The map $L^2(\Omega^*) \to \mathscr{L}^2(T_\Omega)$ *defined by*

$$\varphi \to f(z_1) = \int_{\Omega^*} e^{2\pi i\langle\lambda, z_1\rangle}\varphi(\lambda)\, M^*(2\lambda)^{-\frac{1}{2}}d\lambda$$

is a Hilbert space isomorphism. The integral converges absolutely for all $z_1 \in T_\Omega$.

The proof can be found in [5].

Lemma 6.1 is an analogue of Lemma 2.2. Similarly it is easy to prove analogues of Lemmas 2.1 and 3.1, and finally one obtains the following results. (The notation is the same as in Section 4.)

THEOREM 6.1. *The map* $L^2(\Omega^* \times \boldsymbol{R}^{n_2}) \to \mathscr{L}^2(D)$ *which carries* $\varphi \in L^2(\Omega^* \times \boldsymbol{R}^{n_2})$ *to*

$$F(z) = \int_{\Omega^* \times \boldsymbol{R}^{n_2}} \varphi(\lambda, a)\overline{\chi_z(\lambda, a)}\, M^*(2\lambda)^{-\frac{1}{2}}d\lambda\, da \tag{6.1}$$

and the map $\hat{L}^2 \to H^2(D)$ *which carries* $A \in \hat{L}^2$ *to*

$$F(z) = \int_{\Omega^*} e^{2\pi i\langle\lambda, z_1\rangle} A(\lambda, z_2)\, M^*(2\lambda)^{-\frac{1}{2}}d\lambda \tag{6.2}$$

are Hilbert space isomorphisms. The integrals (6.1), (6.2) *converge absolutely for every fixed* $z = (z_1, z_2) \in D$.

THEOREM 6.2. *The Bergman kernel* K *of* D, *i.e. the reproducing kernel of* $\mathscr{L}^2(D)$, *is given by*

$$K(z, w) = \int_{\Omega^*} e^{-2\pi\langle\lambda, \varrho(z,w)\rangle}\frac{(\det B_\lambda)}{M^*(2\lambda)}d\lambda.$$

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The initial value problem for parabolic equations with data in $L^p(R^n)$

by

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Abstract. Suppose $u(x, t)$ belongs to the class of functions having derivatives, $D_x^a u(x,t)$, $|a| \leqslant 2b$, and $D_t u(x,t)$ in $L^p(R^n \times (0, T))$. Assume that $Lu(x,t) = 0$ where $L = \sum_{|a|\leqslant 2b} a_a(x,t)D_x^a - D_t$ is a parabolic operator with coefficients bounded and measurable and for $|a| = 2b$ uniformly continuous. Let $\omega(s)$ denote the modulo of continuity of a coefficient of order $2b$. If $\int_0^1 \frac{\omega(s)^{4/3}}{s}ds < \infty$, then we show that for $1 < p < \infty$ $\|u(\cdot, t)\|_{L^p(R^n)} \leqslant c\|u(\cdot, 0)\|_{L^p(R^n)}$. This á priori estimate is used to resolve uniquely the initial value problem, $Lu(x,t) = 0$, $t > 0$, and $u(x, 0) = g(x)$ where $g(x) \in L^p(R^n)$.

1. Introduction. In this paper we consider the initial value problem for the uniformly parabolic operator $L = \sum_{|a|\leqslant 2b} a_a(x,t)D_x^a - D_t$ when the initial data, $g(x)$, belongs to $L^p(R^n)$, $1 < p < \infty$, and when the coefficients, $a_a(x,t)$, are bounded, measurable, and for $|a| = 2b$, uniformly continuous over the strip $S_T = R^n \times (0, T)$. As usual b is a positive integer, x is a point in R^n, $t \in (0, T)$, $a = (a_1, \ldots, a_n)$ is an n-tuple of non-negative integers, $D_x^a = \partial^{a_1}/\partial x_1^{a_1} \ldots \partial^{a_n}/\partial x_n^{a_n}$, and $|a| = \sum_{i=1}^n a_i$. By the uniform parabolicity of L we mean that the real part of the form, $A(x, t; \xi) = \sum_{|a|=2b} a_a(x,t)(i\xi)^a$, satisfies the condition, $\mathrm{Re}\big(A(x,t;\xi)\big) \leqslant -\eta|\xi|^{2b}$, with $\eta > 0$ and independent of $(x, t) \in S_T$.

Given a function $g(x) \in L^p(R^n)$, $1 < p < \infty$, we consider the problem of finding a unique function $u(x, t)$ such that

(I)
(i) for every δ, $0 < \delta < T$, $D_x^a u$, $|a| \leqslant 2b$, and $D_t u$ exist in the sense of distributions over $S_{\delta,T} = R^n \times (\delta, T)$ and belong to $L^p(S_{\delta,T})$;
(ii) $Lu = 0$ in S_T;
(iii) $\lim_{t\to 0+}\|u(\cdot, t) - g(\cdot)\|_{L^p(R^n)} = 0$.

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Given a strip $S_{T_0,T_1} = R^n \times (T_0, T_1)$ we will denote by $W^p_{2b,1}(S_{T_0,T_1})$ the closure of $C_0^\infty(R^{n+1})$ in the norm,

$$\|u\|_{W^p_{2b,1}(S_{T_0,T_1})} = \sum_{|\alpha|\leqslant 2b} \|D_x^\alpha u\|_{L^p(S_{T_0,T_1})} + \|D_t u\|_{L^p(S_{T_0,T_1})}.$$

By $\mathring{W}^p_{2b,1}(S_{T_0,T_1})$ we mean the closure of $C_0^\infty(R^n \times (T_0, \infty))$ in the above norm. (As noted above S_T denotes the strip $S_{(0,T)}$.)

To solve the initial value problem, (I), we consider the problem of estimating the L^p-norm over R^n at time t of a solution $u(x,t) \in \mathring{W}^p_{2b,1}(S_T)$, $1 < p < \infty$, to the problem $Lu = 0$ by the same norm on $u(x, 0)$. That is, we consider the question: If $u \in W^p_{2b,1}(S_T)$, $1 < p < \infty$, satisfies the equation $Lu = 0$, then is it true that

$$\text{(II)} \qquad \|u(\cdot,t)\|_{L^p(R^n)} \leqslant C_{p,L} \|u(x,0)\|_{L^p(R^n)},$$

$C_{p,L}$ independent of $t \in (0, T)$?

OBSERVATION 1. *An affirmative answer to problem* (II) *implies there exists a unique solution to problem* (I) *when* $1 < p < \infty$.

Proof. Given $g(x) \in L^p(R^n)$, $1 < p < \infty$, let $\{g_k\}$ denote a sequence of functions, $g_k \in C_0^\infty(R^n)$, such that $g_k \to g$ in $L^p(R^n)$. Let u denote the unique solution in $W^p_{2b,1}(S_T)$ to the problem $Lu_k = 0$, $u_k(x, 0) = g_k(x)$ (see [5], [6]) and consider the sequence $\{tu_k\}$. The function $tu_k(x,t) \in \mathring{W}^p_{2b,1}(S_T)$ and L is an isomorphism from the space $\mathring{W}^p_{2b,1}(S_T)$ to $L^p(S_T)$ (see [6]). Hence

$$\|t(u_k - u_j)\|_{W^p_{2b,1}(S_T)} \leqslant C_p \big\|L\big(t(u_k - u_j)\big)\big\|_{L^p(S_T)} = C_p \|(u_k - u_j)\|_{L^p(S_T)}.$$

Using the estimate (II) we see that

$$\|u_k - u_j\|_{L^p(S_T)} \leqslant C_p T^{1/p} \|g_k - g_j\|_{L^p(R^n)} \to 0$$

as $k, j \to \infty$. This means that the sequence $\{tu_k\}$ is Cauchy in $W^p_{2b,1}(S_T)$ and hence there exists a function, $u(x,t)$, such that for every δ, $0 < \delta < T$, $u_k \to u$ in $W^p_{2b,1}(S_{\delta,T})$. Clearly u satisfies (i) and (ii) of problem (I). Now,

$$\|u(\cdot,t) - g(\cdot)\|_{L^p(R^n)} \leqslant \|u(\cdot,t) - u_k(\cdot,t)\|_{L^p(R^n)} + \|u_k(\cdot,t) - g_k(\cdot)\|_{L^p(R^n)} + \\ + \|g_k - g\|_{L^p(R^n)}.$$

$\|u(\cdot,t) - u_k(\cdot,t)\|_{L^p(R^n)} = \lim_{m\to\infty} \|u_m(\cdot,t) - u_k(\cdot,t)\|_{L^p(R^n)}$ and using (II), this last limit $\leqslant C_p \lim_{m\to\infty} \|g_m - g_k\|_{L^p(R^n)} = C_p \|g - g_k\|_{L^p(R^n)}$. Hence $\|u(\cdot,t) - g(\cdot)\|_{L^p(R^n)} \leqslant C_p \|g - g_k\|_{L^p(R^n)} + \|u_k(\cdot,t) - g_k(\cdot)\|_{L^p(R^n)}$. The first term on the left-hand side is small with k, independent of t. With k fixed the second term tends to zero as $t \to 0+$. Hence u satisfies condition (iii) of (I).

To begin discussing problem (II) we need to introduce various moduli of continuity of $a_\alpha(x,t)$ for $|\alpha| = 2b$. Let

$$\omega_1(\delta) = \sup_{\substack{\alpha \\ |x-y|\leqslant\delta \\ t\in(0,T)}} |a_\alpha(x,t) - a_\alpha(y,t)|, \qquad \omega_2(\delta) = \sup_{\substack{\alpha \\ x\in R^n \\ 0<t-s<\delta \\ t,s\in(0,T)}} |a_\alpha(x,t) - a_\alpha(x,s)|,$$

and $\omega(\delta) = \omega_1(\delta^{1/2b}) + \omega_2(\delta)$. $\omega(\delta)$ can be considered the modulus of continuity in the variables (x,t) while ω_1 and ω_2 are respectively the moduli of continuity in x, uniformly in t, and in t, uniformly in x.

It is known that if $\omega_1(\delta) \leqslant c\delta^\gamma$, $0 < \gamma \leqslant 1$, or if $\omega_2(\delta) \leqslant c\delta^\gamma$, $0 < \gamma \leqslant 1$, then the á priori estimate, (II), is valid. The former is due to the work of S. D. Eidelman (see [3] and [7], Chapter IX) and the latter due to the work of T. Kato (see [10]). However, if neither ω_1 or ω_2 satisfy a Dini condition, i.e. if $\int_0^1 \frac{\omega_i(\delta)}{\delta} d\delta = \infty$, $i = 1, 2$, then an example of A. M. Illin ([9]) shows that one cannot in general obtain inequality (II) in the case $p = 1$ even for very good data with a constant c uniform for all parabolic operators with the same parameter of parabolicity, η, and with the same modulus of continuity, $\omega(\delta)$, for the highest order coefficients. Nevertheless, for the case $1 < p < \infty$, the estimate, (II), remains an open problem and the primary object of this paper is to give a general condition on $\omega(\delta)$ for which it is certainly possible that $\int_0^1 \frac{\omega(\delta)}{\delta} d\delta = \infty$ but for which the à priori estimate, (II), is valid for $1 < p < \infty$, and hence for which problem (I) is solvable. (See Theorem 1.)

Before proceeding with a statement and proof of our main result, two more observations regarding the à priori estimate are necessary.

OBSERVATION 2. *Estimate* (II) *is valid in the strip* S_T *if there exists* T_0, $0 < T_0 \leqslant T$, *independent of* u, *for which it is valid for* $t \in (0, T_0)$.

Proof. We are now assuming that for each $u \in W^p_{2b,1}(S_T)$ for which $Lu = 0$, the estimate, $\|u(\cdot,t)\|_{L^p(R^n)} \leqslant C_{p,L} \|u(\cdot,0)\|_{L^p(R^n)}$ is valid for every $t \in (0, T_0)$ with C_p independent of t but possibly depending on T_0. Now let $\psi(t) \in C^\infty[0,\infty)$ be a function which equals 0 in $[0, T_0/2]$ and 1 in $[T_0, \infty)$. For $T_0 \leqslant t < T$,

$$\|u(\cdot,t)\|_{L^p(R^n)} = \|(u\psi)(\cdot,t)\|_{L^p(R^n)} \leqslant C_T \|u\psi\|_{W^p_{2b,1}(S_T)} \leqslant C_{p,T} \|L(u\psi)\|_{L^p(S_T)} \\ \leqslant C_{p,T,T_0} \big\|(\|u(\cdot,t)\|_{L^p(R^n)})\big\|_{L^\infty(0,T_0)} \leqslant C_{p,T,T_0} \|u(\cdot,0)\|_{L^p(R^n)}.$$

Given numbers $M > 0$, $\eta > 0$ we denote by $\wp_{M,\eta}$ the class of all uniformly parabolic operators $L = \sum_{|\alpha|\leqslant 2b} a_\alpha(x,t) D_x^\alpha - D_t$ such that

$$\sup_{\substack{\alpha \\ (x,t)\in S_T}} |a_\alpha(x,t)| \leqslant M, \qquad \operatorname{Re}\Big(\sum_{|\alpha|=2b} a_\alpha(x,t)(i\xi)^\alpha\Big) \leqslant -\eta|\xi|^{2b},$$

and whose coefficients $a_\alpha(x,t)$ for $|\alpha| = 2b$ are uniformly continuous over S_T.

For a given operator $L \in \wp_{M,\eta}$ we denote by $\omega_{1,L}, \omega_{2,L}$ and ω_L, the various moduli of continuity of the highest order coefficients of L as defined above. When no confusion arises we will drop the subscript L from these functions. We will call a coefficient of L smooth over an open set Ω if it belongs to $C^\infty(\Omega)$ and has any order derivative bounded there.

OBSERVATION 3. *Suppose $L \in \wp_{M,\eta}$. Assume that for each $\tilde{L} \in \wp_{M,\eta}$ with smooth coefficients of order $2b$ over R^{n+1} and with $\omega_{\tilde{L}} \leqslant \omega_L$ estimate* (II) *is valid with $C_{p,\tilde{L}} = C(p, \omega_{\tilde{L}})$ an increasing function of $\omega_{\tilde{L}}$, i.e. if $\omega(\delta) \leqslant \omega_*(\delta)$ for every δ, then $C(p,\omega) \leqslant C(p,\omega_*)$. Then* (II) *is valid for L provided $C(p,\omega_L) < \infty$.*

Proof. Let $L = \sum_{|\alpha|\leqslant 2b} a_\alpha(x,t) D_x^\alpha - D_t$. For $|\alpha| = 2b$, $a_\alpha(x,t)$ is uniformly continuous over $\overline{S_T}$. By defining

$$a_\alpha(x,t) = \begin{cases} a_\alpha(x,0) & \text{for } t \leqslant 0, \\ a_\alpha(x,T) & \text{for } t \geqslant T, \end{cases}$$

we may assume that for $|\alpha| = 2b$, a_α is uniformly continuous over R^{n+1} and that the new modulus of continuity of L satisfies the condition $C(p,\omega_L) < \infty$. Now let $\varphi(x,t) \in C_0^\infty(R^{n+1})$, $\varphi \geqslant 0$, $\int \varphi\, dx\, dt = 1$, and for k a positive integer set $a_{\alpha,k}(x,t) = k^{n+1} \int_{R^{n+1}} \varphi\big(k(y,s)\big) a_\alpha(x-y, t-s)\, dy\, ds$. Also set $L_k = \sum_{|\alpha|=2b} a_{\alpha,k}(x,t) D_x^\alpha - D_t + \sum_{|\alpha|<2b} a_\alpha(x,t) D_x^\alpha$.

$L_k \in \wp_{M,\eta}$ and the coefficients of order $2b$ are smooth over R^{n+1}. Let u and u_k denote respectively solutions in $W^p_{2b,1}(S_T)$ of $Lu = 0$ and $L_k u_k = 0$ with $u_k(x,0) = u(x,0)$. $u - u_k \in \mathring{W}^p_{2b,1}(S_T)$ and hence

$$\|u-u_k\|_{W^p_{2b,1}(S_T)} \leqslant C_p \|L_k(u-u_k)\|_{L^p(S_T)}$$

with C_p independent of k. Therefore

$$\|u-u_k\|_{W^p_{2b,1}(S_T)} \leqslant C_p \|(L-L_k)u\|_{L^p(S_T)} \to 0$$

as $k \to \infty$. If $\omega_k(\delta)$ denotes the modulus of continuity for the coefficients of highest order for L_k, then $\omega_k(\delta) \leqslant \omega_L(\delta)$ and from the hypothesis

$$\|u(\cdot,t)\|_{L^p(R^n)} = \lim_k \|u_k(\cdot,t)\|_{L^p(R^n)} \leqslant \big(\overline{\lim_{k\to\infty}} C(p,\omega_k)\big) \|u(x,0)\|_{L^p(R^n)} \leqslant C(p,\omega_L) \|u(x,0)\|_{L^p(R^n)}.$$

2. The à priori estimate.

THEOREM 1. *Suppose $L \in \wp_{M,\eta}$ and let $\omega(\delta) = \omega_1(\delta^{1/2b}) + \omega_2(\delta)$ denote the modulus of continuity of the coefficients of highest order. If $\int_0^1 \frac{\omega^{4/3}(\delta)}{\delta} d\delta < \infty$, then* (II) *is valid for L; i.e. if $g(x) \in C_0^\infty(R^n)$ and $u(x,t) \in W^p_{2b,1}(S_T)$ is the solution to the problem $Lu = 0$ in S_T, $u(x,0) = g(x)$, then*

$$\|u(\cdot,t)\|_{L^p(R^n)} \leqslant C(p,\omega) \|g\|_{L^p(R^n)} \qquad (1 < p < \infty),$$

where

$$C(p,\omega) = A\Big[1 + \int_0^t \frac{\omega^{4/3}(\delta)}{\delta} d\delta + \omega(\delta)\Big] \quad \textit{and} \quad A = A(p,n,M,\eta,T).$$

Proof. Consider $L = \sum_{|\alpha|\leqslant 2b} a_\alpha(x,t) D_x^\alpha - D_t$ $\wp_{M,\eta}$, and as before set $A(y,s;\xi) = \sum_{|\alpha|=2b} a_\alpha(y,s)(i\xi)^\alpha$. Let $\Gamma_{y,s}(x,t) = \mathscr{F}_\xi(e^{A(y,s;\xi)t})(x)$, where $\mathscr{F}_\xi$ denotes Fourier transform in the ξ-variable. Set

$$Pf(x,t) = \int_0^t \int_{R^n} \Gamma_{y,0}(x-y, t-s) f(y,s)\, dy\, ds,$$

where $f \in L^p(S_T)$. It is known that P is continuous from $L^p(S_T) \to \mathring{W}^p_{2b,1}(S_T)$ for $1 < p < \infty$ and that $LPf = -f + J_L f$, where J_L: $L^p(S_T) \to L^p(S_T)$ is given by

$$J_L f(x,t) = \sum_{|\alpha|=2b} \lim_{\varepsilon\to 0} \int_0^{t-\varepsilon} \int_{R^n} [a_\alpha(x,t) - a_\alpha(y,0)] D_x^\alpha \Gamma_{y,0}(x-y, t-s) f(y,s)\, dy\, ds + \sum_{|\alpha|<2b} a_\alpha(x,t) \int_0^t \int_{R^n} D_x^\alpha \Gamma_{y,0}(x-y, t-s) f(y,s)\, dy\, ds.$$

The above limit is to be interpreted as a limit in $L^p(S_T)$ (see [4]). It is also true that

$$\|J_L f\|_{L^p(S_T)} \leqslant A\Big[\omega_L(T) + \sum_{j=0}^{2b-1} T^{1-j/2b}\Big] \|f\|_{L^p(S_T)},$$

where $A = A(p,M,\eta,n)$ and ω_L is the modulus of continuity of the highest order coefficients. We will prove the à priori inequality in an interval $(0,T)$ for which

$$\Big[\omega_L(T) + \Big(\int_0^T \frac{\omega_L^{4/3}(s)}{s} ds\Big) + \sum_{j=0}^{2b-1} T^{1-j\,2b}\Big]$$

is sufficiently small for L and then use observation 2 to obtain the estimate for L over an arbitrary finite interval. Also by Observation 3 we may assume that the highest order coefficients of L are smooth over R^{n+1} and that over the strip S_T, $\|J_L\| \leqslant 1/2$ as an operator from $L^p(S_T) \to L^p(S_T)$.

In this case then we can write the solution $u(x,t)\in W^p_{2b,1}(S_T)$ to the problem $Lu=0$ in S_T, $u(x,0)=g(x)\in C_0^\infty(R^n)$, in the form

$$u(x,t)=\Gamma(g)(x,t)-P\Big(\sum_{k=0}^{\infty}J^k\big(L\Gamma(g)\big)\Big),$$

where $\Gamma(g)(x,t)=\int_{R^n}\Gamma_{y,0}(x-y,t)g(y)\,dy$, $J=J_L$, and

$$L\Gamma(g)(x,t)=\sum_{|\alpha|\leqslant 2b}a_\alpha(x,t)D_x^\alpha\Gamma(g)(x,t)-D_t\Gamma(g)(x,t).$$

It is not difficult to see that $|\Gamma_{y,s}(x,t)|\leqslant\psi(x/t^{1/2b})t^{-n/2b}$, where $\int\psi(x)\,dx<\infty$ and ψ only depends on M and η. Hence

$$\|\Gamma(g)(\cdot,t)\|_{L^p(R^n)}\leqslant C\|g\|_{L^p(R^n)}.$$

Our problem then is to consider the L^p-norm over R^n of the function $P(\sum_{k=0}^{\infty}J^k(Lg))(x,t)$. The succession of lemmas that follow proves that under the hypotheses of Theorem 1,

$$\Big\|P\Big(\sum_{k=0}^{\infty}J^k(L\Gamma g)\Big)(\cdot,t)\Big\|_{L^p(R^n)}\leqslant A\Big[[\omega_L(t)+\int_0^t\frac{\omega_L^{4/3}(\delta)}{\delta}\,d\delta\Big]\|g\|_{L^p(R^n)},$$

where $A=A(p,M,\eta)$.

Lemma 1. *For $f\in L^p(S_T)$, the potential*

$$Pf(x,t)=\int_0^t\int_{R^n}\Gamma_{y,0}(x-y,t-s)f(y,s)\,dy\,ds$$

satisfies the estimate,

$$\|Pf(\cdot,t)\|_{L^p(R^n)}\leqslant C\Big[\sup_{0<r<t}\Big\|\int_0^r f(\cdot,s)\,ds\Big\|+\sup_{0<r<t}\frac{1}{r^{1/p}}\|sf\|_{L^p(S_t)}\Big].$$

Proof.

$$\begin{aligned}Pf(x,t)&=\int_{t/2}^{t}\int_{R^n}\Gamma_{y,0}(x-y,t-s)f(y,s)\,dy\,ds+\\&\quad+\int_0^{t/2}\int_{R^n}[\Gamma_{y,0}(x-y,t-s)-\Gamma_{y,0}(x-y,t)]f(y,s)\,dy\,ds+\\&\quad+\int_{R^n}\Gamma_{y,0}(x-y,t)\Big[\int_0^{t/2}f(y,s)\,ds\Big]dy\\&=P_1f+P_2f+P_3f.\end{aligned}$$

$$\|P_1f(\cdot,t)\|_{L^p(R^n)}\leqslant C\int_{t/2}^{t}\|f(\cdot,s)\|_{L^p(R^n)}\,ds\leqslant C\frac{1}{t}\int_0^t\|f(\cdot,s)\|s\,ds\leqslant\frac{C}{t^{1/p}}\|sf\|_{L^p(S_t)}.$$

For P_2f we use the mean-value theorem on the difference in the integrand and we obtain

$$\|P_2f(\cdot,t)\|_{L^p(R^n)}\leqslant\frac{C}{t}\int_0^{t/2}s\|f(\cdot,s)\|_{L^p(R^n)}\,ds\leqslant\frac{C}{t^{1/p}}\|sf\|_{L^p(S_t)}.$$

Again recalling that

$$|\Gamma_{y,0}(x-y,t)|\leqslant C\psi\Big(\frac{x-y}{t^{1/2b}}\Big)t^{-n/2b},$$

we see that

$$\|P_3f(\cdot,t)\|_{L^p(R^n)}\leqslant C\Big\|\int_0^{t/2}f(\cdot,s)\,ds\Big\|_{L^p(R^n)}.$$

Let us now introduce the norms

$$N_1(f)(t)=\sup_{0<r<t}\frac{1}{r^{1/p}}\|sf\|_{L^p(S_r)},\qquad N_2(f)(t)=\sup_{0<r<t}\Big\|\int_0^r f(\cdot,s)\,ds\Big\|_{L^p(R^n)},$$

and $N(f)(t)=N_1(f)(t)+N_2(f)$. From Lemma 1 we have proved that

$$\Big\|P\Big(\sum_{k=0}^{\infty}J^k(L\Gamma g)\Big)(\cdot,t)\Big\|_{L^p(R^n)}\leqslant C\sum_{k=0}^{\infty}N(J^kL\Gamma g)(t).$$

Lemma 2. *For $f\in L^p(S_T)$, $1<p<\infty$,*

$$N_1(Jf)(t)\leqslant A(p,M,\eta)\Big[\omega_1(t^{1/2b})+\omega_2(t)+\sum_{j=0}^{2b-1}t^{1-j/2b}\Big]N(f)(t).$$

Proof.

$$\begin{aligned}Jf(x,t)&=\sum_{|\alpha|=2b}\int_0^t\int_{R^n}[a_\alpha(x,0)-a_\alpha(y,0)]D_x^\alpha\Gamma_{y,0}(x-y,t-s)f(y,s)\,dy\,ds+\\&\quad+\sum_{|\alpha|=2b}[a_\alpha(x,t)-a_\alpha(x,0)]\lim_{\varepsilon\to0}\int_0^{t-\varepsilon}\int_{R^n}D_x^\alpha\Gamma_{y,0}(x-y,t-s)f(y,s)\,dy\,ds\\&\quad+\sum_{|\alpha|<2b}a_\alpha(x,t)\int_0^t\int_{R^n}D_x^\alpha\Gamma_{y,0}(x-y,t-s)\,dy\,ds\\&=J_1f+J_2f+J_3f,\end{aligned}$$

$$sJ_3f(x,s)=J_3(uf)(x,s)+sJ_3(f)(x,s)-J_3(uf)(x,s),$$

$$\|J_3(uf)\|_{L^p(S_r)}\leqslant C\Big(\sum_{j=0}^{2b-1}r^{1-j/2b}\Big)\|uf\|_{L^p(S_r)}.$$

Now

$$\begin{aligned}&s(J_3f)(x,s)-J_3(uf)(x,s)\\&\quad=\sum_{|\alpha|<2b}a_\alpha(x,s)\int_0^s\int_{R^n}D^\alpha\Gamma_{y,0}(x-y,s-u)(s-u)f(y,s)\,dy\,ds.\end{aligned}$$

Following the same procedure as in Lemma 1, we see that for $0 < s < r$

$$\|sJ_3f(\cdot, s)-J_3(uf)(\cdot, s)\|_{L^p(R^n)} \leqslant C_pM\Big(\sum_{j=0}^{2b-1} r^{1-j/2b}\Big)\Big[\frac{1}{s}\int_0^s \|f(\cdot, u)\|_{L^p(R^n)}u\,du+N_2(f)(s)\Big].$$

Now using Hardy's lemma ([8]),

$$\|sJ_3f(x, s)-J_3(uf)(x, s)\|_{L^p(S_r)} \leqslant C_pM\Big(\sum_{j=0}^{2b-1} r^{1-j/2b}\Big)[\|uf\|_{L^p(S_r)}+r^{1/p}N_2(f)(r)].$$

From this follows immediately that

$$N_1(J_3f)(t) \leqslant A\Big(\sum_{j=0}^{2b-1} t^{1-j/2b}\Big)N(f)(t).$$

Once again,

$$sJ_2f(x, s) = J_2(uf)(x, s)+sJ_2f(x, s)-J_2(uf)(x, s)$$

and we have

$$\|J_2(uf)\|_{L^p(S_r)} \leqslant \omega_2(r)\|uf\|_{L^p(S_r)}.$$

Again following Lemma 1

$$\|sJ_2f(\cdot, s)-J_2(uf)(\cdot, s)\|_{L^p(R^n)} \leqslant C\omega_2(s)\Big(\frac{1}{s}\int_0^s u\|f(\cdot, u)\|_{L^p(R^n)}du+N_2(f)(s)\Big).$$

From the last two inequalities we conclude that

$$N_1(J_2f)(t) \leqslant A\omega_2(t)N(f)(t).$$

Finally we need to consider $N_1(J_1f)(t)$. Let

$$a_{a,\lambda}(x) = \lambda^{-n}\int_{R^n} a_a(y, 0)\,\varphi\Big(\frac{x-y}{\lambda}\Big)dy,$$

where $\varphi\in C_0^\infty(R^n)$, $\varphi \geqslant 0$, and $\int_{R^n}\varphi = 1$. Set

$$K_a(f) = \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{R^n} D_x^a\Gamma_{y,0}(x-y, t-s)f(y, s)\,dy\,ds$$

for $|a| = 2b$ with the above limit understood in $L^p(S_T)$.

$$J_1f = \sum_{|a|=2b}[a_a(x, 0)-a_{a,\lambda}(x)]K_a(f)+\sum_{|a|=2b}K_a([a_{a,\lambda}(y)-a_a(y, 0)]f)+$$
$$+\sum_{|a|=2b}\int_0^t\int_{R^n}[a_{a,\lambda}(x)-a_{a,\lambda}(y)]D_x^a\Gamma_{y,0}(x-y, t-s)f(y, s)\,dy\,ds.$$

Using the fact $|a_{a,\lambda}-a_a| \leqslant \omega_1(\lambda)$ and proceeding as for J_2, we see that N_1 applied to each term of the first two summations and evaluated at t is $\leqslant C_p\omega_1(\lambda)N(f)(t)$. Using the fact that $|a_{a,\lambda}(x)-a_a(y)| \leqslant C\dfrac{\omega_1(\lambda)}{\lambda}|x-y|$ and proceeding as for J_3, we see that N_1 applied to each term of the last summation and evaluated at t is $\leqslant C_p\dfrac{\omega_1(\lambda)}{\lambda}t^{1/2b}N(f)(t)$. We conclude that

$$N_1(J_1f)(t) \leqslant C_p\Big[\frac{t^{1/2b}}{\lambda}\omega_1(\lambda)+\omega_1(\lambda)\Big]N(f)(t).$$

Choosing $\lambda = t^{1/2b}$ we obtain our final result.

Lemma 3. *Suppose $0 \leqslant |a| < 2b$. For $f\in L^p(S_T)$ set*

$$J_a(f)(x, t) = a_a(x, t)\int_0^t\int_{R^n} D_x^a\Gamma_{y,0}(x-y, t-s)f(y, s)\,dy\,ds.$$

Then

$$N_2(J_af)(t) \leqslant A(p, M, \eta)t^{1-|a|/2b}N(f)(t).$$

Proof.

$$\int_0^r J_af(x, s)ds = \int_{r/2}^r\int_{R^n} f(y, u)\Big[\int_u^r D_x^a\Gamma_{y,0}(x-y, s-u)a_a(x, s)ds\Big]dy\,du$$
$$+\int_0^{r/2}\int_{R^n} f(y, u)\Big[\int_{2u}^r D_x^a\Gamma_{y,0}(x-y, s-u)a_a(x, s)ds\Big]dy\,du+$$
$$+\int_0^{r/2}\int_{R^n} f(y, u)\Big[\int_u^{2u} D_x^a\Gamma_{y,0}(x-y, s-u)a_a(x, s)ds\Big]dy\,du$$
$$= F_1(x, r)+F_2(x, r)+F_3(x, r).$$

$$\|F_1(\cdot, r)\|_{L^p(R^n)} \leqslant C\int_{r/2}^r \|f(\cdot, u)\|_{L^p(R^n)}(r-u)^{1-|a|/2b}$$
$$\leqslant Cr^{-|a|/2b}\frac{1}{r}\int_0^r u\|f(\cdot, u)\|du \leqslant Cr^{1-|a|/2b}N_1(f)(r),$$

$$F_2(x, r) = \int_0^{r/2}\int_{R^n}\frac{d}{du}\Big(\int_0^u f(y, \theta)d\theta\Big)\Big(\int_{2u}^r D_x^a\Gamma_{y,0}(x-y, s-u)a_a(x, s)ds\Big)dy\,du$$
$$=\int_0^{r/2}\int_{R^n}\Big(\int_0^u f(y, \theta)d\theta\Big)\Big(D_x^a\Gamma_{y,0}(x-y, u)a_a(x, 2u)+\int_{2u}^r D_sD_x^a\Gamma_{y,0}(x-y, s-u)ds\Big).$$

Hence

$$\|F_2(\cdot, r)\|_{L^p(R^n)} \leqslant C\int_0^{r/2}\Big\|\int_0^u f(\cdot, \theta)d\theta\Big\|_{L^p(R^n)}\Big(u^{-|a|/2b}+\int_{2u}^\infty\frac{ds}{s^{1+|a|/2b}}\Big)du.$$
$$\leqslant CN_2(f)(r)r^{1-|a|/2b}.$$

Finally

$$\|F_3(\cdot, r)\|_{L^p(R^n)} \leqslant C\int_0^{r/2} \|f(\cdot, u)\|_{L^p(R^n)} u^{1-|\alpha|/2b}\,du$$

$$\leqslant C\int_0^{r/2}\left[\frac{d}{du}\int_0^u \theta\|f(\cdot, \theta)\|_{L^p(R^n)}\,d\theta\right]u^{-|\alpha|/2b}\,du$$

$$\leqslant C\Big\{r^{-|\alpha|/2b}\int_0^r \theta\|f(\cdot, \theta)\|_{L^p(R^n)}\,d\theta +$$

$$+\int_0^{r/2}\frac{1}{u^{1+|\alpha|/2b}}\int_0^u \theta\|f(\cdot, \theta)\|_{L^p(R^n)}\,d\theta\,du\Big\}$$

$$\leqslant CN_1(f)(r)r^{1-|\alpha|/2b}.$$

This concludes the proof of Lemma 3.

The proofs of the following two lemmas appear in the appendix.

LEMMA 4. $N_2(L\Gamma g)(t) \leqslant A\Big[\omega_1(t^{1/2b}) + \Big(\int_0^t \frac{\omega_2^2(s)}{s}\,ds\Big)^{1/2} + \sum_{j=0}^{2b-1} t^{1-j/2b}\Big]\|g\|_{L^p(R^n)}$ $(1 < p < \infty)$.

LEMMA 5. *For* $|\alpha| = 2b$ *and for* $f \in L^p(S_T)$, $1 < p < \infty$, *set*

$$J_\alpha(f)(x, t) = \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{R^n}[a_\alpha(x, t) - a_\alpha(x, 0)]D_x^\alpha\Gamma_{y,0}(x-y, t-s)f(y, s)\,dy\,ds,$$

the limit understood as a limit in $L^p(S_T)$. *Then*

$$N_2(J_\alpha(f))(t) \leqslant A\left(\omega_1(t^{1/2b}) + \omega_2(t) + \Big(\int_0^t \frac{\omega_2^2(s)}{s}\,ds\Big)^{1/2}\right)N(f)(t) +$$

$$+A\int_0^t \frac{N_1(f)(t)}{r}\Big(\int_0^r \frac{\omega_2^2(s)}{s}\,ds\Big)^{1/2}dr.$$

We return now to finish the proof of Theorem 1. After Lemma 1 we noted that

$$\Big\|P\Big(\sum_{k=0}^{\infty} J^k(L\Gamma g)\Big)(\cdot, t)\Big\|_{L^p(R^n)} \leqslant C\sum_{k=0}^{\infty} N\big(J^k(L\Gamma g)\big)(t).$$

From Lemmas 2, 3, and 5 we see that for $k \geqslant 2$

$$N\big(J^k(L\Gamma g)\big)(t)$$

$$\leqslant A\Big\{\omega_1(t^{1/2b}) + \omega_2(t) + \sum_{j=0}^{2b-1} t^{1-j/2b} + \Big(\int_0^t \frac{\omega_2^2(s)}{s}\,ds\Big)^{1/2}\Big\}N\big(J^{k-1}(L\Gamma g)\big)(t) +$$

$$A\int_0^t \frac{N_1\big(J^{k-1}(L\Gamma g)\big)(r)}{r}\Big(\int_0^r \frac{\omega_2^2(s)}{s}\,ds\Big)^{1/2}dr.$$

By Lemma 2,

$$N_1\big(J^{k-1}(L\Gamma g)\big)(r) \leqslant A\Big[\omega_1(r^{1/2b}) + \omega_2(r) + \sum_{j=0}^{2b-1} r^{1-j/2b}\Big]N\big(J^{k-2}(L\Gamma g)\big)(r).$$

Setting $\Phi(t) = \Big[\omega(t) + \int_0^t \frac{\omega^{4/3}(s)}{s}\,ds + \sum_{j=0}^{2b-1} t^{1-j/2b}\Big]$, we conclude that

$$N\big(J^k(L\Gamma)g\big)(t) \leqslant A\Phi(t)\big[N\big(J^{k-1}(L\Gamma g)\big)(t) + N\big(J^{k-2}(L\Gamma g)\big)(t)\big], \quad k \geqslant 2.$$

It is easy to see that

$$N_1(L\Gamma g)(t) \leqslant A\Big[\omega_1(t^{1/2b}) + \omega_2(t) + \sum_{j=0}^{2b-1} t^{1-j/2b}\Big]\|g\|_{L^p(R^n)}.$$

From Lemmas 2, 4, and 5 this implies that

$$N\big(J(L\Gamma g)\big)(t) + N(L\Gamma g)(t) \leqslant \big[\big(A\Phi(t)\big)^2 + A\Phi(t)\big]\|g\|_{L^p(R^n)}.$$

Since $\Phi(t) \to 0$ as $t \to 0+$, there exists T_0 depending on Φ such that for each $t \in (0, T_0)$,

$$\sum_{k=0}^{\infty} N(J^k L\Gamma g)(t) \leqslant A\Phi(t)\|g\|_{L^p(R^n)}.$$

The proof of Theorem 1 is now complete.

3. Appendix. Suppose $k(x, t) = \Omega\Big(\frac{x}{t^{1/2b}}\Big)t^{-n/2b-1}$, where $\Omega(x) \in S(R^n)$, the space of rapidly decreasing functions over R^n, and $\int \Omega(x)\,dx = 0$. Fix $t > 0$ and $0 < \varepsilon < t$. Set $K_{t,\varepsilon}(g)(x) = \int_{R^n} g(y)\int_\varepsilon^t k(x-y, s)\,ds$.

Below $\mathscr{F}$ and $\mathscr{F}_\xi$ both denote Fourier transform.

THEOREM (A.1). *For* $f \in L^p(R^n)$, $1 < p < \infty$,

$$\|K_{t,\varepsilon}(f)\|_{L^p(R^n)} \leqslant C\Big[\int (1 + |x|)\big(|\Omega(x)| + |\nabla\Omega(x)|\big)\,dx\Big]\|f\|_{L^p(R^n)},$$

where C *depends only on* p *and* n. *Moreover*, $\lim_{\varepsilon\to 0} K_{t,\varepsilon}(f)$ *exists in* $L^p(R^n)$.

Proof. We first observe that

$$\Big|\mathscr{F}_\xi\Big(\int_\varepsilon^t k(\xi, s)\,ds\Big)(x)\Big| = \Big|\int_\varepsilon^t \frac{\mathscr{F}(\Omega)(xs^{1/2b})}{s}\,ds\Big| \leqslant C\int_0^\infty \frac{|\mathscr{F}(\Omega)(x's)|}{s}\,ds,$$

where $|x'| = 1$. Since $\mathscr{F}(\Omega)(0) = 0$ we see that this last integral $\leqslant C\sup_{R^n}\big[|D_{x_i}\mathscr{F}(\Omega)| + |x_i\mathscr{F}(\Omega)|\big]$. Hence the result is valid for $p = 2$.

Moreover,

$$\int_{|x|>4|y|}\left|\int_s^t [k(x-y,s)-k(x,s)]\,ds\right| dx \leqslant C|y|\int_0^\infty \frac{1}{s^{1+1/2b}}\int_{|x|>|y|/s^{1/2b}} |\nabla\Omega(x)|\,dx\,ds$$
$$\leqslant C|y|\int |\nabla\Omega(x)|\Big(\int_{(|y|/|x|)^{2b}} s^{-1-1/2b}\,ds\Big)\,dx$$
$$\leqslant C\int |x|\,|\nabla\Omega(x)|.$$

We can now apply the result in [1] to finish the proof of the first part. The proof of the second part is fairly standard and we will omit details.

Suppose now that $b(x,s)$ is a smooth function over R^{n+1} with $b(x,0)=0$. Set

$$\omega_{2,b}(\delta) = \sup_{\substack{x\\ 1<s<\delta}} |b(x,s)|.$$

Set

$$K_{t,b}(f)(x) = \int_{R^n} f(y)\int_0^t k(x-y,s)\,b(x,s)\,ds,$$

where $k(x,s)$ is described above.

Before proceeding with the next theorem, we would like to review for the reader what we will call the Calderón–Zygmund decomposition or C–Z decomposition of a function $f(x)\geqslant 0$ corresponding to a given number $\lambda>0$. Precisely, given $f\geqslant 0$, $\epsilon L^1(R^n)$, and $\lambda>0$, we can find a sequence, $\{I_j\}$, of non-overlapping cubes such that

(i) $$\lambda < \frac{1}{|I_j|}\int_{I_j} f \leqslant 2^n\lambda;$$

(ii) if $D_\lambda = \bigcup_j I_j$, then $f\leqslant\lambda$ almost everywhere in D_λ^c, the complement of D_λ.

We now set

$$g(x) = \begin{cases} \dfrac{1}{|I_k|}\displaystyle\int_{I_k} f & \text{if } x\in I_k,\\ f & \text{if } x\in D_\lambda^c,\end{cases}$$

and $h=f-g$. h and g have the following properties:

(iii) $\|g\|^2_{L^2(R^n)} \leqslant C_n\lambda\|f\|_{L^1(R^n)}$ and $\|g\|_{L^1(R^n)}\leqslant\|f\|_{L^1(R^n)}$;

(iv) $\int_{I_j} h = 0$ and $\|h\|_{L^1(R^n)}\leqslant 2\|f\|_{L^1(R^n)}$.

(See [2].)

THEOREM (A.2). (a) *For* $f\epsilon L^2(R^n)$,

$$\|K_{t,b}(f)\|_{L^2(R^n)} \leqslant CA\Big(\int_0^t \frac{\omega_{2,b}^2(s)}{s}\,ds\Big)^{1/2}\|f\|_{L^2(R^n)},$$

where

$$A = \int (1+|x|)\big(|\Omega(x)|+|\nabla\Omega(x)|\big)\,dx.$$

(b) *For* $f\epsilon L^1(R^n)$,

$$|\{x\colon |K_{t,b}(f)(x)|>\lambda\}| \leqslant CA\Big[\Big(\int_0^t \frac{\omega_{2,b}^2(s)}{s}\,ds\Big)^{1/2}+\omega(t)\Big]\lambda^{-1}\|f\|_{L^1(R^n)}$$

with A *as in part* (a) ($|\ \{\ \}|$ *denotes Lebesgue measure of the set* $\{\ \}$.)

Proof. (a) $K_{t,b}(f)(x) = \int_0^t b(x,s)\big(\int_{R^n} k(x-y,s)f(y)\,dy\big)\,ds$. Hence

$$|K_{t,b}(f)(x)|^2 \leqslant \Big(\int_0^t \frac{\omega_{2,b}(s)^2}{s}\,ds\Big)\int_0^t s\,\Big|\int_{R^n} k(x-y,s)f(y)\,dy\Big|^2 ds.$$

Using Parseval's theorem,

$$\int_{R^n} |K_{t,b}(f)(x)|^2\,dx \leqslant C\Big(\int_0^t \frac{\omega_{2,b}(s)^2}{s}\,ds\Big)\int_{R^n} |\mathscr{F}(f)(x)|^2 \int_0^\infty \frac{|\mathscr{F}(\Omega)(xs^{1/2b})|^2}{s}\,ds\,dx.$$

Proceeding now as in Theorem A.1, we obtain part (a) of Theorem (A.2).

(b) We first observe, as in A.1, that

$$\int_{|x|>4|y|}\left|\int_0^t b(x,s)[k(x-y,s)-k(x,s)]\,ds\right| dx \leqslant C\omega_{2,b}(t)\int |x|\,|\nabla\Omega(x)|\,dx.$$

We now use the C–Z decomposition of f corresponding to the number, $\lambda/\Phi(t)$, where $\Phi(t) = \Big(\int_0^t \frac{\omega_{2,b}^2(s)}{s}\,ds\Big)^{1/2}$. These observations together with part (a) are all that are needed to carry through the usual weak-type proof as found in [1] or [2]. We leave this to the reader.

Theorem (A.2) insures that for $1<p\leqslant 2$,

$$\|K_{t,b}(f)\|_{L^p(R^n)} \leqslant CA\Big[\Big(\int_0^t \frac{\omega_{2,b}(s)^2}{s}\,ds\Big)^{1/2}+\omega_{2,b}(t)\Big]\|f\|_{L^p(R^n)},$$

where

$$A = \int_{R^n} (1+|x|)\big(|\Omega(x)|+|\nabla\Omega(x)|\big)\,dx.$$

To obtain this result for all p, $1 < p < \infty$, we need to consider the operator

$$\bar{K}_{t,b}(f)(x) = \int_{R^n} f(y)\Big(\int_0^t k(x-y,s)\,b(y,s)\,ds\Big)\,dy.$$

THEOREM (A.3). (a) *For* $f \in L^2(R^n)$,

$$\|\bar{K}_{t,b}(f)\|_{L^2(R^n)} \leqslant CA\Big(\int_0^t \frac{\omega_{2,b}^2(s)}{s}\,ds\Big)^{1/2}\|f\|_{L^2(R^n)}$$

with A *is in Theorem* (A.2).

(b) *For* $f \in L^1(R^n)$,

$$|\{x\colon |\bar{K}_{t,b}(f)(x)| > \lambda\}| \leqslant CA\Big[\Big(\int_0^t \frac{\omega_{2,b}^2(s)}{s}\,ds\Big)^{1/2} + \omega_{2,b}(t)\Big]\lambda^{-1}\|f\|_{L^1(R^n)}.$$

Proof. Part (a) follows easily from Theorem (A.2) part (a). For part (b) we begin by writing $f = g+h$, the C–Z decomposition of f corresponding to the number $\lambda/\Phi(t)$. Consider the set $D = D_{\lambda/\Phi(t)} = \bigcup_j I_j$ discussed above. We expand each side of I_j symmetrically about its center to a length of say six times the original. We denote by I_j^* the resulting cube and by $D^* = \bigcup_j I_j^*$. Again following the usual proof for "weak-type" as found in [1], we see that to prove part (b) it is sufficient to prove that

$$|\{x\colon |\bar{K}_{t,b}(h)(x)| > A\lambda\} \cap D^{*c}| \leqslant \frac{C}{\lambda}\big(\Phi(t)+\omega_{2,b}(t)\big)\int |f|,$$

where C depends only on n. (Recall that $\Phi(t) = \Big(\int_0^t \frac{\omega_{2,b}^2(s)}{2}\,ds\Big)^{1/2}$ and D^{*c} = complement of D^*.) Set

$$H(y,s) = \begin{cases} \dfrac{1}{|I_j|}\displaystyle\int_{I_j} h(z)\,b(z,s)\,dz & \text{for } y \in I_j,\\ 0 & \text{for } y \in D^c_{\lambda/\Phi(t)}. \end{cases}$$

Let y_j denote the center of I_j

$$\begin{aligned}\bar{K}_{t,b}(h)(x) &= \sum_j \int_0^t \int_{I_j} [k(x-y,s)-k(x-y_j,s)][h(y,s)b(y,s)-H(y,s)]\,dy\,ds + \\ &\quad + \int_0^t \int_{R^n} k(x-y,s)\,H(y,s)\,dy\,ds = \tilde{h}_1(x)+\tilde{h}_2(x).\end{aligned}$$

$$|\{x\colon |\tilde{h}_1(x)| > A\lambda\} \cap D^{*c}| \leqslant \frac{1}{A\lambda}\int_{D^{*c}} |\tilde{h}_1(x)|\,dx,$$

$$\begin{aligned}\int_{D^{*c}} |\tilde{h}_1(x)|\,dx &\leqslant \sum_j \int_0^t \int_{I_j}\Big(\int_{D^{*c}} |k(x-y,s)-k(x-y_j,s)|\,dx\Big)[|h(y)b(y,s)|+|H(y,s)|]\,dy\,ds\\ &\leqslant \sum_j \int_{I_j} |h(y)|\Big(\int_0^t \omega_{2,b}(s)\int_{|x|>4|y|} |k(x-y,s)-k(x,s)|\,dx\,ds\Big)\,dy\\ &\leqslant C\omega_{2,b}(t)\Big[\int |x|\,|\Delta\Omega(x)|\,dx\Big]\int |h| < CA\omega_{2,b}(t)\int |f|.\end{aligned}$$

$$|\{x\colon |\tilde{h}_2(x)| > A\lambda\}| \leqslant \frac{1}{(A\lambda)^2}\int |\tilde{h}_2(x)|^2\,dx.$$

From Parseval's theorem,

$$\begin{aligned}\|\tilde{h}_2\|_{L^2(R^n)} &= C\Big\|\int_0^t \frac{\mathscr{F}(\Omega)(xs^{1/2})}{s}\mathscr{F}_\xi\big(H(\xi,s)\big)(x)\,ds\Big\|^2_{L^2(R^n)}\\ &\leqslant C\int_{R^n}\Big(\int_0^t \frac{|\mathscr{F}(\Omega)(xs^{1/2})|^2}{s}\,ds\Big)\Big(\int_0^t \frac{|\mathscr{F}_\xi\big(H(\xi,s)\big)(x)|^2}{s}\,ds\Big)\\ &\leqslant CA^2\int_0^t\int_{R^n} |\mathscr{F}_\xi\big(H(\xi,s)\big)|^2(x)\,dx\frac{ds}{s}\\ &\leqslant CA^2\int_0^t\int_{R^n}\frac{|H(x,s)|^2}{s}\,dx\,ds.\end{aligned}$$

Observe now that if $x \in I_j$,

$$|H(x,s)| \leqslant \frac{\omega_{2,b}(s)}{|I_j|}\int_I |h| \leqslant C\omega_{2,b}(s)\frac{\lambda}{\Phi(t)}.$$

Hence

$$\int |H(x,s)|^2\,dx \leqslant C\omega_{2,b}(s)^2\frac{\lambda^2}{\Phi(t)^2}|D_{\lambda/\Phi(t)}| \leqslant C\omega_{2,b}(s)^2\frac{\lambda}{\Phi(t)}\int |f|.$$

We conclude that $|\{x\colon |\tilde{h}_2(x)| > A\lambda\}| \leqslant C\dfrac{\Phi(t)}{\lambda}\displaystyle\int |f|$, and hence the proof of part (b) is complete.

THEOREM (A.4). *Suppose* $f \in L^p(R^n)$, $1 < p < \infty$. *Then*

$$\|K_{t,b}(f)\|_{L^p(R^n)} \leqslant C_{p,n}A\,\|f\|_{L^p(R^n)},$$

where

$$A = \int (1+|x|)\big(|\Omega(x)|+|\nabla\Omega(x)|\big)\,dx$$

and $C_{p,n}$ *depends only on* p *and* n.

Proof. The proof follows in the usual manner from Theorems (A.2) and (A.3) and the Marcinkiewicz interpolation theorem.

Now suppose that

$$k(y; x, t) = \frac{\Omega(y; x/t^{1/2b})}{t^{n/2b+1}},$$

where for each y, $\Omega(y, \cdot) \in S(R^n)$, $\int_{R^n} \Omega(y, x)\,dx = 0$, and for each γ and β,

$$\sup_{y \in R^n} \|x^\gamma D_x^\beta \Omega(y; x)\|_{L^2(R^n)} < \infty.$$

Set

$$\tilde{K}_{t,\varepsilon}(f)(x) = \int_{R^n} f(y) \Big(\int_\varepsilon^t k(y; x-y, s)\,ds \Big)\,dy$$

and

$$\tilde{K}_{t,b}(f)(x) = \int_{R^n} f(y) \Big[\int_0^t k(y; x-y, s)\,b(x, s)\,ds \Big]\,dy.$$

Once again $b(x, s)$ is assumed to be smooth over R^{n+1} with $b(x, 0) = 0$. Again set $\omega_{2,b}(\delta) = \sup_{\substack{0<s<\delta \\ x \in R^n}} |b(x, s)|$.

THEOREM (A.5). *For* $1 < p < \infty$,

$$\|\tilde{K}_{t,\varepsilon}(f)\|_{L^p(R^n)} \leqslant C \|f\|_{L^p(R^n)},$$

$$\|K_{t,b}(f)\|_{L^p(R^n)} \leqslant C \left[\omega_{2,b}(t) + \left(\int_0^t \frac{\omega_{2,b}^2(s)}{s}\,ds \right)^{1/2} \right] \|f\|_{L^p(R^n)},$$

where C *is independent of* ε.

Proof. For $j = (j_1, \ldots, j_n)$, $j_i \geqslant 0$ an integer, set $H_j(x) = \prod_{i=1}^{i} H_{j_i}(x_i)$, where $H_{j_i}(x_i)$ is the Hermite polynomial of order j_i. The family $\{H_j(x) e^{-|x|^2/2}\}$ is a complete orthogonal family with respect to Lebesgue measure on R^n and we can write

$$\Omega(y; x) = \sum_j a_j(y) H_j(x) e^{-|x|^2/2},$$

$$a_j(y) = \frac{C_n}{2^{|j|} \prod_{i=1}^n j_i!} \int \Omega(y; x) H_j(x) e^{-|x|^2/2}\,dx.$$

The series has the property that, for each γ and β,

$$\sum_j \sup |a_j| \sup |x^\gamma D^\beta (H_j(x) e^{-|x|^2/2})| < \infty.$$

Since $\int \Omega(y; x)\,dx = 0$, we can also write $\Omega(y; x) = \sum_j a_j(y) \Omega_j(x)$, where $\Omega_j(x) = [H_j(x) - C_j] e^{-|x|^2/2}$, with $C_j = \int H_j(x) e^{-|x|^2/2}\,dx / \int e^{-|x|^2/2}\,dx$. The series $\sum_j a_j(y) \Omega_j(x)$ still maintains the property that for each γ and β

$$\sum_j \sup_y |a_j(y)| \sup_x |x^\gamma D_x^\beta \Omega_j(x)| < \infty.$$

(For a detailed discussion of the above see [5].) Hence

$$\tilde{K}_{t,\varepsilon}(f)(x) = \sum_j \int_{R^n} a_j(y) f(y) \int_\varepsilon^t k_j(x-y, s)\,ds\,dy,$$

where

$$k_j(x, s) = \Omega_j(x/s^{1/2b}) s^{-n/2b-1}.$$

Now from (A.1) we see that

$$\|\tilde{K}_{t,\varepsilon}(f)\|_{L^p(R^n)} \leqslant C \sum_j \Big[\int (1+|x|)\big(|\Omega_j(x)| + |\nabla \Omega_j(x)|\big)\,dx \Big] \|f a_j\|_{L^p(R^n)}$$

$$\leqslant C \Big[\sum_j \sup |a_j| \int (1+|x|)\big(|\Omega_j(x)| + |\nabla \Omega_j(x)|\big)\,dx \Big] \|f\|_{L^p(R^n)}.$$

In the same manner, using (A.4), we see that

$$\|\tilde{K}_{t,b}(f)\|_{L^p(R^n)} \leqslant C \left[\omega_{2,b}(t) + \left(\int_0^t \frac{\omega_{2,b}^2(s)}{s}\,ds \right) \right]^{1/2} \|f\|_{L^p(R^n)}.$$

THEOREM (A.6). (Formerly Lemma 4 of section 2). *For* $1 < p < \infty$,

$$N_2(L\Gamma g)(t) \leqslant C \left[\omega_1(t^{1/2}) + \omega_2(t) + \left(\int_0^t \frac{\omega_2^2(s)}{s}\,ds \right)^{1/2} + \sum_{j=0}^{2b-1} t^{1-j/2b} \right] \|g\|_{L^p(R^n)}.$$

Proof.

$$\int_0^r L(\Gamma g)(x, s)\,ds = \sum_{|\alpha|=2b} \int_{R^n} [a_\alpha(x, 0) - a_\alpha(y, 0)] g(y) \int_0^r D_x^\alpha \Gamma_{y,0}(x-y, s)\,ds\,dy +$$

$$+ \sum_{|\alpha|=2b} \int_{R^n} g(y) \int_0^r D_x^\alpha \Gamma_{y,0}(x-y, s) [a_\alpha(x, s) - a_\alpha(x, 0)]\,ds\,dy +$$

$$+ \sum_{|\alpha|<2b} \int_{R^n} g(y) \int_0^r D_x^\alpha \Gamma_{y,0}(x-y, s) a_\alpha(x, s)\,ds\,dy.$$

It is clear that the L^p-norm over R^n of the last summation is $\leqslant C\big(\sum_{j=0}^{2b-1} r^{1-j/2b}\big) \times \|g\|_{L^p(R^n)}$. Using the second part of (A.5) we see that the L^p-norm over R^n of each term in the second summation is

$$\leqslant C \left[\left(\int_0^r \frac{\omega_2^2(s)}{s}\,ds \right)^{1/2} + \omega_2(r) \right] \|g\|_{L^p(R^n)}.$$

To consider the first summation, we first take $a_{a,\lambda}(x) = \lambda^{-n}\int \varphi(y/\lambda)\times a_a(x-y,0)\,dy$, where $\varphi \in C_0^\infty(R^n)$, $\varphi \geqslant 0$, $\int \varphi = 1$, and we observe using the first part of (A.5) that the L^p-norm over R^n of a general term in the first summation is bounded by

$$C\Big[\omega_1(\lambda)\|g\|_{L^p(R^n)} + \Big\|\int_{R^n} g(y)[a_{a,\lambda}(x)-a_{a,\lambda}(y)]\int_0^r D_x^a\Gamma_{y,0}(x-y,s)\,ds\,dy\Big\|_{L^p(R^n)}\Big].$$

Since $|a_{a,\lambda}(x)-a_{a,\lambda}(y)| \leqslant C\dfrac{\omega_1(\lambda)}{\lambda}|x-y|$, the last term above

$$\leqslant C\frac{\omega_1(\lambda)}{\lambda}r^{1/2b}\|g\|_{L^p(R^n)}.$$

Hence we have shown that

$$\Big\|\int_0^r L\Gamma(g)(\cdot,s)\,ds\Big\|_{L^p(R^n)}$$
$$\leqslant C\Big[\frac{\omega_1(\lambda)}{\lambda}r^{1/2b}+\omega_2(r)+\Big(\int_0^r\frac{\omega_2^2(s)}{s}ds\Big)^{1/2}+\sum_{j=0}^{2b-1}r^{1-j/2b}\Big]\|g\|_{L^p(R^n)}.$$

We conclude the proof of (A.6) by choosing $\lambda = r^{1/2b}$.

Now suppose once again that

$$k(y;x,t) = \frac{\Omega(y;x/t^{1/2b})}{t^{n/2b+1}}$$

satisfies the conditions stated just prior to (A.5). For $f\in L^p(S_T)$, $1<p<\infty$, set

$$K(f)(x,t) = \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{R^n} k(y;x-y,t-s)f(y,s)\,dy\,ds,$$

the limit understood in the sense of $L^p(S_T)$. Assume $b(x,t)$ is smooth over R^{n+1}, $b(x,0)=0$, and again set $\omega_{2,b}(\delta) = \sup\limits_{\substack{0<s<\delta\\ x\in R^n}}|b(x,s)|$. Let

$$R_b(f)(x,t) = \int_0^t b(x,s)K(f)(x,s)\,ds.$$

Lemma 5 of Section 2 is an immediate corollary of the following theorem.

THEOREM (A.7). *For* $1<p<\infty$,

$$\|R_b(f)(\cdot,t)\|_{L^p(R^n)} \leqslant C\Big[\omega_{2,b}(t)+\Big(\int_0^t\frac{\omega_{2,b}^2(s)}{s}ds\Big)^{1/2}\Big]N(f)(t)+$$
$$+C\int_0^t\frac{N_1(f)(r)}{r}\Big[\Big(\int_0^r\frac{\omega_{2,b}^2(s)}{s}ds\Big)^{1/2}+\omega_{2,b}(r)\Big]dr.$$

Proof.

$$R_b(f)(x,t) = \int_0^t\frac{b(x,s)}{s}K(rf)(x,s)\,ds+$$
$$+\int_0^t\frac{b(x,s)}{s}\Big[\int_0^s\int_{R^n}k(y;x-y,s-r)(s-r)f(y,r)\,dy\,dr\Big]ds.$$

Hence $R_b(f)(x,t) = R_{b,1}(f)(x,t)+R_{b,2}(f)(x,t)$,

$$\|R_{b,1}(f)(\cdot,t)\|_{L^p(R^n)} \leqslant \int_0^t\frac{\omega_{2,b}(s)}{s}\|K(rf)(\cdot,s)\|_{L^p(R^n)}\,ds$$
$$\leqslant \int_0^t\frac{\omega_{2,b}(s)}{s}\frac{d}{ds}\int_0^s\|K(rf)(\cdot,u)\|_{L^p(R^n)}\,du\,ds$$
$$\leqslant \frac{\omega_{2,b}(t)}{t}\int_0^t\|K(rf)(\cdot,u)\|_{L^p(R^n)}\,du+$$
$$+\int_0^t\Big(\frac{\omega_{2,b}(s)}{s^2}-\frac{\omega'_{2,b}(s)}{s}\Big)\int_0^s\|K(rf)(\cdot,u)\|_{L^p(R^n)}\,du\,ds.$$

Since $\omega'_{2,b}(s)\geqslant 0$ we can drop the term involving this derivative, and we see that

$$\|R_{b,1}(f)(\cdot,t)\|_{L^p(R^n)} \leqslant C\omega_{2,b}(t)t^{-1/p}\|K(rf)\|_{L^p(S_t)}+$$
$$+C\int_0^t\frac{\omega_{2,b}(s)}{s}\frac{1}{s^{1/p}}\|K(rf)\|_{L^p(S_s)}\,ds$$
$$\leqslant C\Big[\omega_{2,b}(t)N_1(f)(t)+\int_0^t\frac{\omega_{2,b}(s)}{s}N_1(f)(s)\,ds\Big],$$

$$R_{b,2}(f)(x,t) = \int_0^t\frac{b(x,s)}{s}\int_{s/2}^s\int_{R^n}k(y;x-y,s-r)(s-r)f(y,r)\,dy\,dr\,ds+$$
$$+\int_0^t\frac{b(x,s)}{s}\int_0^{s/2}\int_{R^n}[k(y;x-y,s-r)(s-r)-k(y;x-y,s)s]f(y,r)\,dy\,dr\,ds+$$
$$+\int_0^t b(x,s)\int_{R^n}k(y;x-y,s)\int_0^{s/2}f(y,r)\,dr\,dy\,ds$$
$$= A(x,t)+B(x,t)+C(x,t),$$

$$\|A(\cdot,t)\|_{L^p(R^n)} \leqslant C\int_0^t\frac{\omega_{2,b}(s)}{s}\int_{s/2}^s\|f(\cdot,r)\|_{L^p(R^n)}\,dr \leqslant C\int_0^t\frac{\omega_{2,b}(s)}{s}N_1(f)(s)\,ds.$$

Using the mean-value on the difference in B, we see that

$$\|B(\cdot, t)\|_{L^p(R^n)} \leqslant C \int_0^t \frac{\omega_{2,b}(s)}{s} N_1(f)(s)\,ds.$$

Finally we have

$$C(x, t) = \int_{R^n} \Big(\int_0^t f(y, r)\,dr\Big)\Big(\int_0^t k(y; x-y, s)\,b(x, s)\,ds\Big)\,dy -$$
$$- \int_0^t b(x, s) \int_{R^n} k(y; x-y, s) \int_{s/2}^t \frac{1}{r}\frac{d}{dr}\Big(\int_0^r f(y, u)\,u\,du\Big)\,dr\,dy\,ds$$
$$= \int_{R^n} \Big(\int_0^t f(y, r)\,dr\Big)\Big(\int_0^t k(y; x-y, s)\,b(x, s)\,ds\Big) -$$
$$- \frac{1}{t}\int_{R^n} \Big(\int_0^t f(y, u)\,u\,du\Big)\Big(\int_0^t b(x, s)\,k(y; x-y, s)\,ds\Big)\,dy +$$
$$+ 2\int_0^t \frac{b(x, s)}{s} \int_{R^n} k(y; x-y, s) \int_0^{s/2} f(y, u)\,u\,du\,dy\,ds +$$
$$+ \int_0^t \frac{1}{r^2} \int_{R^n} \Big(\int_0^r f(y, u)\,u\,du\Big) \int_0^{2r} k(y; x-y, s)\,b(x, s)\,ds\,dy\,dr.$$

Using the second part of (A.5) we see that

$$\|C(\cdot, t)\|_{L^p(R^n)} \leqslant C\Big[\omega_{2,b}(t) + \Big(\int_0^t \frac{\omega_{2,b}^2(s)}{s}\Big)^{1/2}\Big] N(f)(t) +$$
$$+ C\int_0^t \frac{\omega_{2,b}(s)}{s} N_1(f)(s)\,ds + C\int_0^t \frac{N_1(f)(r)}{r}\Big(\int_0^r \frac{\omega_{2,b}^2(s)}{s}\,ds\Big)^{1/2} dr.$$

This concludes the proof of (A.7).

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