STUDIA MATHEMATICA T. XLIV. (1972)

H^2 spaces of generalized half-planes

by

A. KORÁNYI (New York) and E. M. STEIN (Jerusalem)

Abstract. Paley–Wiener type characterization of the H^2 and Bergman spaces on Siegel domains of type II. Formulas for their reproducing kernels.

Introduction. The classical result of Paley and Wiener characterizing the H^2-class of the upper half-plane was extended to tube domains over cones by Bochner [2](1) and to all generalized half-planes by Gindikin [3]. Gindikin, however, only gives a sketch of a proof which does not seem easy to complete. In this paper we give a different proof of Gindikin's result, and also give another characterization of the class H^2; the latter has been announced without proof in [4].

We will also deduce the existence of L^2-boundary values of H^2-functions and find an explicit formula for the Szegö kernel in a way different from Gindikin's. Finally we will indicate how the analogous results about the Bergman space and the Bergman kernel can be proved.

1. Definitions and notation. A *regular cone* in a real vector space V is a non-empty open convex cone Ω with vertex at 0 and containing no entire straight line. The *dual cone* Ω^* is the set of all linear functionals λ on V such that $\langle\lambda, x\rangle > 0$ for all $x \in \Omega$.

Given a regular cone Ω in $\boldsymbol{R}^{n_1}$, we say that a Hermitian bilinear map $\Phi\colon \boldsymbol{C}^{n_2}\times\boldsymbol{C}^{n_2}\to\boldsymbol{C}^{n_1}$ is *Ω-positive* if $\Phi(z_2, z_2)\in\overline{\Omega}$ for all $z_2\in\boldsymbol{C}^{n_2}$ and if $\Phi(z_2, z_2) = 0$ implies $z_2 = 0$.

Given such an Ω and Φ, the associated *generalized half-plane* is defined by $D = \{(z_1, z_2)\in\boldsymbol{C}^{n_1}\times\boldsymbol{C}^{n_2} \mid \operatorname{Im} z_1 - \Phi(z_2, z_2)\in\Omega\}$. This definition includes the case where $n_2 = 0$; then D is the tube over Ω and is also denoted by T_Ω. (Gindikin calls D a Siegel domain of type I if $n_2 = 0$, of type II if $n_2 > 0$.)

The Bergman–Šilov boundary of D is given by

$$B = \{(z_1, z_2)\in\boldsymbol{C}^{n_1}\times\boldsymbol{C}^{n_2} \mid \operatorname{Im} z_1 - \Phi(z_1, z_2) = 0\}.$$

B is parametrized in a natural way by $\boldsymbol{R}^{n_1}\times\boldsymbol{C}^{n_2}$. The natural Euclidean structure induces measures on $\boldsymbol{R}^{n_1}$, $\boldsymbol{C}^{n_2}$ which we denote by dx_1 and

(1) See also reference [7], chapter III.

$dx_2 dy_2$, respectively. The corresponding measure on B will be denoted by β. $L^2(B)$ will be the space of square-integrable functions on B with respect to β.

Given a function $F\colon D\to C$ and any $t\in\Omega$, we define F_t on $\bar{D}$ by $F_t(z)=F_t(z_1,z_2)=F(z_1+it,z_2)$.

Now $H^2(D)$ is defined as the set of all holomorphic $F\colon D\to C$ such that

$$\|F\|=\sup_{t\in\Omega}\|F_t\|_{L^2(B)}<\infty.$$

(Of course, $\|F_t\|_{L^2(B)}$ means the $L^2(B)$-norm of the restriction of F_t to B.) $H^2(D)$ is clearly a normed linear space, our main theorem will show among other things that it is a Hilbert space.

2. Recall of some results. For convenient reference we list as lemmas certain known results

LEMMA 2.1. *If $F\in H^2(D)$ and $\varepsilon\in\Omega$, then, for all fixed $z_2\in C^{n_2}$, the function $z_1\to F\big(z_1+i\varepsilon+i\Phi(z_2,z_2),z_2\big)$ is in $H^2(T_\Omega)$. If (z_2,ε) stay in a compact subset of $C^{n_2}\times\Omega$, then the corresponding $H^2(T_\Omega)$-norms stay bounded.*

The first assertion is a lemma in [6]. The proof given there also yields the second assertion immediately.

LEMMA 2.2. *The map $L^2(\Omega^*)\to H^2(T_\Omega)$ given by*

$$\varphi\to f(z_1)=\int_{\Omega^*}e^{2\pi i\langle\lambda,z_1\rangle}\varphi(\lambda)\,d\lambda$$

is a Hilbert space isomorphism. The integral converges absolutely for all fixed $z_1\in T_\Omega$.

This is Bochner's extension of the Paley–Wiener Theorem ([2], cf. also [7]).

LEMMA 2.3. *Let m be a natural number. Let B be a complex bilinear form on C^m whose restriction to R^m is positive definite. Let $\mathscr{L}^2_B$ be the space of holomorphic functions f on C^m such that*

$$\|f\|^2=\int_{C^m}|f(z)|^2e^{-\pi B(z,\bar z)}\,dx\,dy<\infty.$$

Then the map $L^2(R^m)\to\mathscr{L}^2_B$ given by $\varphi\to f(z)=\int_{R^m}a(z,a)\varphi(a)\,da$, where

$$a(z,a)=(\det B)^{3/4}e^{-\frac{\pi}{2}(B(z,z)+B(a,a))+\pi\sqrt{2}B(z,a)},\tag{2.1}$$

is a Hilbert space isomorphism. The inverse map is given by

$$\varphi(a)=\lim_{r\to1-0}\int_C f(rz)\overline{a(z,a)}e^{-\pi B(z,\bar z)}\,dx\,dy,\tag{2.2}$$

where lim *means limit in the topology of $L^2(R^m)$.*

Diagonalizing B and making some changes of variables this is immediately reduced to results of Bargmann [1].

We also note here that for B as in Lemma 2.3 we have

$$\int_{R^m}e^{-\pi B(x,x)}\,dx=(\det B)^{-1/2},\qquad\int_{C^m}e^{-\pi B(z,\bar z)}\,dx\,dy=(\det B)^{-1}.\tag{2.3}$$

If the Fourier transform is defined with the aid of B, i.e.

$$\tilde f(y)=\int_{R^m}f(a)e^{-2\pi iB(y,a)}\,da,$$

then the inversion formula and the Plancherel theorem take the following form:

$$f(a)=(\det B)\int_{R^m}\tilde f(y)e^{-2\pi iB(y,a)}\,dy,\tag{2.4}$$

$$\|\tilde f\|^2=\frac{1}{\det B}\|f\|^2.\tag{2.5}$$

3. Further lemmas.

LEMMA 3.1. *Let $F\in H^2(D)$, and let $K_2\subset C^{n_2}$ be compact. Let $\delta\in\Omega$ be such that $\delta-\Phi(z_2,z_2)\in\Omega$ for all $z_2\in K_2$ (such δ exists). Then the function $F_0\colon K_2\to H^2(T_\Omega)$ defined by $F_0(z_2)(z_1)=F(z_1+i\delta,z_2)$ is holomorphic.*

Proof. Lemma 2.1 implies that $F_0(z_2)\in H^2(T_\Omega)$ and $\|F_0(z_2)\|<M$ for all $z_2\in K_2$, with some M.

It is an immediate consequence of Lemma 2.2 that each $F_0(z_2)$ has boundary values on the Bergman–Šilov boundary of T_Ω, i.e. on R^{n_1}, and the map assigning to $F_0(z_2)$ its boundary function is an isomorphism $H^2(T_\Omega)\to L^2(R^{n_1})$. By the holomorphy of F this boundary function is given simply by $x_1\to F_0(z_2)(x_1)=F(x_1+i\delta,z_2)$.

Therefore, to show that F_0 is holomorphic we must show that for all $\varphi\in L^2(R^{n_1})$ the function $\psi\colon K_2\to C$ defined by

$$\psi(z_2)=\int_{R^{n_1}}F_0(z_2)(x_1)\varphi(x_1)\,dx_1=\int_{R^{n_1}}F(x_1+i\delta,z_2)\varphi(x_1)\,dx_1$$

is holomorphic. This is clear for continuous φ with compact support by the holomorphy of F. Given an arbitrary φ we approximate it in $L^2(R^{n_1})$ by a sequence $\{\varphi_n\}$ of continuous functions with compact support; the corresponding ψ_n are holomorphic, since $\|F_0(z_2)\|<M$, the Schwarz inequality shows that the functions ψ_n are uniformly bounded on K_2. Also by the Schwarz inequality, ψ_n tends to ψ pointwise. Hence ψ is holomorphic, finishing the proof.

LEMMA 3.2. *Let $U\subset C^m$ be a domain, let M be a measure space and let $f\colon U\to L^2(M)$ be holomorphic. Then for each $z\in U$ one can define $f(z)(p)$*

for a.a. $p \in M$, so that for a.a. $p \in M$, $z \to f(z)(p)$ is a holomorphic function $U \to C$ and $(z, p) \to f(z)(p)$ is jointly measurable on $U \times M$. For each sub-domain U_0 whose closure is compact in U, there exists a function $\psi \in L^2(M)$, so that $|f(z)(p)| \leqslant \psi(p)$ for each $z \in U_0$, and $p \in M$.

Proof. It is clearly enough to consider the case where U is a polydisc. For simplicity of notation we assume that U is the unit disc in one variable; the general case is no more difficult than this.

We make first the obvious remark that, for any sequence of numbers $\{a_n\}$ the property $\sum_0^\infty r^n |a_n| < \infty$ $(\forall 0 \leqslant r < 1)$ is equivalent with $\sum_0^\infty r^n |a_n|^2 < \infty$ $(\forall 0 \leqslant r < 1)$.

Now we develop f into a power series at the origin: $f(z) = \sum z^n f_n$ $(f_n \in L^2(M))$. We have $\sum r^n \|f_n\| < \infty$ $(\forall 0 \leqslant r < 1)$, and hence by the remark,

$$\int \sum r^n |f_n(p)|^2 dp = \sum r^n \int |f_n(p)|^2 dp = \sum r^n \|f_n\|^2 < \infty$$

for all $0 \leqslant r < 1$. This implies that, for a.a. $p \in M$ we have $\sum r^n |f_n(p)|^2 < \infty$. Hence, again by the remark, $\sum r^n |f_n(p)| < \infty$. Defining $f(z)(p) = \sum z^n f_n(p)$ we have all the desired properties. Clearly we may take $\psi(p) = \sum r_0^n |f_n(p)|$, if the subdomain U_0 is contained in the disc of radius r_0, $r_0 < 1$.

4. The main result. Let D be a generalized half-plane as described in Section 1. We shall use the notation

$$B_\lambda(z_2, w_2) = 4\langle \lambda, \Phi(z_2, \bar w_2)\rangle$$

for $\lambda \in \Omega^*$; $z_2, w_2 \in C^{n_2}$. So for fixed $\lambda \in \Omega^*$, B_λ is a complex bilinear form on C^{n_2} whose restriction to R^{n_2} is positive definite.

For all $z = (z_1, z_2) \in D$ we define the function χ_z: $\Omega^* \times R^{n_2} \to C$ by

$$\overline{\chi_z(\lambda, a)} = (\det B_\lambda)^{3/4} e^{2\pi\langle \lambda, iz_1 + 2\sqrt{2}\Phi(z_2, a) - \Phi(z_2, \bar z_2) - \Phi(a, a)\rangle}$$

$$= (\det B_\lambda)^{3/4} e^{2\pi i \langle \lambda, z_1\rangle} e^{-\frac{\pi}{2}(B_\lambda(z_2, z_2) + B_\lambda(a, a)) + \pi\sqrt{2} B_\lambda(z_2, a)}.$$

Finally, we define the space $\hat{L}^2$ as the set of all functions A: $\Omega^* \times C^{n_2} \to C$ such that (i) for all compact sets $K_1 \subset \Omega^*$, $K_2 \subset C^{n_2}$ the map $z_2 \to A(\cdot, z_2)$ is a holomorphic function $K_2 \to L^2(K_1)$, and (ii)

$$\|A\|_{\hat{L}^2}^2 = \int_{\Omega^* \times C^{n_2}} e^{-B_\lambda(z_2, \bar z_2)} |A(\lambda, z_2)|^2 d\lambda\, dx_2\, dy_2 < \infty.$$

(Note that, by Lemma 3.2, condition (i) implies that we can modify A on sets of measure 0 in λ for every fixed z_2 so that A is jointly measurable in (λ, z_2). Hence the integral in (ii) is meaningful).

THEOREM 4.1. *The map $L^2(\Omega^* \times R^{n_2}) \to H^2(D)$ which carries $\varphi \in L^2(\Omega^* \times R^{n_2})$ to*

(4.1) $$F(z) = \int_{\Omega^* \times R^{n_2}} \varphi(\lambda, a) \overline{\chi_z(\lambda, a)}\, d\lambda\, da$$

and the map $\hat{L}^2 \to H^2(D)$ which carries $A \in \hat{L}^2$ to

(4.2) $$F(z) = \int_{\Omega^*} e^{2\pi i\langle \lambda, z_1\rangle} A(\lambda, z_2)\, d\lambda$$

are Banach space isomorphisms. The integrals in (4.1) and (4.2) converge absolutely for every fixed $z = (z_1, z_2) \in D$. The limit $\lim_{\substack{t \to 0 \\ t \in \Omega}} F_t|_B$ exists in $L^2(B)$ for every $F \in H^2(D)$ and defines an isometric imbedding of $H^2(D)$ into $L^2(B)$.

Proof. We will describe three maps, i_1: $L^2(\Omega^* \times R^{n_2}) \to H^2(D)$, i_2: $H^2(D) \to \hat{L}^2$ and i_3: $\hat{L}^2 \to L^2(\Omega^* \times R^{n_2})$. We will show that each of them is isometric and that $i_3 \circ i_2 \circ i_1 = id$ on a dense subset of $L^2(\Omega^* \times R^{n_2})$. i_1 will be the map defined by (4.1), and i_2^{-1} will turn out to be the map defined by (4.2). This will prove all the assertions about isomorphisms; th other assertions will be established along the way.

Let i_1 be defined by the integral (4.1). We first show that if $K \subset D$ is compact then there exists $\chi_K \in L^2(\Omega^* \times R^{n_2})$ such that $|\chi_z| \leqslant |\chi_K|$ for all $z \in K$. By the Schwarz inequality this will immediately show that (4.1) converges absolutely and uniformly for $z \in K$, and hence represents a holomorphic function.

The map $z \to \operatorname{Im} z_1 - \Phi(z_2, z_2)$ is continuous from D to Ω, hence carries K to a compact subset of Ω. It follows that there exists $t_0 \in \Omega$ such that $\operatorname{Im} z_1 - \Phi(z_2, z_2) - t_0 \in \Omega$ for all $z \in K$. An easy computation using theidentity

$$\operatorname{Re} B_\lambda(z_2, z_2) = -B_\lambda(z_2, \bar z_2) + 2B_\lambda(x_2, x_2)$$

shows that if we define m by

$$m(\lambda, a) = (\det B_\lambda)^{3/4} e^{-2\pi\langle \lambda, t_0\rangle} e^{-\frac{\pi}{2} B_\lambda(a - \sqrt{2}x_2, a - \sqrt{2}x_2)}$$

then $|\chi_z| \leqslant m$, for $z \in K$. Using this and the fact that $t_0 \in \Omega$ one sees at once that

$$\sup_{z \in K} \int_{\Omega^*} \int_{R^{n_2}} |\chi_z(\lambda, a)|^2 da\, d\lambda < \infty.$$

Now the integral (4.1) is clearly a holomorphic function of z whenever φ is bounded and has compact support in $\Omega^* \times R^{n_2}$. The uniformity of the $L^2(\Omega^* \times R^{n_2})$ norms of $\bar\chi_z$, which we just observed, extends this assertion to all $\varphi \in L^2(\Omega^* \times R^{n_2})$ (see the argument in Lemma 3.1). Hence the mapping $z \to \bar\chi_z$ is a holomorphic mapping from D to $L^2(\Omega^* \times R^{n_2})$, and thus Lemma 3.2 is applicable, and therefore the required majorant χ_K exists.

We still have to show that the function F represented by (4.1) belongs to $H^2(D)$ and $\|F\| = \|\varphi\|$. So let $t \in \Omega$; we have to compute the $L^2(B)$-norm of $F_t|_B$. A simple computation gives

$$(4.3)\quad F_t|_B = F(x_1 + i\Phi(z_2, z_2) + it, z_2) = \int_{\Omega^* \times \boldsymbol{R}^{n_2}} e^{2\pi i\langle\lambda, x_1\rangle} e^{\pi i\sqrt{2}B(y_2, a)} \times$$

$$\times \varphi(\lambda, a)(\det B_\lambda)^{\frac{3}{4}} e^{-2\pi\langle\lambda, t\rangle} e^{-\pi i B_\lambda(x_2, y_2)} e^{-\frac{\pi}{2}B_\lambda(a - \sqrt{2}x_2, a - \sqrt{2}x_2)} d\lambda\, da.$$

We know from the first part of the argument that this integral converges absolutely. Hence, for a.a. $\lambda \in \Omega^*$, the integral with respect to a exists. Also for a.a. λ, φ is square-integrable with respect to a. It follows that for a.a. λ, our a-integral is the Fourier transform of a function in $L^1 \cap L^2$. By (2.5) the square of the L^2-norm of this function is

$$(4.4)\quad n_\lambda^2 = (\det B_\lambda)^{\frac{3}{2}} e^{-4\pi\langle\lambda, t\rangle} \frac{2^{n_1/2}}{\det B_\lambda} \int_{\boldsymbol{R}^{n_2}} |\varphi(\lambda, a)|^2 e^{-\pi B_\lambda(a - \sqrt{2}x_2, a - \sqrt{2}x_2)} da.$$

It is clear that $\int_{\Omega^*} n_\lambda^2 d\lambda < \infty$. Therefore (4.3) can be regarded as the Fourier transform of a vector-valued function of λ (while x_2 is still being kept fixed). The vector-valued form of Plancherel's theorem now gives

$$\int_{\boldsymbol{R}^{n_1}} \int_{\boldsymbol{R}^{n_2}} |F(x_1 + i\Phi(z_2, z_2) + it)|^2 dy_2\, dx_1 = \int_{\Omega^*} n_\lambda^2 d\lambda.$$

Finally, it follows that

$$\|F_t\|_{L^2(B)}^2 = \int_{\Omega^* \times \boldsymbol{R}^{n_2}} n_\lambda^2 d\lambda\, dx_2.$$

Using (4.4), Fubini's theorem, and (2.3) one computes that

$$\|F_t\|_{L^2(B)}^2 = \int_{\Omega^* \times \boldsymbol{R}^{n_2}} e^{-4\pi\langle\lambda, t\rangle} |\varphi(\lambda, a)|^2 d\lambda\, da.$$

From this formula it is clear that $F \in H^2(D)$ and $\|F\| = \|\varphi\|$.

At this point it is also easy to prove the last statement of the theorem, at any rate for F representable in the form (4.1) (we have not yet proved that every $F \in H^2(D)$ is representable like this). In fact, let $t, t' \in \Omega$. Writing down formula (4.3) for $(F_t - F_{t'})|_B$ and going through the same steps as above, we find

$$\|F_t - F_{t'}\|_{L^2(B)}^2 = \int_{\Omega^* \times \boldsymbol{R}^{n_2}} [e^{-2\pi\langle\lambda, t\rangle} - e^{-2\pi\langle\lambda, t'\rangle}]^2 |\varphi(\lambda, a)|^2 d\lambda\, da.$$

From the Lebesgue dominated convergence theorem it follows now that $\lim_{t\to 0} F_t$ exists in $L^2(B)$ and its norm equals $\|\varphi\|$.

Now we will define $i_2\colon H^2(D) \to \hat{L}^2$ and prove that it is an isometry.

Let $F \in H^2(D)$. For fixed $z_2 \in \boldsymbol{C}^{n_2}$ and δ such that $\delta - \Phi(z_2, z_2) \in \Omega$, Lemmas 2.1 and 2.2 show the existence of a unique function $\lambda \to A_\delta(\lambda, z_2)$ in $L^2(\Omega^*)$ such that

$$F(z_1 + i\delta, z_2) = \int_{\Omega^*} e^{2\pi i\langle\lambda, z_1\rangle} A_\delta(\lambda, z_2)\, d\lambda.$$

Defining A by $A_\delta(\lambda, z_2) = e^{-2\pi\langle\lambda, \delta\rangle} A(\lambda, z_2)$ it is clear that A is independent of the choice of δ and

$$F(z_1, z_2) = \int_{\Omega^*} e^{2\pi i\langle\lambda, z_1\rangle} A(\lambda, z_2)\, d\lambda$$

for z such that $\operatorname{Im} z_1 - \Phi(z_2, z_2) \in \Omega$, i.e. for all $z \in D$.

Now let $K_2 \subset \boldsymbol{C}^{n_2}$ be compact and δ such that $\delta - \Phi(z_2, z_2) \in \Omega$ for all $z_2 \in K_2$. Using Lemma 3.1 and the isomorphism of $H^2(T_\Omega)$ and $L^2(\Omega^*)$ (Lemma 2.2) it follows that the map $K_2 \to L^2(\Omega^*)$ defined by $z_2 \to A_\delta(\cdot, z_2)$ is holomorphic.

If $K_1 \in \Omega^*$ is compact, it follows by restriction and by the definition of A that the map $K_2 \to \hat{L}^2(K_1)$ defined by $z_2 \to A(\cdot, z_2)$ is holomorphic.

To compute the $\hat{L}^2$-norm of A, we let $t \in \Omega$ and compute $\|F_t\|_{L^2(B)}$. First of all, for fixed z_2 we have

$$\int_{\boldsymbol{R}^{n_1}} |F(x_1 + i(t + \Phi(z_2, z_2)), z_2)|^2 dx_1$$

$$= \int_{\Omega^*} |A_{t+\Phi(z_2, z_2)}(\lambda, z_2)|^2 d\lambda = \int_{\Omega^*} e^{-4\pi\langle\lambda, t\rangle} e^{-4\pi\langle\lambda, \Phi(z_2, z_2)\rangle} |A(\lambda, z_2)|^2 d\lambda.$$

Integrating both sides and taking $\sup_{t\in\Omega}$ we find that $\|F\|_{H^2(D)}^2 = \|A\|_{\hat{L}^2}^2$.

Next, we have to describe the isometry $i_3\colon \hat{L}^2 \to L^2(\Omega^* \times \boldsymbol{R}^{n_2})$.

Let $A \in \hat{L}^2$. By Lemma 3.2, given a compact $K_2 \subset \boldsymbol{C}^{n_2}$, A can be redefined so that $A(\lambda, z_2)$ is a holomorphic function of z_2 on K_2 for a.a. λ. Exhausting $\boldsymbol{C}^{n_2}$ by countably many compact sets we see that $A(\lambda, z_2)$ is an entire function of $z_2 \in \boldsymbol{C}^{n_2}$ for a. a. λ. Also, by definition of $\hat{L}^2$ and by Fubini's theorem,

$$\int_{\boldsymbol{C}^{n_2}} e^{-B_\lambda(z_2, \bar{z}_2)} |A(\lambda, z_2)|^2 dx_2\, dy_2 < \infty$$

for a. a. λ. For these λ, by Lemma 2.3 we have

$$A(\lambda, z_2) = (\det B_\lambda)^{\frac{3}{4}} e^{-\frac{\pi}{2}B_\lambda(z_2, z_2)} \int_{\boldsymbol{R}^{n_2}} \varphi(\lambda, a) e^{-\frac{\pi}{2}B_\lambda(a, a) + \pi\sqrt{2}B_\lambda(z_2, a)} da$$

and

$$\int_{\boldsymbol{C}^{n_2}} e^{-\pi B_\lambda(z_2, \bar{z}_2)} |A(\lambda, z_2)|^2 dx_2\, dy_2 = \int_{\boldsymbol{R}^{n_2}} |\varphi(\lambda, a)|^2 da,$$

where $\varphi(\lambda, a)$ is an L^2-function of a for a. a. λ.

If we know that φ is jointly measurable in (λ, a), we can integrate our last equality with respect to λ and find $\|A\|^2_{\hat{L}^2} = \|\varphi\|^2$, which shows that i_3 defined by $i_3(A) = \varphi$ is an isometry as desired. Therefore we have to prove now the measurability of φ.

For $0 < r < 1$ we define A_r by $A_r(\lambda, z_2) = A(\lambda, rz_2)$ and φ by $i_3(A_r)$. Let us denote by $a_\lambda(z_2, a)$ the function defined as in (2.1) with the aid of the bilinear form B_λ. Then, by Lemma 2.3

$$\varphi_r(\lambda, a) = \int_{C^{n_2}} A_r(\lambda, z_2)\overline{a_\lambda(z_2, a)}\, e^{-\pi B_\lambda(z_2,\bar{z}_2)}\, dx_2\, dy_2.$$

The integral here is jointly measurable in (λ, a, z_2) by Lemma 3.2. Hence, by Fubini's theorem, the integral taken over a compact subset of C^{n_2} is measurable in (λ, a). Since the integral converges absolutely, it follows that φ_r, as a pointwise limit of measurable functions, is measurable.

By the argument given before, it follows now that $\|A_r\|_{\hat{L}^2} = \|\varphi_r\|$. To prove that i_3 is an isometry it is enough to show that the functions of the form A_r are dense in $\hat{L}^2$. This is best done by showing that $\lim_{r\to 1-0} A_r = A$ in $\hat{L}^2$.

A change of variable gives

$$\|A_r\|^2_{\hat{L}^2} = r^{2n_2}\int |A(\lambda, z_2)|^2 e^{-\frac{\pi}{r^2}B_\lambda(z_2,\bar{z}_2)}\, d\lambda\, dx_2\, dy_2.$$

By the dominated convergence theorem this shows that $\lim_{r\to 1-0}\|A_r\| = \|A\|$. It is enough therefore to show that A_r tends to A weakly. To do this, we imbed $\hat{L}^2$ into the Hilbert space of all functions on $\Omega^*\times C^{n_2}$ square-integrable with respect to the weight function $e^{-\pi B_\lambda(z_2,\bar{z}_2)}$. In this space the continuous functions b with compact support are dense. It is therefore enough to show that

$$\lim_{r\to 1-0}\int A_r(\lambda, z_2)\overline{b(\lambda, z_2)}\, e^{-\pi B_\lambda(z_2,\bar{z}_2)}\, d\lambda\, dx_2\, dy_2 = \int A(\lambda, z_2)\overline{b(\lambda, z_2)}\, e^{-\pi B_\lambda(z_2,\bar{z}_2)}\, d\lambda\, dx_2\, dy_2.$$

This, however, follows from the dominated convergence theorem by changing the variable from rz_2 to z_2 and taking into account that b is bounded and has compact support.

We have shown that i_1, i_2, i_3 are isometric maps. Now we show that $i_3\circ i_2\circ i_1$ is the identity on the set of smooth functions with compact support in $\Omega^*\times \mathbf{R}^{n_1}$. In fact, let φ be such a function and let $F = i_1(\varphi)$. F is given by integral (4.1). Let $A = i_1(F)$; then A is uniquely determined by (4.2); comparison with (4.1) gives that

$$A(\lambda, z_1) = (\det B_\lambda)^{\frac{3}{4}}\int_{\mathbf{R}^{n_2}} \varphi(\lambda, a)\, e^{-\frac{\pi}{2}(B_\lambda(z_2,z_2)+B_\lambda(a,a))+\pi\sqrt{2}B_\lambda(z_2,a)}\, da$$

(all integrals involve only smooth functions with compact support). By the definition of i_3 we have now $i_3(A) = \varphi$, as we had to show.

We know now that i_1, i_2, i_3 are Hilbert space isomorphisms. Hence the map defined by (4.2) is the inverse of i_2. This finishes the proof of all the statements of the Theorem.

5. The Szegö kernel of D. Given a Hilbert space H of functions on a set E one says that a function $K\colon E\times E\to \mathbf{C}$ is a reproducing kernel of H if, for all $w\in E$, $K_w\colon E\to \mathbf{C}$ defined by $K_w(z) = K(z, w)$ is in H, and $(f, K_w) = f(w)$ for all $f\in H$ and $w\in E$.

We will show that $H^2(D)$ has a reproducing kernel, called the Szegö kernel and denoted S, and we will find an explicit formula for it.

LEMMA 5.1. *Let E be a set, H a Hilbert space of functions on E, $\mathscr{H}$ another Hilbert space. Suppose that for every $z\in E$ there exists an element $\chi_z\in\mathscr{H}$ such that defining $\hat{\varphi}\colon E\to \mathbf{C}$ for all $\varphi\in E$ by $\hat{\varphi}(z) = (\varphi, \chi_z)$, the mapping $\varphi\to\hat{\varphi}$ is an isomorphism $\mathscr{H}\to H$. Then H has a reproducing kernel K, and $K(z, w) = (\chi_w, \chi_z)$ for all $z, w\in E$.*

Proof. Let K be defined by $K(z, w) = (\chi_w, \chi_z)$ and K_w by $K_w(z) = K(z, w)$. Then by our definitions we have $K_w = \hat{\chi}_w$. Now, for all $\varphi\in\mathscr{H}$, $w\in\mathscr{H}$,

$$(\hat{\varphi}, K_w) = (\hat{\varphi}, \hat{\chi}_w) = (\varphi, \chi_w) = \hat{\varphi}(w)$$

which proves that K is a reproducing kernel.

THEOREM 5.1. *The Szegö kernel S of the generalized half-plane D is given by*

$$S(z, w) = \int_{\Omega^*} e^{-2\pi\langle\lambda, \varrho(z,w)\rangle}(\det B_\lambda)\, d\lambda,$$

where

$$\varrho(z, w) = i(\bar{w}_1 - z_1) - 2\Phi(z_2, w_2).$$

Proof. Use Lemma 5.1 with $\mathscr{H} = L^2(\Omega^*\times\mathbf{R}^{n_1})$, $E = D$, $H = H^2(D)$ and χ_z defined as in Section 4. The assertion follows by a simple computation involving (2.3).

6. The Bergman kernel of D. Let $\mathscr{L}^2(D)$ be the Hilbert space of square-integrable holomorphic functions on the generalized half-plane D. All the theory of $H^2(D)$ that we have developed can be extended to $\mathscr{L}^2(D)$ by a close analogy.

LEMMA 6.1. *Let $M^*\colon \Omega^*\to\mathbf{R}$ be defined by*

$$M^*(\lambda) = \int_{\Omega} e^{-2\pi\langle\lambda, y_1\rangle}\, dy_1.$$

The map $L^2(\Omega^*) \to \mathscr{L}^2(T_\Omega)$ *defined by*

$$\varphi \to f(z_1) = \int_{\Omega^*} e^{2\pi i \langle \lambda, z_1 \rangle} \varphi(\lambda)\, M^*(2\lambda)^{-\frac{1}{2}} d\lambda$$

is a Hilbert space isomorphism. The integral converges absolutely for all $z_1 \in T_\Omega$.

The proof can be found in [5].

Lemma 6.1 is an analogue of Lemma 2.2. Similarly it is easy to prove analogues of Lemmas 2.1 and 3.1, and finally one obtains the following results. (The notation is the same as in Section 4.)

THEOREM 6.1. *The map* $L^2(\Omega^* \times \boldsymbol{R}^{n_2}) \to \mathscr{L}^2(D)$ *which carries* $\varphi \in L^2(\Omega^* \times \boldsymbol{R}^{n_2})$ *to*

$$F(z) = \int_{\Omega^* \times \boldsymbol{R}^{n_2}} \varphi(\lambda, a)\, \overline{\chi_z(\lambda, a)}\, M^*(2\lambda)^{-\frac{1}{2}} d\lambda\, da \tag{6.1}$$

and the map $\hat{L}^2 \to H^2(D)$ *which carries* $A \in \hat{L}^2$ *to*

$$F(z) = \int_{\Omega^*} e^{2\pi i \langle \lambda, z_1 \rangle} A(\lambda, z_2)\, M^*(2\lambda)^{-\frac{1}{2}} d\lambda \tag{6.2}$$

are Hilbert space isomorphisms. The integrals (6.1), (6.2) *converge absolutely for every fixed* $z = (z_1, z_2) \in D$.

THEOREM 6.2. *The Bergman kernel* K *of* D, *i.e. the reproducing kernel of* $\mathscr{L}^2(D)$, *is given by*

$$K(z, w) = \int_{\Omega^*} e^{-2\pi \langle \lambda, \varrho(z,w) \rangle} \frac{(\det B_\lambda)}{M^*(2\lambda)} d\lambda.$$

References

[1] V. Bargmann, *On a Hilbert space analytic functions and an associated integral transform*, I, Comm. Pure Appl. Math. 14 (1961), pp. 187–214.

[2] S. Bochner, *Group invariance of Cauchy's formula in several variables*, Ann. of Math. 45 (1944), pp. 686–707.

[3] S. G. Gindikin, *Analysis in homogeneous domains*, Uspekhi Mat. Nauk 19 (1964), pp. 3–92 (in Russian).

[4] A. Korányi, C. I. M. E. notes, 1967.

[5] — *The Bergman kernel function for tubes over convex cones*, Pacific J. Math. 12 (1962), pp. 1355–1359.

[6] E. M. Stein, *Note on the boundary values of holomorphic functions*, Ann. of Math. 82 (1965), pp. 351–353.

[7] — and G. Weiss *Introduction to Fourier analysis in Euclidean spaces*, Princeton 1971.

Received September 27, 1971 (403)

STUDIA MATHEMATICA T. XLIV. (1972)

The initial value problem for parabolic equations with data in $L^p(R^n)$

by

Eugene FABES* (Minneapolis, Minn.)

Abstract. Suppose $u(x,t)$ belongs to the class of functions having derivatives, $D_x^a u(x,t)$, $|a| \leqslant 2b$, and $D_t u(x,t)$ in $L^p(R^n \times (0,T))$. Assume that $Lu(x,t) = 0$ where $L = \sum_{|a| \leqslant 2b} a_a(x,t) D_x^a - D_t$ is a parabolic operator with coefficients bounded and measurable and for $|a| = 2b$ uniformly continuous. Let $\omega(s)$ denote the modulo of continuity of a coefficient of order $2b$. If $\int_0^1 \frac{\omega(s)^{4/3}}{s} ds < \infty$, then we show that for $1 < p < \infty$ $\|u(\cdot, t)\|_{L^p(R^n)} \leqslant c \|u(\cdot, 0)\|_{L^p(R^n)}$. This á priori estimate is used to resolve uniquely the initial value problem, $Lu(x,t) = 0$, $t > 0$, and $u(x,0) = g(x)$ where $g(x) \in L^p(R^n)$.

1. Introduction. In this paper we consider the initial value problem for the uniformly parabolic operator $L = \sum_{|a| \leqslant 2b} a_a(x,t) D_x^a - D_t$ when the initial data, $g(x)$, belongs to $L^p(R^n)$, $1 < p < \infty$, and when the coefficients, $a_a(x,t)$, are bounded, measurable, and for $|a| = 2b$, uniformly continuous over the strip $S_T = R^n \times (0,T)$. As usual b is a positive integer, x is a point in R^n, $t \in (0,T)$, $a = (a_1, \ldots, a_n)$ is an n-tuple of non-negative integers, $D_x^a = \partial^{a_1}/\partial x_1^{a_1} \ldots \partial^{a_n}/\partial x_n^{a_n}$, and $|a| = \sum_{i=1}^n a_i$. By the uniform parabolicity of L we mean that the real part of the form, $A(x,t;\xi) = \sum_{|a|=2b} a_a(x,t)(i\xi)^a$, satisfies the condition, $\operatorname{Re}(A(x,t;\xi)) \leqslant -\eta|\xi|^{2b}$, with $\eta > 0$ and independent of $(x,t) \in S_T$.

Given a function $g(x) \in L^p(R^n)$, $1 < p < \infty$, we consider the problem of finding a unique function $u(x,t)$ such that

(I)
(i) for every δ, $0 < \delta < T$, $D_x^a u$, $|a| \leqslant 2b$, and $D_t u$ exist in the sense of distributions over $S_{\delta,T} = R^n \times (\delta, T)$ and belong to $L^p(S_{\delta,T})$;
(ii) $Lu = 0$ in S_T;
(iii) $\lim_{t \to 0+} \|u(\cdot, t) - g(\cdot)\|_{L^p(R^n)} = 0$.

* This research was partially supported by National Science Foundation grant number GP-15832.