 7.4. (iv).
It is trivial that (b) implies (c). Suppose now that (c) holds. Consider any function $u$ in $\mathcal{S}_0(P)$ and let $\beta = \text{sgn} u$. Given a $\mu$ satisfying (iii) for this $\beta$, we can find a finite symmetric subset $\Sigma$ of $P$ and a real number $d$ such that $0 < d < 1$ for which the inequality

$$|\beta(x) - \mu^*(x)| < 1 - d$$

holds for all $x \in P \setminus \Sigma$. For a given positive integer $n$, write $\sum_{n=1}^\infty$ for a sum over $P \setminus \Sigma$, and $\sum_{n=1}^m$ for a sum over $P_n \cap \Sigma$, and $\sum_n$ for a sum over $P_n$. Then we have

$$\sum_n |w(x)| = \sum_n \beta(x) w(x) \leq \sum_n \mu^*(x) w(x) + (1 - d) \sum_n |u(x)|,$$

which implies that

$$d \sum_n |w(x)| \leq \sum_n \mu^*(x) w(x).$$

We also have

$$\sum_n \mu^*(x) w(x) = \int_{(P \setminus \Sigma)} \sum_n u(x) \overline{w(x)} \mu(t) dt - \sum_n \mu^*(x) w(x).$$

We can estimate the integral in (3) as follows:

$$\int_{(P \setminus \Sigma)} \sum_n u(x) \overline{w(x)} \mu(t) dt = \int \sum_n u(x) \overline{w(x)} d\mu(t)$$

$$= \int s_n u(t) d\mu_t - \sum_n |s_n| u(x)| = n_{n=1} |f_{n-1} - f_n| u(x).$$

Combining (2), (3) and (4), we find

$$d \sum_n |w(x)| \leq \sum_n |f_{n-1} - f_n| u(x) + \sum_n \mu^*(x) w(x).$$

Now suppose that $\sup \{|f_{n-1} - f_n| u(x)| < \infty$. Letting $n \to \infty$ in (5), we see that $\mu \in \mathcal{S}_0(P)$; that is, $P$ possesses the $\mathcal{F}(\mathcal{S}, \mathcal{W})$ property. This proves that (c) implies (a).

8.8. Corollary. If $\mathcal{H} = (X_n)_{n=1}^\infty$ and $\mathcal{H}^* = (X_n^*)_{n=1}^\infty$ are sequences as in 3.4, and if $P$ is a symmetric subset of $X$ satisfying $P \subseteq \bigcup_{n=1}^\infty X_n$ and $P = \bigcup_{n=1}^\infty X_n$, then $P$ possesses property $\mathcal{F}(\mathcal{H}, \mathcal{W})$ if and only if it possesses property $\mathcal{F}(\mathcal{H}^*, \mathcal{W})$. Likewise $P$ possesses the full $\mathcal{F}(\mathcal{H}, \mathcal{W})$ property if and only if it possesses the full $\mathcal{F}(\mathcal{H}^*, \mathcal{W})$ property.

Proof. The left side of 8.7 (ii) is actually equal to

$$\inf \sup_{x \in P \setminus \Sigma} \{|\beta(x) - \mu^*(x)|\},$$

where $\Sigma$ runs over all finite subsets of $P$. Hence the validity of statement (b) in 8.7 depends only on $P$ and not on any particular sequence $\mathcal{F}$. ■

8.9. Definition. Let $P$ be a countably symmetric subset of $X$. We write $\mathcal{F}(P, W)$ provided $\mathcal{F}(P, \mathcal{S}, W)$ holds for some (and hence for every) $\mathcal{S}$ of the union of whose elements contains $P$; we say that $P$ has the $\mathcal{F}(W)$ property or that $P$ is an $\mathcal{F}(W)$-set. Similarly, $P$ has the full $\mathcal{F}(W)$ property and is called a full $\mathcal{F}(W)$-set if it has the full $\mathcal{F}(\mathcal{S})$ property for some (and hence for every) $\mathcal{S}$ whose union contains $P$. We now obtain a matching property strongly reminiscent of F. Bries's matching property for Sidonicity.

8.10. Theorem. Let $P$ be a countably symmetric subset of $X$. The property $\mathcal{F}(P, \mathcal{G})$ obtains if and only if every function $u$ in $\mathcal{S}_0(P)$ can be represented in the form

$$\beta(x) = \mu^*(x) \quad \text{for all} \quad x \in P \setminus \{1\},$$

$\mu^*$ being a measure in $\mathcal{M}_0(\mathcal{G})$.

Proof. Suppose that (i) holds for all $\beta$ as described. Then we can write $\beta$ in the form $\beta = s_n - a\Sigma_n$, where $s_n = 0$, and $\Sigma_n = \beta(1) - \mu^*(1) \Sigma_n$ if $1 \in P$ and $\Sigma_n = 0$ if $1 \notin P$. By Theorem 7.4, $\mathcal{F}(P, \mathcal{G})$ holds.

To prove the necessity of our condition, we cite Theorem 8.4 with $V = \mathcal{G}$. Given $\beta \in \mathcal{S}_0(P)$, we find $\mu \in \mathcal{S}_0(\mathcal{G})$ and $\mu^* \in \mathcal{M}_0(\mathcal{G})$ such that

$$\beta = \mu^* + \mu^* \mathcal{G} - \mu^*.$$

Now write $\mu = |\mu| = |\mu| + \mu^*$. Plainly $\mu$ belongs to $\mathcal{M}_0(\mathcal{G})$, and from (1), $\mu^*$ matches $\beta$ on $P \setminus \{1\}$, as expected.

8.11. Remarks. (a) The foregoing theorem clarifies the relation of the Fatou–Zygmund property to Sidonicity. A subset $P$ of $X$ is of course a Sidon set if and only if every bounded complex-valued function on $P$ is matchable on $P$ by a Fourier–Stieltjes transform $\mu$, where $\mu$ is some complex measure in $\mathcal{M}(\mathcal{G})$ (see for example [35], Vol. II, Theorem 37.2). A countably symmetric subset $P$ of $X$ is an $\mathcal{F}(\mathcal{G})$-set if and only if every bounded Hermitian function on $P$ is matched except perhaps at 1 by the Fourier–Stieltjes transform of a non-negative measure.

(b) Consider a countably infinite symmetric subset of $X$ that is an $\mathcal{F}(\mathcal{G})$-set. A bounded function $\beta$ on $P$ such that $\beta(x^{-1}) = \beta(x)$ can be redefined at 1 (if $1 \in P$) so that $\beta$ admits a positive-definite extension over all of $X$. This follows from 8.10 since $\mu^*$ is positive-definite for $\mu \in \mathcal{M}_0(\mathcal{G})$. Since $\epsilon_0$ is positive-definite for $t \geq 0$ and sums of positive-definite functions are positive-definite, we see that $\mu + \epsilon_0$ admits a positive-definite extension over $X$ for all $t \geq 0$.

(c) Now let $P$ be a finite subset of $X$, and consider the finite symmetric set $\Phi P^{-1} = P$. Plainly $\Phi$ is an $\mathcal{F}(\mathcal{G})$-set and so every symmetric...
function on \( \Psi \) can be redefined at 1 so as to admit a positive-definite extension over \( X \). For previous results in this direction, see (14).

We next establish an analogue of 8.10 for functions in \( c_{00}(P) \), suggested by a well known fact for Sidon sets. We precede this by a technical lemma.

8.12. Lemma. Let \( \Phi \) be a finite subset of \( X \setminus \{1\} \) and let \( \varepsilon \) be a positive real number. There exists a function \( f \in \mathcal{I}_\varepsilon (G) \) such that \( f'(x) = 1 \) for all \( x \in \Phi \) and \( \|f\|_\varepsilon \leq 1 + \varepsilon \).

Proof. With no loss of generality, we may suppose that \( \Phi \) is symmetric. By [5], Vol. II, Theorem 28.57, there is a sequence \( \{K_n\}_{n=1}^{\infty} \) of functions in \( \mathcal{I}_\varepsilon (G) \) such that \( \|K_n\|_\varepsilon \leq 1 \) and

\[
\lim_{n \to \infty} K_n(x) = 1 \quad \text{for all } x \in \Phi.
\]

Define

\[
f_n = K_1 + \sum_{n=2}^{\infty} (1 - K_n(x)) x + \sum_{n=2}^{\infty} |1 - K_n(x)|,
\]

Plainly \( f_n \) is in \( \mathcal{I}_\varepsilon (G) \), and since \( 1 \in \Phi \), we have \( f_n(1) = 1 \) for all \( x \in \Phi \).

Also we have

\[
\|f_n\|_\varepsilon = f_n(1) = K_n(1) + \sum_{n=2}^{\infty} |1 - K_n(x)| = 1 + \sum_{n=2}^{\infty} |1 - K_n(x)|,
\]

and this is less than \( 1 + \varepsilon \) for \( n \) sufficiently large.

8.13. Theorem. The property \( EF(P; G) \) holds if and only if every function \( \beta \in c_{00}(P) \) can be represented in the form

\[\beta(x) = f'(x) \quad \text{for all } x \in P \setminus \{1\},\]

where \( f \) is a function in \( \mathcal{I}_\varepsilon (G) \).

Proof. A glance at 8.10. (i) and (ii) shows that we lose no generality in supposing that \( 1 \in P \).

Suppose that (i) holds. We will apply Lemma 2.5 with \( P = \mathcal{I}_\varepsilon (G) \), \( P = c_{00}(P) \) (both with the usual norms) and \( T \) the mapping \( f \to f' \). For \( A_n \) (\( n \in \{1, 2, 3, \ldots \} \)) we take \( f \in \mathcal{I}_\varepsilon (G) \) such that \( \|f\|_\varepsilon = n \). The mapping \( T \) is linear and continuous, hence closed. Plainly \( A_n + A_m \) is contained in \( A_{n+m} \), and \( A_{n+m} \) is closed, convex, contains 0, and is sequentially complete. Also the equality \( q A_n = A_n \) holds for all positive integers \( q \) and \( n \). The matching property (i) is just the assertion \( T(\bigcup_{n=1}^{\infty} A_n) = c_{00}(P) \). Hence the conclusion of Lemma 2.5 holds. Rewritten slightly, this asserts that there is a positive real number \( x \) such that for all \( \beta \in c_{00}(P) \), the function \( f \) in (i) may be chosen so that

\[
\|f\|_\varepsilon \leq x \|\beta\|_{\infty}.
\]

Now given a function \( \gamma \) in \( B_\varepsilon (P) \), we can trivially find a sequence \( \{\beta_j\}_{j=1}^{\infty} \) such that

\[
\lim_{j \to \infty} \beta_j(x) = \gamma(x) \quad \text{for all } x \in P,
\]

and \( |\beta_j_1| = \|\beta\|_{\infty} + \|\gamma\|_{\infty}. \)

For each \( j \), choose \( f_j \in \mathcal{I}_\varepsilon (G) \) for which

\[
|f_j(1)| = 1 \quad \text{and } \|f_j\|_\varepsilon \leq \frac{\|\beta_j\|_{\infty}}{x},
\]

and this is less than \( 1 + \varepsilon \) for \( n \) sufficiently large. \( \square \)

Consider any \( \beta \in c_{00}(P) \). For every positive integer \( j \), define

\[P_j = \{x \in P : 2^{-j} \|\beta\|_{\infty} < |\beta(x)| \leq 2^{-(j+1)} \|\beta\|_{\infty}\}.
\]

Plainly \( P_j \) is a finite symmetric subset of \( P \). Define \( \beta_j \) by \( \beta_j x \). It is clear that

\[
\|\beta_j\|_{\infty} \leq 2^{-j+1} \|\beta\|_{\infty}.
\]

Using (4) and (5), we find measures \( \mu_j \cdot M_\varepsilon (G) \) such that

\[
\mu_j |P = \beta_j, \quad \|\mu_j\| \leq 2^{-j+1} \|\beta\|_{\infty},
\]

Now use Lemma 8.12 to choose polynomials \( p_j \) in \( \mathcal{I}_\varepsilon (G) \) such that

\[
\|p_j f_j \|_1 < \frac{1}{2} \quad \text{and } \quad p_j(1) = 1.
\]

Define the function \( f \) as \( \sum_{j=1}^{\infty} p_j j \cdot \mu_j \). It is easy to check from (6) and (7) that \( f \in \mathcal{I}_\varepsilon (G) \) (and incidentally that \( \|f\|_1 \leq 3 \|\beta\|_{\infty} \)). From (6) and (7) it is also clear that

\[
|f(1)| \leq \|f\|_1 \leq 3 \|\beta\|_{\infty}.
\]

8.14. Remarks. We do not know whether or not every symmetric countably infinite Sidon set \( P \) is an \( EF(G; \mathbb{G}) \)-set. Nevertheless, we can set
down some conditions equivalent to $E^2(P, G)$ that look considerably stronger than Sidonness. For a real-valued function $f$ on $G$, we write $\max(f)$ and $\min(f)$ for $\max(f(x); x \in G)$ and $\min(f(x); x \in G)$, respectively.

8.15. Theorem. Let $P$ be a countably infinite symmetric subset of $X$ not containing 1. The following conditions are equivalent:

(i) $P$ satisfies the property $E^2(P, G)$;

(ii) every function $\beta$ in $B_1(P)$ is matchable on $P$ by $\mu^*$ for some $\mu \in M_+(G)$;

(iii) there is a positive constant $\kappa$ such that $\|f\|_1 \leq \kappa \cdot \max(f)$ for all $f \in L_1^\mu(P, G)$;

(iv) there is a positive constant $\kappa$ such that $\|f\|_1 \leq \kappa \cdot \text{esssup } f$ for all $f \in L_1^\mu(P, G)$;

(v) $f \in L_1^\mu(P, G)$ and $\text{esssup}(f) < \infty$ imply $f \in L^1(X)$;

(vi) $\mu \in M_+(G)$ and $\mu \ll c\lambda$ for some real number $c$ (which depends on $\mu$) imply $\mu \in L^1(X)$;

(vii) there is a positive constant $\kappa$ such that $\mu \in M_+(G)$ and $\mu \ll c\lambda$ (depending on $\mu$) imply $\|\mu\|_1 \leq \kappa c$.

The constants in (iii), (iv) and (vii) may be taken equal.

Proof. The equivalence of (i) and (ii) is Theorem 8.10. It is simple to establish directly that (iii) implies (i), and we will now do this. Suppose that (iii) holds, and that $\beta$ is any function in $B_1(P)$. If the functions $s_n = \sum_{n \geq 0} s_n(x)\beta(x)$ are bounded above by say $\alpha$ on $G$, then from (iii) we have

$$\|s_n\|_1 = \sum_{n \geq 0} \alpha |s_n(x)| \leq \kappa \cdot \max(s_n) \leq \kappa \alpha,$$

and so taking the limit as $n \to \infty$, we find that $\alpha \beta \in L_1(P, G)$, i.e., property $E^2(P, G)$ holds.

We now prove that (ii) implies (iii). We first apply Lemma 2.5 with $E = M_+(G), F = \mathbb{S}_1(P), \mathbb{S}_1(P, G), \mathbb{S}_1(P, G) = \|\mu\|_{1, P},$ and $A_\mu = \{\mu \in M_+(G); \|\mu\|_{1, P} \leq \delta\}$. Lemma 2.5 shows that there exists a positive integer $n$ such that $\|\mathbb{S}_1(P, G)\|_{1, P} \leq \delta = (\Lambda_\mu)^{-1}$. In other terms, there is a positive constant $\kappa$ such that for every $\beta \in \mathbb{S}_1(P), \|\mu\|_{1, P} \leq \kappa$, and for which $\mu \perp P$. Now let $f$ be a polynomial in $L_1^\mu(P, G)$, define $\beta$ as $\text{sgn } f$, and choose $\mu \in M_+(G)$ such that $\|\mu\|_1 \leq \kappa$ and $\mu \perp P$. Then we have

$$\|f\|_1 = \sum_{n \geq 0} \|f(x)\beta(x)\|_1 = \sum_{n \geq 0} \|f(x)\mu^*(x)\|_1 \leq \sum_{n \geq 0} \|f\|_1 \cdot \mu^*(x) \cdot \kappa \cdot \min(f) \cdot \kappa \cdot \min(f).$$

This is exactly (iii). We have established the equivalence of (i), (ii), and (iii).

For $\mu \in M_+(G)$, we have $0 = \mu^*(1) = \int_0^1 d\mu$, and so any $\alpha$ as in (vi) or (vii) must be nonnegative. Condition (vii) trivially implies (iv) with the same value of $\alpha$, and (iv) trivially implies (iii) with the same value of $\alpha$. Suppose now that (iii) holds; we prove (vii). Let $\mu \in M_+(G)$ have the property that $\mu \ll c\lambda$. By (8.5), Vol. II, 32.3, there is a sequence $(K_\chi)_{\chi \in \mathcal{F}}$ of functions in $\mathbb{S}_1(G)$ such that $\int_\mathcal{F} K_\chi \, d\lambda = 1$ and $\lim_{\chi \to \infty} K_\chi = 1$ for all $\chi \in \mathcal{F}$. Consider the polynomials $f_\chi = \mu \cdot K_\chi$. For each of them and for all $x \in G$, we have

$$f_\chi(x) = \int_\mathcal{F} K_\chi(y^{-1}x) \, d\mu(y) = \int_\mathcal{F} K_\chi(y^{-1}x) \, d\lambda(y) = \int_\mathcal{F} K_\chi \, d\lambda = 0.$$

By (iii) and the evident fact that $f_\chi \in L_1^\mu(P, G)$, we have

$$\|f_\chi\|_1 \leq \kappa \alpha.$$

By Fatou's lemma for series, we have

$$\|\mu\|_{1, G} = \sum_{\chi \in \mathcal{F}} \|K_\chi \|_{1, G} \mu^*(\chi) = \liminf_{\chi \to \infty} \left[ \sum_{\chi \in \mathcal{F}} \|K_\chi \|_{1, G} \mu^*(\chi) \right],$$

Thus (vii) holds with the same value of $\alpha$.

We have thus proved the equivalence of (i), (ii), (iii), (iv) and (vii). Since (vii) obviously implies (vi) and (vi) obviously implies (v), it suffices for us to prove that (v) implies (iii). First we show that

$$\|f\|_1 \leq 2\max(f) \quad \text{for } f \in L_1^\mu(P, G).$$

We have $f = f^+ - f^-$ where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$. Since $1$ is not in $P$, we have

$$0 = \int f \, d\lambda = \int f^+ \, d\lambda - \int f^- \, d\lambda = \|f^+\|_1 - \|f^-\|_1,$$

and hence

$$\|f\|_1 = \|f^+\|_1 + \|f^-\|_1 = 2\|f^+\|_1 \leq 2\max(f).$$

This establishes (1). Now we apply Lemma 2.3 with

$$E = L_1^\mu(G), \quad \varphi(f) = \|f\|_1 \quad \text{and} \quad \psi(f) = \text{esssup}(f).$$

Hypothesis 2.3. (i) holds in view of (v). We conclude that there is a constant $\kappa$ such that

$$\|f\|_1 \leq \kappa \cdot \max(\|f\|_1, \text{esssup}(f))$$
for all \( f \in \mathcal{L}^1_P(G) \). Inequalities (1) and (2) yield
\[
\|f\|_1 \leq 2\kappa \max(f) \quad \text{for } f \in \mathcal{L}^1_P(G),
\]
and so (iii) holds. \( \blacksquare \)

We list yet another property equivalent to \( \text{FZ}(P,G) \).

8.16. Theorem. Let \( P \) be a countable symmetric subset of \( X \) not containing 1. In order that \( P \) be an \( \text{FZ}(G) \)-set it is necessary and sufficient that
\( \text{(a) } P \) be a Sidon set and
\( \text{(b) there exists a real number } \kappa > 1 \text{ such that either} \)
\[
\kappa^{-1} \leq \max(f) \leq \kappa \quad \text{for every } f \in \mathcal{L}^1_P(G) \text{ satisfying } \min(f) = -1,
\]
or (what is equivalent)
\[
-\kappa \leq \min(f) \leq -\kappa^{-1} \quad \text{for every } f \in \mathcal{L}^1_P(G) \text{ satisfying } \max(f) = 1.
\]

Proof. The equivalence of (i) and (ii) is trivial. Suppose that (ii) holds, and let \( f \) be any nonzero polynomial in \( \mathcal{L}^1_P(G) \). Since \( \|f\|_1 \equiv 0 \), we have \( \beta = \max(f) > 0 \). For the function \( f/\beta \), we have by (ii) that \( \min(f/\beta) \geq -\kappa \), hence \( \min(f) \geq -\kappa \beta \), and so
\[
\|f\|_1 = \max(f) - \min(f) \leq \kappa \max(f).
\]
If (a) also holds, we select a constant \( \kappa' \) such that \( \|f\|_1 \leq \kappa' \|f\|_1 \) and find
\[
\|f\|_1 \leq \kappa' \max(f)
\]
for all \( f \in \mathcal{L}^1_P(G) \). That is, the property \( \text{FZ}(P,G) \) holds, by Theorem 8.15.

Conversely, suppose that \( P \) is an \( \text{FZ}(G) \)-set. By 8.17, \( P \) is a Sidon set. By 8.15, there is a positive constant \( \kappa \) such that
\[
\|f\|_1 \leq \kappa \max(f)
\]
for all \( f \in \mathcal{L}^1_P(G) \). It is trivial that \( \|f\|_1 \leq \|f\|_1 \), and so we have
\[
\max(f) \leq \max(f) - \min(f) \leq \kappa \max(f)
\]
for all \( f \in \mathcal{L}^1_P(G) \). Similarly we have
\[
\max(-f) \leq \max(-f) - \min(-f) \leq \kappa \max(-f),
\]
and combining (1) and (2), we find
\[
\max(f) - \min(f) \leq \kappa \max(f) - \min(f).
\]
Again note that if \( f \neq 0 \), then both \( a = -\min(f) \) and \( b = \max(f) \) are positive numbers. We rewrite (3) as
\[
\frac{1}{2}(a + b + |a - b|) \leq \frac{1}{2}(a + b + |a - b|),
\]
or equivalently
\[
|a - b| \leq \theta(a + b),
\]
where
\[
\theta = \frac{\kappa - 1}{\kappa + 1},
\]
such that \( 0 < \theta < 1 \).

The inequality (4) translates readily into
\[
(a \leq b) \text{ if } a \geq b,
\]
and (7) becomes
\[
b \leq \kappa a \text{ if } a \leq b.
\]
Thus if \( a \) has a fixed value, say 1, then (6) becomes
\[
b \geq \frac{1}{\kappa} \text{ if } b \leq 1
\]
and (7) becomes
\[
b \leq \kappa \text{ if } b \geq 1
\]
That is, (i) holds. \( \blacksquare \)

We next set down some matching properties enjoyed by certain Sidon sets, which so far as we know are new.

8.17. Theorem. Let \( P \) be a symmetric Sidon set containing no elements \( \lambda \) such that \( \lambda^2 = 1 \). Let \( \kappa \) be as in (8). Then we have
\[
\|f\|_1 \leq \kappa \|f\|_1 \quad \text{for all } f \in \mathcal{L}^1_P(G).
\]
Let \( \beta \) be any function in \( \mathcal{B}_P \) such that
\[
\beta(\lambda) = -\beta(\lambda^{-1}) \quad \text{for all } \lambda \in P.
\]
Then there is a measure \( \mu \neq 0 \) such that:
\[ \text{(i) } \mu(\lambda) = \kappa \beta(\lambda), \]
\[ \text{(ii) } \Im \mu(\lambda) = -\Re \beta(\lambda) \quad \text{for all } \lambda \in P. \]

Proof. Let \( \mathcal{E}_P(\lambda) \) be the real linear space of real-valued functions on \( G \) spanned by all functions \( i(\lambda - \lambda^{-1}) \) for \( \lambda \in P \). Plainly if \( f \in \mathcal{E}_P(\lambda) \), we have
\[
f(\lambda^{-1}) = -f(\lambda) \quad \text{for all } \lambda \in \lambda \Lambda G,
\]
and we say that \( f \) is an odd function. Associated with our function \( \beta \) is a real linear functional \( \mathcal{M}_\beta \) on \( \mathcal{E}_P(\lambda) \), defined by
\[
\mathcal{M}_\beta(f) = \frac{1}{2} \sum_{\lambda \in \Lambda} \beta(\lambda)f(\lambda).
\]
If \( f = \sum_{k=1}^{n} \alpha_k (\lambda_k - \lambda_k^{-1}) \) with real \( \alpha_k \), then we have
\[
\mathcal{M}_\beta(f) = \sum_{k=1}^{n} \alpha_k (\beta(\lambda_k) - \beta(\lambda_k^{-1})) = \sum_{k=1}^{n} \alpha_k 2\beta(\lambda_k).
Since $P$ is a Sidon set, we have

\[ |M_{\delta}(f)| \leq \|f\|_{L^1} \leq \varepsilon \|\delta\|_{L^1}/\|f\|_{L^1}. \]

Thus the number

\[ \gamma = \|M_{\delta}\| = \sup\{M_{\delta}(f): f \in L^1(G), \|f\|_{L^1} \leq 1\} \]

is a nonnegative real number, equal to zero if and only if $\delta$ is the zero function.

The function $1$ is not in $L^1(G)$. We extend $M_{\delta}$ to a functional $M_{\delta}$ on $L^1(G)$ by the rule

\[ M_{\delta}(f + \alpha 1) = M_{\delta}(f) + \alpha \gamma. \]

We compute the norm of $M_{\delta}$. Suppose that $\|f + \alpha 1\|_{L^1} \leq 1$, with $f \in L^1(G)$ and $\alpha \in \mathbb{R}$. Then we have

\[ -1 - \alpha \leq f \leq 1 - \alpha. \]

Since $f$ is an odd function, we must have $-1 \leq \alpha \leq 1$, and also

\[ \min(1 - \alpha, 1 + \alpha) \leq f \leq \min(1 + \alpha, 1 - \alpha). \]

That is, we have $\|f + \alpha 1\|_{L^1} \leq 1 - |\alpha|$, and so

\[ |M_{\delta}(f + \alpha 1)| \leq \|M_{\delta}(f) + \alpha \gamma\| \leq \|\gamma\|_{L^1} \leq 1 - |\alpha|. \]

This proves that $\|M_{\delta}\| = \gamma = \|M_{\delta}\|_{L^1}$. Now use the Hahn–Banach theorem to extend $M_{\delta}$ to a real linear functional $M_{\delta}$ on $L^1(G)$ such that $\|M\| = \|M_{\delta}\| = \gamma$. Since $M(1) = \|M\|$, $M_{\delta}$ is a nonnegative linear functional (see e.g. [8], Vol. II, (34.4.8.b)). Thus there is a measure $\mu$ in $M_{\delta}(G)$ for which

\[ \int_G \phi \, d\mu = M(\phi) \quad \text{for all } \phi \in C_0(G). \]

Taking $\phi = \chi - \chi^{-1}$ with $\chi \in P$, and using (3), we find

\[ M_{\delta}(\mu) = \delta(\chi) - \delta(\chi^{-1}) = 2\delta(\chi), \]

and also

\[ M_{\delta}(\mu) = \int_G \phi \, d\mu = \int_G \mu^*(\chi - \chi^{-1}) = 2\Im \mu^*(\chi). \]

Equalities (6) and (7) imply (ii). The inequality (i) follows from (4) and the definition of $\gamma$. \hfill \Box

8.18. COROLLARY. Let $P$ be a Sidon set such that $\delta \in P_0$ implies $\chi^{-1} \in P$. Let $\beta$ be any bounded real-valued function on $P_0$. There is a measure $\mu \in M_{\delta}(G)$ such that $\Im \mu^*(\chi) = \beta(\chi)$ for all $\chi \in P$. \hfill \Box

Proof. By Drury’s theorem [2], the set $P = P_0 \cup P_0^{-1}$ is a Sidon set. Define $\beta$ on $P_0$ and as $-\beta_0$ on $P_0^{-1}$, and apply 8.17. \hfill \Box

To conclude this section, we give an analogue of 8.10 for the full $FZ$ property; see 8.9 and 5.3.

8.19 THEOREM. Let $P$ be a countably infinite subset of $X$. Then $P$ has the full $FZ$ property if and only if for every $\beta \in \mathcal{B}_0(P)$ and every compact symmetric neighborhood $U$ of $e$, there exist $\mu \in M_{\delta}(U)$ and $g \in L_0^\infty(G)$ such that

\[ \beta = (\mu^* - g^*)|P|. \]

Proof. Supposing that (i) holds for all $\beta \in \mathcal{B}_0(P)$, consider a fixed but arbitrary $\beta \in \mathcal{B}_0(P)$ and choose $\mu \in M_{\delta}(U)$ and $g \in L_0^\infty(G)$ for which (i) holds. By the Riemann–Lebesgue lemma, $g|P$ belongs to $c_0(P)$. Hence for $m$ large enough and $\chi \in P^m \mathcal{P}_m$, we have $|g^*| < \frac{1}{m}$, so that

\[ |\beta(\chi) - \mu^*(\chi)| < \frac{1}{m} \]

for all $\chi \in P^m \mathcal{P}_m$. Thus 8.7. (iii) holds and so by Theorem 8.7, $P$ is an $FZ(U)$-set.

Now suppose that $W$ is any set in $\mathcal{F}$, and let $\chi$ be a function in $P_0^\infty(P)$ such that $s_n\chi(x) = \sum_{\phi \in \mathcal{F}} \phi(x)$ is bounded above for all $x$ and for all $x \in W$. Let $a_n$ be in int $(W)$ and $U$ a compact symmetric neighborhood of $e$ such that $U \in \mathcal{F}$ and $a_n U \subset W$. Then for all $x \in U$, we have

\[ \sum_{\phi \in \mathcal{F}} \phi(x) \in U \]

since $P$ is an $FZ(U)$-set, the function $\chi \mapsto x_n \chi(x)$ is in $P(P)$ and so therefore is $\mu$ itself. That is, $P$ is an $FZ(W)$-set for all $W \in \mathcal{F}$, which is to say, $P$ has the full $FZ$ property.

The converse is simple. If $P$ has the full $FZ$ property, we merely cite 8.5 to see that the matching property (i) holds. \hfill \Box

§ 9. Drury’s theorem for $FZ(G)$-sets. Our aim here is to prove an analogue of Drury’s theorem [2].

9.1. THEOREM. Let $P_0$ and $P_0$ be countably infinite (symmetric) $FZ(G)$-sets in $X$. Then the union $P_0 \cup P_0$ is also an $FZ(G)$-set.

The proof is broken up into several parts. We follow Drury’s construction, with some simplifications permitted by our current situation. We will suppose throughout that 1 is in none of our sets $P_0$, $P_0$, or $P_0$.

9.2. DEFINITION. Let $\alpha$ be a real number $> 1$, and let $P$ be a symmetric countable subset of $X$ not containing 1. Suppose that for every function $\beta \in \mathcal{B}_0(P)$ (see 8.6), there is a measure $\mu \in M_{\delta}(G)$ such that $|\mu| = |\beta| < \alpha$. Then $P$ is called an $FZ(\alpha)$ set.

9.3. THEOREM. Every $FZ(\alpha)$-set $P$ not containing 1 is an $FZ$ set for some $\alpha > 1$. \hfill \Box
Proof. We use Lemma 2.5, with \( E = M_1(G), F = B_\alpha(P), \) and \( T(\mu) \) defined as \( \mu^* \mid P \) for all \( \mu \in M_1(G) \). Theorem 5.10 implies that \( T(M_1(G)) = B_\alpha(P) \). Defining \( A_n \) as the set \( \{ \mu \in M_1(G) : ||\mu|| \leq n \} \) for \( n \in \{1, 2, 3, \ldots, n\} \), we see that 2.5 (i) and 2.5 (ii) hold, with \( B_\alpha(P) = A_n \). By Lemma 2.5, there is a positive integer \( m \) such that for all \( \mu \in B_\alpha(P) \) with \( ||\mu|| = 1 \), there is \( \mu \in M_1(G) \) such that \( ||\mu|| \leq m \) and \( \mu^* \mid P = \beta \). Thus we may take \( c = m \).

9.4. Theorem. Let \( P \) be an \( E \)-c set such that either (a) \( \chi^2 = 1 \) for all \( \chi \in E \) or (b) \( \chi^2 = 1 \) for all \( \chi \in E \). Let \( \alpha \) be a real number in \( [0, 1] \). There is a measure \( \mu \in M_1(G) \) such that \( \mu(\chi) = 1 \) for all \( \chi \in E \), \( ||\mu|| = \alpha \), and \( \mu^* \mid P = \beta \). Thus we have \( \mu = \mu_1 \mu_2 \) where \( \mu_1 \) is a \( \alpha \)-c measure and \( \mu_2 \) is a \( \beta \)-c measure.

The proof of 9.4 is not really very complicated, but involves several small computations. We first make a reduction.

9.5. If the conclusion of Theorem 9.4 holds for all finite \( E \)-c sets, then it holds for all \( E \)-c sets.

Proof. Given a finite symmetric subset \( \Phi \) of \( E \), let \( \mu_0 \) be such that \( \mu_0(\chi) = 1 \) for all \( \chi \in \Phi \), \( ||\mu_0|| \leq 1 \) for all \( \chi \in X \), and \( \mu_0(\chi) \leq \alpha \). Here \( A = 2 \) in Case (a) and \( A = 4 \) in Case (b). Under inclusion the family \( \{\Phi\} \) is a directed set. The set of measures \( \{\mu_0\} \) lies in the \( \alpha \)-c set \( \text{conv} \{P \mid \chi \in X \} \) and so the set \( \{\mu_0\} \) admits a \( \alpha \)-c convergent subnet with \( \mu = \mu_0 \). It is clear that \( \mu(\chi) \) is a cluster point of the complex-valued net \( \{\mu_0(\chi)\} \) for all \( \chi \in X \), and so the conclusion of Theorem 9.4 holds for \( P \).

9.6. Notation. For a fixed but arbitrary positive integer \( m \), let \( \Omega \) denote the multiplicative group \( \prod_{j=1}^{n} (1, -1)_0 \). For \( \omega \in \Omega \) and \( k \in \{1, 2, \ldots, m\} \) let \( \omega_k \) denote the \( k \)-th coordinate of \( \omega \). Let \( \Gamma \) denote the character group of \( \Omega \). Note that all \( \gamma \in \Gamma \) are real valued and that \( \omega = -\omega^{-1} \) for all \( \omega \in \Omega \).

A subset \( P \) of \( X \) is called asymmetric if \( \chi \in P \) and \( \chi^{-1} \in P \) imply \( \chi = \chi^{-1} \).

9.7. Lemma. Let \( \{X_1, \ldots, X_m\} \) be an asymmetric set in \( X \) such that \( P = \bigcup \{X_1, \ldots, X_m\} \) is an \( E \)-c set. For each \( \alpha \in \Gamma \), there is a measure \( \nu_\alpha \in M_1(G) \) such that:

(i) \( \nu_\alpha(\chi_k) = \omega_k \) for \( k \in \{1, 2, \ldots, m\} \).

(ii) \( ||\nu_\alpha|| = \nu_1(1) \leq \alpha \).

For \( \chi \in X \), let \( f_\chi \) be the function on \( \Omega \) such that \( f_\chi(\omega) = \nu_\alpha(\chi) \). Then we have

\[ \sum_{\omega \in \Omega} |f_\chi(\omega)| \leq \alpha^{\chi} \] for all \( \chi \in X \).

Proof. For \( \alpha \in \Omega \), let \( \mu_0 \) be a measure in \( M_1(G) \) such that

(1) \( \mu_0(\chi_k) = \omega_k \) for \( k \in \{1, 2, \ldots, m\} \)

and

(2) \( ||\mu_0|| \leq \alpha \).

For each \( \alpha \in \Omega \), define \( \nu_\alpha \) as

\[ \nu_\alpha = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \mu_0. \]

Plainly \( \nu_\alpha \) is in \( M_1(G) \), and (i) is immediate. To check (ii), we write

\[ \nu_\alpha(\chi_k) = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \mu_0(\chi_k) = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \omega_k = 2^{-\sum_{\omega \in \Omega} \omega_k} = \omega_k. \]

We now prove (iii). We have

\[ f_\chi(\omega) = \nu_\alpha(\chi) = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \mu_0(\chi), \]

and so for \( \gamma \in \Gamma \),

\[ f_\chi(\gamma) = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \gamma(\omega) = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \mu_0(\chi) \gamma(\omega) = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \nu_\gamma(\chi) \gamma(\omega). \]

This equality implies

\[ |f_\chi(\gamma)| = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} |\gamma(\omega)| \]

and so

\[ \sum_{\gamma} |f_\chi(\gamma)| = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \sum_{\gamma} |\nu_\gamma(\chi)| \gamma(\omega) = 2^{-\sum_{\omega \in \Omega} |\mu_0(\omega)|} \gamma(\omega). \]

The final sum in (3) is equal to \( 2^m \) if \( \alpha = \alpha \) and zero otherwise (see e.g. [8], Vol. I, (23.19)). Thus from (3) we have

\[ \sum_{\gamma} |f_\chi(\gamma)| = 2^{-m} \sum_{\omega \in \Omega} |\mu_0(\omega)| \chi \leq 2^{-m} \sum_{\omega \in \Omega} |\mu_0(\omega)||\chi| \leq \alpha^{\chi}. \]

This is (iii).

9.8. Proof of Theorem 9.4. As noted in 9.5, we may suppose that our set \( P \) is finite, as in 9.7. Define \( \sigma \) by

(1) \( \sigma = \frac{1}{2} \alpha^{\chi}. \)
We construct the measure \( \mu \) actually as a trigonometric polynomial times Haar measure on \( G \). To do this, define for every \( \omega \in \Omega \) the Riesz polynomial
\[
p_{\omega}(t) = \prod_{\alpha \in \Delta} (1 + \omega_{\alpha} \sigma(z_{\alpha} + x_{\alpha}^{\alpha})).
\]
It is trivial that \( p_{\omega} \geq 0 \). Finally we define \( p \) by
\[
p(t) = 2^{-m} \sum_{\omega \in \Omega} p_{\omega}(t) v_{\omega}(\theta),
\]
where the measures \( v_{\omega} \) are as in Lemma 9.7. Plainly \( p \) is nonnegative and real. We estimate the norm \( \|p\| \), using 9.7: (ii):
\[
\|p\|_{1} = 2^{-m} \sum_{\omega \in \Omega} \int_{G} p_{\omega}(y^{-1}z) d\nu_{\omega}(y) dx = 2^{-m} \sum_{\omega \in \Omega} \int_{G} p_{\omega}(t) dt v_{\omega}(\theta)
\leq c^{2} \int_{G} \left| \sum_{\omega \in \Omega} p_{\omega}(t) \right| dt.
\]
Write \( g_{\omega}(t) = \sigma(z_{\omega}(t) + x_{\omega}^{\omega}(t)) \). From (2), we see that the integrand in (4) can be rewritten as
\[
\int_{G} \left[ 1 + g_{\omega}(t) + 1 - g_{\omega}(t) \right] dt = 1.
\]
Thus (4) implies that
\[
\|p\|_{1} = c^{2}.
\]
We next compute \( p^{*}(x_{\omega}) \), first using 9.7. (i) to write
\[
p^{*}(x_{\omega}) = 2^{-m} \sum_{\omega \in \Omega} p_{\omega}^{*}(x_{\omega}) v_{\omega}^{*}(x_{\omega}) = 2^{-m} \sum_{\omega \in \Omega} \omega_{\omega} p_{\omega}(x_{\omega}).
\]
We multiply out the product (2), obtaining
\[
p_{\omega}(t) = \prod_{\omega \in \Omega} \omega_{\omega} \sigma(z_{\omega}(t) + x_{\omega}^{\omega}(t)).
\]
the sum in (7) being over all of the \( 2^{m} \) subsets \( S \) of \( \{1, 2, \ldots, m\} \), and the void product being taken as 1. Multiplying out the products in (7), we have
\[
p_{\omega}(t) = \sum_{S \in \Omega} \prod_{\omega \in \Omega} \omega_{\omega} \sigma(z_{\omega}(t) + x_{\omega}^{\omega}(t)),
\]
the inner sum in (8) being on all subsets \( T \) of \( S \). The Fourier transform \( p_{\omega}^{*}(x_{\omega}) \) is the coefficient of \( x_{\omega} \) in the polynomial (8). That is, we have
\[
p_{\omega}^{*}(x_{\omega}) = \sum_{T \subseteq \Omega} \omega_{T} \prod_{\omega \in T} \omega_{\omega}^{T},
\]
the sum in (9) being over all pairs \( (T, S) \) such that \( T \subseteq S \subseteq \{1, 2, \ldots, m\} \) and
\[
\prod_{\omega \in \Omega} \int_{G} \prod_{\omega \in \Omega} \omega_{\omega}^{T} = \sigma.
\]
We know no one of the numbers \( p_{\omega}^{*}(x_{\omega}) \), but we can nonetheless evaluate \( p^{*}(x_{\omega}) \). By (8) and (9), we have
\[
p^{*}(x_{\omega}) = \sum_{T \subseteq \Omega} \omega_{T} \prod_{\omega \in T} \omega_{\omega}^{T} v_{\omega}^{T}(x_{\omega}).
\]
Suppose that a given \( S \) contains an element \( j_{0} \neq k \). Then the mapping \( \omega \rightarrow \omega_{k} \prod_{j \neq k} \omega_{j} \) is a character of \( \Omega \) that is not identically 1 and so we have
\[
2^{-m} \sum_{\omega \in \Omega} \omega_{k} \prod_{j \neq k} \omega_{j} = 0.
\]
The only pairs \( (T, S) \) that can make nonzero contributions to the sum (11) are therefore \( (\emptyset, \emptyset) \), \( (\emptyset, \{k\}) \), and \( (\{k\}, \{k\}) \). The corresponding characters on the left side of (10) are 1, \( x_{\omega}^{\emptyset} \), and \( x_{\emptyset}^{\emptyset} \), respectively. Since \( 1 \neq x_{\omega}^{\emptyset} \), the pair \( (\emptyset, \{k\}) \) cannot yield \( x_{\emptyset}^{\{k\}} \). If \( 2 \) has order different from 2, then \( x_{\omega}^{\emptyset} \neq x_{\emptyset}^{\{k\}} \), the pair \( (\emptyset, \{k\}) \) contributes nothing, and only the pair \( (\{k\}, \{k\}) \) contributes to (11). If \( 2 \) has order 2, then both \( (\emptyset, \{k\}) \) and \( (\{k\}, \{k\}) \) contribute to (11). Thus we have:
\[
p^{*}(x_{\omega}) = \sigma 2^{-m} \sum_{\omega \in \Omega} \omega_{k} \prod_{j \neq k} \omega_{j} = \sigma \quad \text{if} \quad x_{\omega}^{\emptyset} \neq 1;
\]
\[
p^{*}(x_{\omega}) = \sigma + \sigma = 2 \sigma \quad \text{if} \quad x_{\omega}^{\emptyset} = 1.
\]
We now estimate \( |p^{*}(x)| \) for \( x \notin \bigcup_{j=1}^{n} \{e_{j} \} \), beginning with the obvious equality
\[
p^{*}(x) = 2^{-m} \sum_{\omega \in \Omega} p_{\omega}^{*}(x_{\omega}) v_{\omega}^{*}(x_{\omega}).
\]
We use the polynomial \( f_{x} \) on \( \Omega \) that was introduced in 9.7. We write \( f_{x} \) in its own Fourier expansion:
\[
v_{\omega}(x) = f_{\omega}(x) = \sum_{j \in \mathbb{Z}} f_{x}(\gamma) g_{\omega}(\gamma)(\gamma)\omega(\gamma).
\]
Substituting this value in (13) and citing 9.7. (iii), we find
\[
|p^{*}(x)| = \sum_{\omega \in \Omega} 2^{-m} \sum_{j \neq k} \left| \omega_{j} g_{\omega}(\gamma)(\gamma) \right| \omega_{k} \prod_{j \neq k} \omega_{j}
\leq \sum_{\omega \in \Omega} |\gamma(x)| \left| 2^{-m} \sum_{\omega \in \Omega} \omega_{k} \prod_{j \neq k} \omega_{j} \right| \cdot \omega(\gamma).
\]
\[
\leq \max \left| 2^{-m} \sum_{\omega \in \Omega} \omega_{k} \prod_{j \neq k} \omega_{j} \right| \cdot c. 
\]
We now estimate the quantity \([\ldots]\) in (14). Each \(\gamma \in \Gamma\) has the form
\[
\gamma(\omega) = \prod_{j=1}^{m} \omega_j, \quad \text{where } S \text{ is a subset of } \{1, 2, \ldots, m\}, \text{ or equivalently,}
\]
\[
\gamma(\omega) = \prod_{j=1}^{m} \omega_j^2, \quad \text{where } \{\omega_j\}_{j=1}^{m} \text{ is a sequence consisting of 0's and 1's. Again, write } g_0 = \sigma(\chi_0 + \chi_0^{-1}). \text{ Then by (2), we have}
\]
\[
\begin{align*}
2^{-m} \sum_{\omega_j} p_\omega(\gamma(\omega)) & = 2^{-m} \sum_{\omega_j} \int_0^\infty \left[ 1 + \omega_j g_0(t) \right] \chi(t) dt \cdot \prod_{j=1}^{m} \omega_j^2 \\
& = \int_0^\infty \left[ 2^{-m} \sum_{\omega_j} \left( \prod_{j=1}^{m} \omega_j^2 \right) \right] \chi(t) dt.
\end{align*}
\]

By Fubini’s theorem for \(n = (1, 2, 1, 3, 4)\), the expression \([\ldots]\) in the last line of (15) is equal to
\[
\prod_{j=1}^{m} \left[ 1 + \omega_j g_0(t) + \left( \omega_j g_0(t) \right)^{-1} \right] = \prod_{j=1}^{m} g_0(t) = \prod_{j=1}^{m} \sigma(\chi_0(t) + \chi_0^{-1}(t)).
\]
Hence, the last line of (15) is equal to
\[
\int_0^\infty \left[ \prod_{j=1}^{m} \sigma(\chi_0(t) + \chi_0^{-1}(t)) \right] \chi(t) dt.
\]
For \(S = \emptyset\), (16) is zero because \(\chi \neq 1\). For \(S = \emptyset\) for some \(i \in \{1, 2, \ldots, m\}\), (16) again vanishes because \(\chi \neq \chi_0\) and \(\chi \neq \chi_0^{-1}\). If card(S) = 2, then the integrand in (16) has absolute value \(\leq 4\sigma^2\), and so we see that for all \(\gamma \in \Gamma\), the inequality
\[
2^{-m} \sum_{\omega_j} p_\omega(\gamma(\omega)) \leq 4\sigma^2
\]
obtains. Going back to (14), we infer that
\[
|p^*(\gamma)| \leq 4\sigma^2
\]
for all \(\gamma \in P \cup \{1\}\).

At this point we distinguish between Cases (a) and (b) in the proof of 9.4. If \(\gamma^* = 1\) for all \(\chi \in P\), then we define \(\mu\) as the absolutely continuous nonnegative measure \(1/2\ \mu\). It is clear from (1) and (3) that
\[
|u| = \mu(1) \leq 2\sigma^2,
\]
from (12), that
\[
\mu^*(\gamma) = 1 \quad \text{for } \chi \in P,
\]
and from (17) that
\[
|\mu^*(\gamma)| \leq \frac{2}{\epsilon} \quad \text{for } \chi \notin X \cup (P \cup \{1\}).
\]
In Case (b), we define \(\mu\) as \(1/2\ \mu\) and obtain the desired results, using (12) instead of (12a) and getting \(\epsilon\) instead of \(\frac{1}{2}\ \epsilon\) in (20).
10.3. Remarks. (a) For \( X = Z \), Stečkin sets were introduced by Stečkin \cite{15}, under the name "\( K \) sets". Stečkin proved that all Stečkin subsets of \( Z \) are Sidon sets.

(b) The notion of Stečkin sets was extended to arbitrary Abelian groups by Rudin \cite{13}, pp. 124–126. Rudin proved, under some mild restrictions, that Stečkin subsets of arbitrary Abelian groups are Sidon sets. Rider \cite{12} has extended Rudin's result somewhat.

(c) The dissociate sets of Hewitt and Zuckerman (see e.g. \cite{8}, Vol. II, (31.13)), are plainly special cases of Stečkin sets.

10.4. Theorem. If \( A \) is a countably infinite Stečkin set, then \( A \cup A^{-1} \) is an \( E \)-set.

Proof. In view of 9.1, we may suppose that \( A \) has property (B). Splitting \( A \) into the subset of elements of order 2 and elements not of order 2, we may also suppose that

1. all or none of the elements of \( A \) have order 2.

2. We may also suppose that \( 1 \notin A \) and that

\[ x \in A \quad \text{and} \quad x^2 \neq 1 \quad \text{imply} \quad x^{-1} \notin A. \]

Let \( \beta \) be an arbitrary function in \( \Delta_0(A) \). According to Theorem 8.7, the present theorem will be proved if we can find a measure \( \nu \in \mathcal{M}_0(G) \) such that

\[ \sup \{ |\beta(x) - \nu(x)| : x \in A \cup A^{-1} \} < \delta < 1. \]

We enumerate \( A \) as a sequence \( (x_1, x_2, \ldots) \). If no element of \( A \) has order 2, we have

\[ A \cup A^{-1} = \{ x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}, \ldots \}; \]

all characters appearing in (4) are distinct by (2). If all elements of \( A \) have order 2, then clearly we have

\[ A \cup A^{-1} = A = \{ x_1, x_1, \ldots, x_n, \ldots \}. \]

In either case, we define \( \Delta_0 = \{ x_1, x_2, \ldots, x_n, \ldots \} \). We define certain Riesz polynomials. Let \( \tau \) be the number

\[ \tau = \frac{1}{A(2A + 1)}, \]

where \( A \) is the constant in 10.3. For every positive integer \( n \), define \( p_n \) as

\[ p_n = \int 1 + \tau \beta(x) x_1 + \tau \beta(x_1^{-1}) x_2^{-1} \]
Note our choice of \( r \) in (3), and infer from (14) that
\begin{equation}
|\mu'(\chi) - r \varphi(\chi)| < \frac{1}{4} r \text{ for all } \chi \in \Delta \cup \Delta^{-1},
\end{equation}
in Case (4). In like fashion, (12) leads to
\begin{equation}
|\mu'(-\chi) - 2r \varphi(\chi)| < \frac{1}{4} r \text{ for all } \chi \in \Delta \cup \Delta^{-1},
\end{equation}
in Case (4). From (15) it is evident that the measure \( r = \frac{1}{2} \mu \) satisfies (3)
with \( d = \frac{1}{2} \). From (15) it is evident that \( r = \frac{1}{2} \mu \) satisfies (3) with
\( d = \frac{1}{4} \).

References


Received September 23, 1971 (424)

STUDIA MATHEMATICA T. XLIV, (1975)

A divergent multiple Fourier series
of power series type

by

J. MARSHALL ASH and LAWRENCE GLUCK (Chicago, Ill.)

We present this paper to honor a great mathematician. The first co-author wishes to think of Professor Zygmund for the personal interest he has taken and the encouragement he has given over the pre- and post-doctoral years.

Abstract. A continuous complex-valued function on the torus whose (double) Fourier series diverges restrictedly rectangularly at every point has been constructed by Charles Fefferman. The present paper presents a function which has the above properties and whose Fourier series is of power series type (\( \alpha_m = 0 \) if \( m < 0 \) or \( n < 0 \)).

Charles Fefferman [2] has given an example of a continuous function \( F(x, y) \) defined on the torus \( T^2 \) with the property that the double Fourier series \( \sum_{m,n} \exp(i(mx + ny)) \) of \( F \) is everywhere restrictedly rectangularly divergent. This means that for each point \((x, y) \) and \( E > 1 \),
\begin{equation}
S_{M,N}(x, y) = \sum_{m,n \in \mathbb{Z}} a_{m,n} e^{i(mx + ny)}
\end{equation}

fails to tend to a limit as \( M \) and \( N \) tend to infinity with \( E^{-1} \leq M, N \leq E \).

In this paper we extend Fefferman's result by proving the following.

Theorem 1. There is a continuous complex-valued function \( H(x, y) \) on the torus whose double Fourier series is of power series type (\( \alpha_m = 0 \) if \( m < 0 \) or \( n < 0 \)) and is restrictedly rectangularly divergent everywhere.

On \([0, 2\pi] \times [0, 2\pi] \) set \( g(x, y) = g(x, y, \lambda) = \varphi(x) \varphi(y) e^{i\lambda xy} \) where \( \varphi \) is a \( C^\infty \) function equal to \( 0 \) if \( 0 \leq t \leq 1/40 \) or if \( 2\pi - 1/40 \leq t \leq 2\pi \) and to 1 if \( 1/20 \leq t \leq 2\pi - 1/20 \) with \( 0 \leq \lambda \leq 1 \) elsewhere on \([0, 2\pi] \). The real parameter \( \lambda \) is greater than 1. Then clearly \( g(x, y) \) is a \( C^\infty \) function on the torus \( T^2 \) obtained from \([0, 2\pi] \times [0, 2\pi] \) by identifications, and \( \|g\|_{\infty} = \sup |g(x, y)| = 1 \).