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On optimal observability of linear systems with infinite-dimensional states

by

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Abstract. The note contains the concept of optimal observability of linear systems expressed in the language of Banach spaces. By linear system with the read-out map we shall understand a triple of Banach spaces and two continuous operators

$$X_0 \xrightarrow{P} X \xrightarrow{G} Y.$$

The problem of the optimal observability of one linear quantity f is reduced to the minimum norm problem for the equation $f = P^*G^*\varphi$. The note contains results concerning existence of optimal observability. There are also similar results for simultaneous observations of systems of linear quantities.

The problem of observability was considered by Kalman ([2]–[4]). N. N. Krasowski in his book [5] has considered the problem of optimal observability for systems described by systems of ordinary differential equations. In paper [7] the following abstract schema of observability and optimal observability was done. By a *linear system with a read-out map* we understand a triple of Banach spaces and two continuous linear operators

$$X_0 \xrightarrow{P} X \xrightarrow{G} Y. \tag{1}$$

By a *linear quantity* f we understand a linear continuous functional. We say that $f \in X_0^*$ is *observable* if there is $\varphi \in Y^*$ such that

$$f = P^*G^*\varphi. \tag{2}$$

We say that f is *optimally observable* if there is φ satisfying (2) with minimal norm. In paper [7] it is shown that f is observable if and only if $0 \notin \overline{GP\{x\colon f(x) = 1\}}$ and that each observable f is optimally observable.

Let us now suppose that we are simultaneously observing a finite system of linear quantities $F = (f_1, \ldots, f_n)$. Of course, we may assume without loss of generality that f_i are linearly independent. The *system* F is called *observable* if each f_i, $i = 1, 2, \ldots, n$, is observable. In paper [7] the following definition of optimal observability of systems is introduced.

We regard $F = (f_1, \ldots, f_n)$ as an operator $F(x) = (f_1(x), \ldots, f_n(x))$ mapping X_0 into an n-dimensional Banach space E. We consider the following diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{P} & X & \xrightarrow{G} & Y \\ & {\scriptstyle F}\searrow & & \swarrow{\scriptstyle S} & \\ & & E & & \end{array} \tag{3}$$

The system $F = (f_1, \ldots, f_n)$ is observable if there is an S such that diagram (3) is commutative. We say that F is optimally observable if there is an operator S with minimal norm such that diagram (3) is commutative.

The following example shows that from the point of view of practice it is interesting to extend these definitions of observability and optimal observability to the case when we take as E an infinite-dimensional Banach space.

Example 1. Let us consider a string described by the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0, t) = u(\pi, t) = 0. \tag{4}$$

We are observing $u(x, 0)$ in the L^2-norm and measuring $u(x, t)$ in the L^2-norm. More precisely:

E is $L^2[0, \pi]$,

X_0 is the space of pairs $q = (q_0, q_1)$, $q_0, q_1 \in L^2[0, \pi]$, $\|q\| = \sqrt{\|q_0\|^2 + \|q_1\|^2}$,

$X = Y$ is $L^2([0, \pi] \times [0, 2\pi])$,

P is an operator which assigns to each pair $q = (q_0, q_1)$ the solution of equation (4) satisfying initial conditions $u(x, 0) = q_0$, $\dfrac{\partial u}{\partial t}(x, 0) = q_1$,

G is the identity,

F is the projection of X_0 onto the first coordinate.

Now we shall formulate certain facts concerning observability and optimal observability.

Proposition 1. *If*

(i) *GPX_0 is a projection of Y,*

(ii) *GP is continuously invertible on GPX_0,*

then each F is observable.

Proof. Let us denote by Q a continuous projection operator mapping Y onto GPX_0. Let us put $S = F(GP)^{-1}Q$. It is easy to verify that the diagram (3) is commutative. ■

Proposition 2. *Let $E = X_0$ and let F be the identity. Then F is observable if and only if* (i) *and* (ii) *hold.*

Proof. Sufficiency follows trivially from Proposition 1.

Necessity. Let us suppose that there is an S such that diagram (3) is commutative. The operator $Q = GPS$ is a projection of Y onto GPX_0. ■

Proposition 3. *Let F be a projection of X_0 onto its subspace X_1. Then F is observable if and only if*

(i) *GPX_1 is a projection of Y,*

(ii) *GP is continuously invertible on GPX_1.*

Proof. Sufficiency. Let Q be a projection of Y onto GPX_1. Let us put $S = (GP)^{-1}Q$. Then diagram (3) is commutative.

Necessity. Let us suppose that there is an S such that the diagram (3) is commutative. Let us put $Q = GPS$. Q is a projection of Y onto GPX_1. ■

There are observable systems which are not optimally observable, as follows from

Example 2. Let $X = Y = L^1[0, 1]$. Let f be a functional $f(x) = \int_0^1 tx(t)\,dt$ defined on X. Let $E = X_0 = \{x \in X: f(x) = 0\}$. Let G and F be the indentities in the respective spaces and let P be the natural embedding of X_0 into X. It is easy to verify that for each positive ε there is a projection of Y onto E with norm $2+\varepsilon$ and there is no projection with norm 2.

Theorem. *Let there be a separating topology τ in E such that the unit ball $K_1 = \{z \in E: \|z\| \leqslant 1\}$ is compact in the τ-topology. Then each observable F is optimally observable.*

Corollary. *If E is either a reflexive space or is the conjugate of a Banach space then each observable F is optimally observable.*

The proof of the theorem is based on the following notions and lemmas.

Let Y and E be two Banach spaces. Let there be given a topology τ in E such that the unit ball in E is compact in the topology τ. We introduce in the space $B(Y \to E)$ of all continuous operators mapping Y into E a topology $\bar{\tau}$ defined by the following family of neighbourhoods of zero: $U = \{B \in B(Y \to E): By_i \in U_i$, where $(y_1, \ldots, y_n)$ is a finite system of elements of Y and $(U_1, \ldots, U_n)$ is a finite system of neighbourhoods of zero in the topology $\tau\}$.

Lemma 1. *The closed ball in $B(Y \to E)$, $K_r = \{B: \|B\| \leqslant r\}$, is compact in the topology $\bar{\tau}$.*

Proof. The proof is going along the same line as the classical proof of the Alaoglu theorem (compare [1], Ch. V § 4).

We consider the product of balls with τ-topology

$$I = \prod_{y \in Y} \{z: \|z\| \leqslant r\,\|y\|\}$$

with the product topology. Since each K_r is by assumption compact in τ-topology, so, by Tichonov's theorem, the set I is compact. We denote by n the natural embedding of K_r into I given by the formula $n(B) = \{B(y)\}$. It is easy to verify that n is a homeomorphism of K_r with $\bar{\tau}$-topology in I. To complete the proof it is enough to show that $n(K_r)$ is closed in I. The projection pr_z onto the z-coordinate is a continuous operator in I, hence

$$A(x, y) = \{B \in I:\ \mathrm{pr}_x B + \mathrm{pr}_y B = \mathrm{pr}_{x+y} B\}$$

and

$$B(a, x) = \{B \in I:\ \mathrm{pr}_{ax} B = a\,\mathrm{pr}_x B,\ a\text{-scalar}\}$$

are closed sets. Therefore the set

$$n(K_r) = \bigcap_{x,y\in Y} A(x, y) \cap \bigcap_{a\text{-scalars}} \bigcap_{y\in Y} B(a, y)$$

is also closed. ■

LEMMA 2. *The set $\mathfrak{A}$ of all S such that diagram* (3) *is commutative is closed in the $\bar{\tau}$-topology.*

Proof. Let $S_0 \in B(Y \to E)$ be an arbitrary operator which does not belong to $\mathfrak{A}$. It means that there is an element $x_0 \in X_0$ such that $Fx_0 \neq S_0 GPx_0$. Let us put $y = GPx_0$ and let U be a neighbourhood of y in the τ-topology such that Fx_0 is not contained in U. The set

$$\mathcal{U} = \{S \in B(Y \to E):\ Sy \in U\}$$

is an open set in the topology $\bar{\tau}$ and moreover $\mathcal{U} \cap \mathfrak{A} = 0$. Now, since S_0 was arbitrary, it follows that the set $\mathfrak{A}$ is closed. ■

Proof of the theorem. Let us suppose that F is observable; it means that the set $\mathfrak{A}$ of these S for which the diagram (3) is commutative is not empty.

Let

$$r = \inf\{\|S\|:\ S \in \mathfrak{A}\}.$$

Let ε be a positive number and let

$$\mathfrak{A}_\varepsilon = \{S:\ \|S\| \leqslant r + \varepsilon,\ S \in \mathfrak{A}\}.$$

Clearly the sets $\mathfrak{A}_\varepsilon$ are non-void and $\mathfrak{A}_\varepsilon \subset \mathfrak{A}_{\varepsilon'}$, $\varepsilon < \varepsilon'$.

By Lemma 1 and Lemma 2 the sets $\mathfrak{A}_\varepsilon$ are compact. Therefore the intersection $\bigcap_{\varepsilon>0} \mathfrak{A}_\varepsilon$ is non-void. Let S_0 be an element of this intersection. It is easy to verify that $S_0 \in \mathfrak{A}$ and $\|S_0\| = r$. ■

The following obvious proposition gives an effective way for finding the element in $\mathfrak{A}$ with minimal norm for certain simple cases.

PROPOSITION 4. *Let F be a projection of norm one of X_0 onto its subspace X_1. Let Q be a projection of norm one of Y onto GPX_1. Then $S = (GP)^{-1}Q$ is an element of $\mathfrak{A}$ of minimal norm.*

Let us apply Proposition 4 to Example 1. It is possible since all spaces in Example 1 are Hilbert spaces and orthogonal projections are projections of norm one.

Let us recall that (see [8], Ch. I § 3) the solution of equation (4) with the initial condition $u(x, 0) = q_0$, $\frac{\partial}{\partial t}u(x, 0) = q_1$, is of the form

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt) \sin nx, \tag{5}$$

where

$$A_n = 2\int_0^{\pi} q_0(x) \sin nx\, dx, \qquad B_n = \frac{2}{n}\int_0^{\pi} q_1(x) \sin nx\, dx.$$

Let us observe that $E = X_1$ is a subset of X_0 consisting of the elements of the form $(q_0, 0)$. Thus GPX_1 is the set of functions of the type

$$\sum_{n=1}^{\infty} A_n \cos nt \sin nx. \tag{6}$$

Let Q denote the orthogonal projection of Y onto GPX_1:

$$Qu = \sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{\pi} \int_0^{\pi} \int_0^{2\pi} u(x, t) \cos nt \sin nx\, dt\, dx\right) \cos nt \sin nx.$$

The operator

$$Su = F(GP)^{-1}Qu = (GP)^{-1}Qu$$
$$= \sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{\pi} \int_0^{\pi} \int_0^{2\pi} u(x, t) \cos nt \sin nx\, dt\, dx\right) \sin nx$$

is an element of $\mathfrak{A}$ of minimal norm.

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О приближении функций класса $\varphi(L)$

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Посвящаеться выдающемуся аналитику Антони Зигмунду к пятидесятилетию его научной работы

Резюме. В статье устанавливаются необходимые и достаточные условия возможности приближения функций классов $\varphi(L)$ и $\varphi^+(L)$ алгебраическими полиномами с рациональными коэффициентами или же непрерывными функциями. Эти найденные условия для класса $\varphi(L)$ принципиально отличны от соответствующих условий для класса $\varphi^+(L)$.

§ 1. Введение. Пусть Φ-совокупность четных, неотрицательных, конечных и неубывающих на полупрямой $[0,\infty)$ функций $\varphi(t)$ с $\lim\limits_{t\to\infty}\varphi(t) = \varphi(\infty) = \infty$. Через $\varphi(L)$ будем обозначать множество всех тех измеримых на отрезке $[0,1]$ функций $f(x)$, для которых

$$\int_0^1 \varphi\big(f(x)\big)\,dx < \infty.$$

Ниже нам понадобится

Определение. Пусть $\mathscr{D} \equiv \{\tau(x)\}$ — некоторый класс конечных и измеримых функций, определенных на отрезке $[0,1]$. Мы говорим, что класс $\mathscr{D}$ обладает свойством W (свойством Вейерштрасса) относительно множества $G\subset\varphi(L)$, если для всякой функции $f\in G$ и всякого числа $\varepsilon>0$ пайдется функция $\tau(x)\in\mathscr{D}$ такая, что

$$\int_0^1 \varphi\big(f(x)-\tau(x)\big)\,dx < \varepsilon.$$

В предлагаемой статье будут указаны необходимые и достаточные условия на функцию $\varphi(t)$, при которых тот или иной класс $\mathscr{D}$ обладает свойством W относительно множества $\varphi(L)$, или же некоторого его подмножества G.

§ 2. Вспомогательные утверждения. В этом параграфе мы установим ряд лемм.