On optimal observability of linear systems with infinite-dimensional states

by

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Abstract. The note contains the concept of optimal observability of linear systems expressed in the language of Banach spaces. By linear system with the read-out map we shall understand a triple of Banach spaces and two continuous operators

\[ X_0 \xrightarrow{F} X \xrightarrow{G} Y. \]

The problem of the optimal observability of one linear quantity \( f \) is reduced to the minimum norm problem for the equation \( f = P^*Gq \). The note contains results concerning existence of optimal observability. There are also similar results for simultaneous observability of systems of linear quantities.

The problem of observability was considered by Kalman ([2],[4]). N. N. Krasovski in his book [5] has considered the problem of optimal observability for systems described by systems of ordinary differential equations. In paper [7] the following abstract schemas of observability and optimal observability was done. By a linear system with a read-out map we understand a triple of Banach spaces and two continuous linear operators

\[ (1) \quad X_0 \xrightarrow{F} X \xrightarrow{G} Y. \]

By a linear quantity \( f \) we understand a linear continuous functional. We say that \( f \) is observable if there is \( \varphi \in Y^* \) such that

\[ f = P^*G\varphi. \]

We say that \( f \) is optimally observable if there is \( \varphi \) satisfying (3) with minimal norm. In paper [7] it is shown that \( f \) is observable if and only if \( 0 \not\in \text{Spec}(P^*G) \) and that each observable \( f \) is optimally observable.

Let us now suppose that we are simultaneously observing a finite system of linear quantities \( F = (f_1, \ldots, f_n) \). Of course, we may assume without loss of generality that \( f_i \) are linearly independent. The system \( F \) is called observable if each \( f_i, i = 1, 2, \ldots, n \), is observable. In paper [7] the following definition of optimal observability of systems is introduced.
We regard $\mathcal{F} = (f_1, \ldots, f_n)$ as an operator $\mathcal{F}(x) = (f_1(x), \ldots, f_n(x))$ mapping $X$ into an $n$-dimensional Banach space $E$. We consider the following diagram

$$
\begin{array}{c}
X \xrightarrow{\pi} X \xrightarrow{\mathcal{F}} Y \xrightarrow{\mathcal{P}} E \xrightarrow{\mathcal{Q}} Y
\end{array}
$$

(3)

The system $\mathcal{F} = (f_1, \ldots, f_n)$ is observable if there is an $S$ such that diagram (3) is commutative. We say that $\mathcal{F}$ is optimally observable if there is an operator $S$ with minimal norm such that diagram (3) is commutative.

The following example shows that from the point of view of practice it is interesting to extend these definitions of observability and optimal observability to the case when we take as $E$ an infinite-dimensional Banach space.

**Example 1.** Let us consider a string described by the equation

$$
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(\pi, t) = 0.
$$

We are observing $u(x, 0)$ in the $L^2$-norm and measuring $u_t(x, t)$ in the $L^2$-norm. More precisely:

- $E$ is $L^2([0, \pi], \mathbb{R})$,
- $X$ is the space of pairs $g = (g_1, g_2), g_1, g_2 \in L^2([0, \pi], \mathbb{R}), ||g|| = ||g_1|| + ||g_2||$,
- $Y = \mathcal{L}([0, \pi] \times [0, 2\pi], \mathbb{R})$,
- $P$ is an operator which assigns to each pair $g = (g_1, g_2)$ the solution of equation (4) satisfying initial conditions $u(x, 0) = g_2, \frac{\partial u}{\partial t}(x, 0) = g_1$,
- $\mathcal{G}$ is the identity,
- $\mathcal{P}$ is the projection of $X$ onto the first coordinate.

Now we shall formulate certain facts concerning observability and optimal observability.

**Proposition 1.** If

(i) $\mathcal{G}X \to Y$ is a projection of $Y$,

(ii) $\mathcal{G}P$ is continuously invertible on $\mathcal{G}X$,

then each $\mathcal{P}$ is observable.

Proof. Let us denote by $Q$ a continuous projection operator mapping $\mathcal{G}X$ onto $\mathcal{G}P$. Let us put $S = \mathcal{P}(\mathcal{G}P)^{-1} Q$. It is easy to verify that the diagram (3) is commutative.

**Proposition 2.** Let $E = X$, and let $\mathcal{F}$ be the identity. Then $\mathcal{F}$ is observable if and only if (i) and (ii) hold.

Proof. Sufficiency follows trivially from Proposition 1.

Necessity. Let us suppose that there is an $S$ such that diagram (3) is commutative. The operator $Q = \mathcal{G}P$ is a projection of $Y$ onto $\mathcal{G}P$. Then

**Proposition 3.** Let $F$ be a projection of $X$ onto its subspace $X_0$. Then $F$ is observable if and only if

(i) $\mathcal{G}FX_0$ is a projection of $Y$,

(ii) $\mathcal{G}F$ is continuously invertible on $\mathcal{G}FX_0$.

Proof. Sufficiency. Let $Q$ be a projection of $Y$ onto $\mathcal{G}PX_0$. Let us put $S = (\mathcal{G}F)^{-1} Q$. Then diagram (3) is commutative.

Necessity. Let us suppose that there is an $S$ such that the diagram (3) is commutative. Let us put $Q = \mathcal{G}PS$. $Q$ is a projection of $Y$ onto $\mathcal{G}PX_0$.

There are observable systems which are not optimally observable, as follows from

**Example 2.** Let $X = Y = L^2([0, 1], \mathbb{R})$. Let $f$ be a functional $f(x) = \int_0^1 t(x(t)) dt$ defined on $X$. Let $E = X_0 = \{ x \in X; f(x) = 0 \}$. Let $G$ and $F$ be the identities in the respective spaces and let $P$ be the natural embedding of $X_0$ into $X$. It is easy to verify that for each positive $\varepsilon$ there is a projection of $Y$ onto $E$ with norm $2 + \varepsilon$ and there is no projection with norm 2.

**Theorem.** Let there be a separating topology $\tau$ in $E$ such that the unit ball $B_1 = \{ x \in E; ||x|| < 1 \}$ is compact in the $\tau$-topology. Then each observable $\mathcal{F}$ is optimally observable.

**Corollary.** If $E$ is either a reflexive space or is the conjugate of a Banach space then each observable $\mathcal{F}$ is optimally observable.

The proof of the theorem is based on the following notions and lemmas.

Let $Y$ and $E$ be two Banach spaces. Let there be given a topology $\tau$ in $E$ such that the unit ball in $E$ is compact in the topology $\tau$. We introduce in the space $B(Y \to E)$ of all continuous operators mapping $Y$ into $E$ a topology $\tilde{\tau}$ defined by the following family of neighbourhoods of zero: $U = (B \in B(Y \to E); B(Y \to E), \{ U_B \}, \{ \tilde{U}_{B, \varepsilon} \}$, where $(g_1, \ldots, g_n)$ is a finite system of elements of $Y$ and $(U_1, \ldots, U_n) is a finite system of neighbourhoods of zero in the topology $\tau$.

**Lemma 1.** The closed ball in $B(Y \to E)$, $B_1 = \{ B; ||B|| < 1 \}$, is compact in the topology $\tilde{\tau}$.

Proof. The proof is along the same line as the classical proof of the Alaoglu theorem (compare [1], Ch. V § 4).

We consider the product of balls with $\tau$-topology

$$
I = \prod_{x \in X} (\{ x; ||x|| < \varepsilon \} ||x||) - \varepsilon.
$$

**Lemma 2.** The set $I$ is compact in the topology $\tilde{\tau}$.
with the product topology. Since each $K_x$ is by assumption compact in $\tau$-topology, so, by Tichonov's theorem, the set $I$ is compact. We denote by $\pi$ the natural embedding of $K_x$ into $I$ given by the formula $\pi(B) = \{y\}$. It is easy to verify that $\pi$ is a homeomorphism of $K_x$ with $\tau$-topology in $I$. To complete the proof it is enough to show that $\pi(K_x)$ is closed in $I$.

The projection $pr_x$ onto the $x$-coordinate is a continuous operator in $I$, hence

$$A(x, y) = \{B \in I: pr_x B + pr_y B = pr_x B\}$$

and

$$B(a, x) = \{B \in I: pr_x B = a pr_x B, a = \text{scalar}\}$$

are closed sets. Therefore the set

$$\pi(K_x) = \bigcap_{x_0 \in \tau} A(x, y) \cap \bigcap_{a \in \mathbb{R}} B(a, y)$$

is also closed. ■

**Lemma 2.** The set $\mathcal{A}$ of all $S$ such that diagram (3) is commutative is closed in the $\tau$-topology.

**Proof.** Let $B(X \to E)$ be an arbitrary operator which does not belong to $\mathcal{A}$. It means that there is an element $x_0 \in X$ such that $F_{x_0} \neq S \circ GP_{x_0}$. Let us put $y = GP_{x_0}$ and let $U$ be a neighbourhood of $y$ in the $\tau$-topology such that $F_{x_0}$ is not contained in $U$. The set

$$U = \{B \in B(Y \to E): S \circ U\}$$

is an open set in the topology $\tau$ and moreover $U \cap \mathcal{A} = 0$. Now, since $S$ was arbitrary, it follows that the set $\mathcal{A}$ is closed. ■

**Proof of the theorem.** Let us suppose that $F$ is observable; it means that the set $\mathcal{A}$ of these $S$ for which the diagram (3) is commutative is not empty.

Let

$$r = \inf\{||S||: S \in \mathcal{A}\}.$$ 

Let $\varepsilon$ be a positive number and let

$$\mathcal{A}_\varepsilon = \{S: ||S|| \leq r + \varepsilon, S \in \mathcal{A}\}.$$ 

Clearly the sets $\mathcal{A}_\varepsilon$ are non-void and $\mathcal{A}_\varepsilon \subset \mathcal{A}, \varepsilon < \varepsilon'$.

By Lemma 1 and Lemma 2 the sets $\mathcal{A}_\varepsilon$ are compact. Therefore the intersection $\bigcap \mathcal{A}_\varepsilon$ is non-void. Let $S_{x_0}$ be an element of this intersection.

It is easy to verify that $S_{x_0} \in \mathcal{A}$ and $||S_{x_0}|| = r$. ■

The following obvious proposition gives an effective way for finding the element in $\mathcal{A}$ with minimal norm for certain simple cases.

**Proposition 4.** Let $F$ be a projection of norm one of $X_1$ onto its subspace $X_1$. Let $Q$ be a projection of norm one of $Y$ onto $GP_{X_1}$. Then $S = (GP)^{-1}Q$ is an element of $\mathcal{A}$ of minimal norm.

Let us apply Proposition 4 to Example 1. It is possible since all spaces in Example 1 are Hilbert spaces and orthogonal projections are projections of norm one.

Let us recall that (see [3], Ch. I § 3) the solution of equation (4)

with the initial condition $u(x, 0) = q_0$, $\partial u(x, 0) = 0$, is of the form

$$u(x, t) = \sum_{k=1}^{\infty} (A_k \cos k t + B_k \sin k t) \sin k x,$$

where

$$A_k = \frac{2}{\pi} \int_0^\pi q_k(x) \sin k x \, dx, \quad B_k = -\frac{2}{\pi} \int_0^\pi q_k(x) \cos k x \, dx.$$

Let us observe that $F = X_1$ is a subset of $X_0$ consisting of the elements of the form $(q_0, 0)$. Thus $GP_{X_1}$ is the set of functions of the type

$$\sum_{k=1}^{\infty} A_k \cos k t \sin k x.$$ 

Let $Q$ denote the orthogonal projection of $X$ onto $GP_{X_1}$:

$$Q = \sum_{k=1}^{\infty} (A_k \cos k t \sin k x) \cos k x.$$

The operator

$$Su = F(GP)^{-1}Qu = (GP)^{-1}Qu$$

$$= \sum_{k=1}^{\infty} \left( \frac{1}{2} \int_0^\pi u(x, t) \cos k t \sin k x \, dx \right) \sin k x$$

is an element of $\mathcal{A}$ of minimal norm.

**References**


О приближении функций класса $\varphi(L)$

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Посвящается выделяемой академику Антонио Даглиру к пятидесятилетию его научной работы

Резюме. В статье устанавливаются необходимые и достаточные условия возможности приближения функций классов $\varphi(L)$ и $\varphi^+(L)$ алгебраическими полиномами с рациональными коэффициентами или непрерывными функциями. Эти найденные условия для класса $\varphi(L)$ принципиально отличны от соответствующих условий для класса $\varphi^+(L)$.

§ 1. Введение. Пусть $\varphi$-свойство четных, неотрицательных, конечных и неубывающих на поддомене $(0, \infty)$ функций $\varphi(t) = \lim_{t \to \infty} \varphi(t) = \varphi(\infty) = \infty$. Через $\varphi(L)$ будем обозначать множество всех тех измеримых на отрезке $[0, 1]$ функций $f(x)$, для которых

$$\int_0^1 \varphi(f(x))dx < \infty.$$ 

Ниже нам понадобится

Определение. Пусть $\mathcal{D} = \{\varphi(x)\}$ — некоторый класс конечных и измеримых функций, определенных на отрезке $[0, 1]$. Мы говорим, что класс $\mathcal{D}$ обладает свойством $W$ (свойством Вейерштрасса) относительно множества $\mathcal{G} \subset \varphi(L)$, если для вейерштрасса функции $f \in \mathcal{D}$ и вейерштрасса числа $\varepsilon > 0$ найдется функция $\tau(x) \in \mathcal{D}$ такая, что

$$\int_0^1 \varphi(f(x) - \tau(x))dx < \varepsilon.$$ 

В предлагаемой статье будут указаны необходимые и достаточные условия на функцию $\varphi(t)$, при которых тот или иной класс $\mathcal{D}$ обладает свойством $W$ относительно множества $\varphi(L)$, или же ненхотного его подмножества $\mathcal{G}$.

§ 2. Вспомогательные утверждения. В этом параграфе мы установим ряд лемм.