STUDIA MATHEMATICA T. XLIV. (1972)

An estimate of the conjugate function

by

Richard A. HUNT* (Lafayette, Ind.)

Abstract. It is shown that the conjugate function of a function $f \in L^1(-\pi, \pi)$ is of exponential type off any set where the Hardy–Littlewood maximal function of f is uniformly bounded. Thus, there is considerable overlap between the set where the conjugate function is large and the set where the Hardy–Littlewood maximal function is large. This general principle is used to show that $S_{n_k}(f, \chi) = o(\log\log n_k)$ a.e., where $S_{n_k}(t, x)$ is the n_kth partial sum of the Fourier series of $f \in L^1(-\pi, \pi)$ and $\{n_k\}$ is any lacunary sequence.

We will show that the conjugate function of a function $f \in L^1(-\pi, \pi)$ is of exponential type off any set where the Hardy–Littlewood maximal function of f is uniformly bounded. Thus, there is considerable overlap between the set where the conjugate function is large and the set where the Hardy–Littlewood maximal function is large. This general principle will be used to obtain a result on the rate of growth of certain partial sums of the Fourier series of f.

$$\tilde{f}(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{2\tan\frac{1}{2}(t-x)}\,dt$$

will denote the conjugate function of $f \in L^1(-\pi, \pi)$,

$$\bar{f}(x) = \sup\left\{\frac{1}{|I|}\int_I |f(t)|\,dt \colon I \text{ an interval with center } x\right\}$$

will denote the Hardy–Littlewood maximal function of f, and

$$S_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac{1}{2})(t-x)}{2\sin\frac{1}{2}(t-x)}\,dt$$

will denote the nth partial sum of the Fourier series of f. mE will denote the Lebesgue measure of the set E and C will denote an absolute constant, although not always the same constant.

---

* This work was partially supported by the National Science Foundation GP–18831 and the Alfred P. Sloan Foundation.

The idea of the proof of our main result is to take a double look at the Calderón–Zygmund proof that $\tilde{f}$ satisfies a weak type $(1,1)$ estimate. For $f \in L^1$ we use the Calderón–Zygmund decomposition $f = g+b$, where g is bounded and b is supported on a set of small measure. (See Calderón and Zygmund [1]). The weak type $(1,1)$ estimate for $\tilde{f}$ is obtained by applying the strong type $(2,2)$ result to $\tilde{g}$ and noting that $\tilde{b}$ is integrable off a set of small measure. The weak type $(1,1)$ estimate for $\tilde{f}$ when $f \in L^1$ leads to an exponential type estimate for $\tilde{f}$ when $f \in L^\infty$. With this in mind we take a second look at the L^1 proof. Since g is bounded, $\tilde{g}$ is of exponential type. Using a lemma of L. Carleson (see Carleson [2, p. 140]) we show that $\tilde{b}$ is also of exponential type when restricted to a set where $\bar{f}$ is bounded. This yields

THEOREM 1. *$m\{x \in (-\pi,\pi): \bar{f}(x) \leqslant y, |\tilde{f}(x)| > \lambda y\} \leqslant Ce^{-C\lambda}$, where $y > 0$, $\lambda > 0$ and $f \in L^1(-\pi,\pi)$.*

The application of our main result to Fourier series is based on the fact that $S_n(f,\cdot)$ is essentially $\tilde{f}_n$, where $f_n(x) = e^{inx}f(x)$. Hence, $S_n(f,\cdot)$ is of exponential type if we avoid the set where $\bar{f}_n$ is large. Since $\bar{f}_n = \bar{f}$, we see there is considerable overlap of the sets where $S_n(f,\cdot)$ is large as n varies. This leads easily to

THEOREM 2. *Given $\{n_k\}_{k\geqslant 1}$ and $a > 1$ with $n_{k+1} > an_k$, we have $S_{n_k}(f,x) = o(\log\log n_k)$ a.e. and*

$$m\left\{x \in (-\pi,\pi): \sup_k \frac{|S_{n_k}(f,x)|}{\log\log n_k} > y\right\} \leqslant C\frac{\|f\|_1}{y},$$

$f \in L^1(-\pi,\pi)$.

Theorem 2 is best possible in the sense there are functions $f \in L^1$ with $S_{n_k}(f,\cdot)$ growing a.e. at a rate arbitrarily close to $\log\log n_k$. (See Chen [3]).

Note Theorem 2 shows that in order to find a function $f \in L^1$ with $S_n(f,\cdot)$ growing on the order of $\log n$, it would be necessary to consider more than a lacunary subsequence of partial sums. (See Zygmund [5; I, p. 308].)

The proofs of Theorems 1 and 2 are based on a collection of intervals which is related to the size of $\bar{f}$.

Given

$$f \in L^1(-\pi,\pi) \quad \text{and} \quad y \geqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\,dt$$

we define $\mathscr{I}$ to be the collection of all dyadic subintervals I_j of $(-\pi,\pi)$ such that

$$\frac{1}{|I_j|}\int_{I_j} |f(t)|\,dt > 4y, \quad \text{but} \quad \frac{1}{|I|}\int_I |f(t)|\,dt \leqslant 4y$$

for every dyadic subinterval $I \supsetneqq I_j$. (Dyadic subintervals are those obtained by dividing $(-\pi,\pi)$ into 2^n intervals each of length $2\pi\cdot 2^{-n}$, $n = 0,1,\ldots$) Note that the intervals $I_j \in \mathscr{I}$ are pairwise disjoint and

$$4y < \frac{1}{|I_j|}\int_{I_j} |f(t)|\,dt \leqslant 8y.$$

Also, if $x \notin \bigcup I_j$, then

$$\frac{1}{|I|}\int_I |f(t)|\,dt \leqslant 4y$$

for every dyadic subinterval I which contains x. This implies $|f(x)| \leqslant 4y$ for a.e. x not in $\bigcup I_j$.

For each $I_j \in \mathscr{I}$, consider I_j as a set on the unit circle, and let I_j^* denote the interval on the unit circle which has the same center as I_j and $|I_j^*| = 3|I_j|$. Set $S^* = \bigcup I_j^*$ and let P denote the complement of S^*. Note

$$mS^* \leqslant \sum |I_j^*| = 3\sum |I_j| \leqslant \frac{3}{4y}\int_{-\pi}^{\pi} |f(t)|\,dt.$$

Also, if $x \in S^*$, say $x \in I_j^*$, let I be the smallest interval with center x and $I \supset I_j$. Then $|I| \leqslant 4|I_j|$. Hence

$$\bar{f}(x) \geqslant \frac{1}{|I|}\int_I |f(t)|\,dt \geqslant \frac{1}{4|I_j|}\int_{I_j} |f(t)|\,dt > y.$$

It follows that $S^* \subset \{\bar{f}(x) > y\}$ or $\{\bar{f}(x) \leqslant y\} \subset P$.

In order to prove Theorem 1 we set

$$g(x) = \begin{cases} \dfrac{1}{|I_j|}\displaystyle\int_{I_j} f(t)\,dt, & x \in I_j, \\ f(x), & x \notin \bigcup I_j \end{cases}$$

and $b(x) = f(x)-g(x)$. Note that $\|g\|_\infty \leqslant 8y$, $\|g\|_1 \leqslant \|f\|_1$, $b \in L^1$, $b(x) = 0$ for $x \notin \bigcup I_j$, $\int_{I_j} b(t)\,dt = 0$, and $\int_{I_j} |b(t)|\,dt \leqslant Cy|I_j|$. We have

$$\tilde{b}(x) = -\frac{1}{\pi}\sum \int_{I_j} b(t)\frac{1}{2\tan\frac{1}{2}(t-x)}\,dt$$

$$= -\frac{1}{\pi}\sum \int_{I_j} b(t)\left[\frac{1}{2\tan\frac{1}{2}(t-x)} - \frac{1}{2\tan\frac{1}{2}(t_j-x)}\right]dt$$

$$= -\frac{1}{\pi}\sum \int_{I_j} b(t)\left[\frac{\sin\frac{1}{2}(t_j-t)}{2\sin\frac{1}{2}(t-x)\sin\frac{1}{2}(t_j-x)}\right]dt,$$

where t_j denotes the center of I_j. Note that $x \notin I_j^*$ implies

$$|x-t_j| \geqslant |I_j| \quad\text{and}\quad (2/3)|x-t| \leqslant |x-t_j| \leqslant (3/2)|x-t| \quad \text{for } t \in I_j.$$

It follows that $x \in P$ implies

$$|\tilde{b}(x)| \leqslant C\sum \int_{I_j} |b(t)|\left[\frac{|I_j|}{(t_j-x)^2+|I_j|^2}\right]dt$$

$$\leqslant Cy\sum \frac{|I_j|^2}{(t_j-x)^2+|I_j|^2} \leqslant Cy\sum \int_{I_j} \frac{|I_j|}{(t-x)^2+|I_j|^2}\,dt.$$

At this point we could complete the proof of Theorem 1 by applying the exponential result to $\tilde{g}$ and using Carleson's exponential estimate of the function

$$\Delta(x) = \sum \frac{|I_j|^2}{(t_j-x)^2+|I_j|^2}.$$

Instead, we assume the strong type $(2,2)$ result for the conjugate function (see Zygmund [5; I, p. 253]) and recall how the Calderón–Zygmund decomposition is used to obtain the weak type $(1,1)$ result. This allows an interesting comparison between the weak type $(1,1)$ proof and the proof of Theorem 1. Also, it is interesting that the weak type $(1,1)$ result implies the exponential result. This illustrates a double use of the Calderón–Zygmund decomposition in the proof of Theorem 1.

We wish to prove the weak type $(1,1)$ result,

$$m\{x \in (-\pi,\pi):\ |\tilde{f}(x)| > y\} \leqslant \frac{C}{y}\|f\|_1.$$

Applying the strong type $(2,2)$ result to g we have

$$m\{|\tilde{g}(x)| > y\} \leqslant \frac{1}{y^2}\int |\tilde{g}|^2 \leqslant \frac{C}{y^2}\int |g|^2 \leqslant \frac{C}{y}\int |g| \leqslant \frac{C}{y}\int |f|.$$

Note $mS^* \leqslant \frac{C}{y}\int |f|$ and

$$m\{x \in P:\ |\tilde{b}(x)| > y\} \leqslant \frac{1}{y}\int_P |\tilde{b}(x)|\,dx$$

$$\leqslant C\sum \int_{I_j}\left(\int \frac{|I_j|}{(t-x)^2+|I_j|^2}\,dx\right)dt$$

$$\leqslant C\sum |I_j| \leqslant \frac{C}{y}\int |f|.$$

The weak type $(1,1)$ result follows.

We will need the exponential result,

$$m\{x \in (-\pi,\pi):\ |\tilde{f}(x)| > y\} \leqq C\exp(-Cy/\|f\|_\infty), \qquad f \in L^\infty(-\pi,\pi)$$

(see Zygmund [5; I, p. 254]). To obtain this from the weak type $(1,1)$ and the L^2 results, we first apply the Marcinkiewicz Interpolation Theorem to obtain $\|\tilde{f}\|_p \leqslant C_p\|f\|_p$, $1 < p < 2$, where $C_p = O\big(1/(p-1)\big)$ $(p \to 1)$. A duality argument and an extrapolation theorem then yield the exponential result (see Zygmund [5; II, p. 112 and p. 119]).

We now return to the proof of Theorem 1.

We have $f = g+b$. Since $\|g\|_\infty \leqslant 8y$ we can use the exponential result in place of the L^2 result to obtain

$$m\{x \in (-\pi,\pi):\ |\tilde{g}(y)| > \lambda y\} \leqslant C\exp(-C\lambda y/\|g\|_\infty) \leqslant Ce^{-C\lambda}.$$

Since $\{\bar{f}(x) \leqslant y\} \subset P$, it is now enough to show

$$m\{x \in P:\ |\tilde{b}(x)| > \lambda y\} \leqslant Ce^{-C\lambda}.$$

(Note there is nothing to prove if $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|\,dt > y$, since this implies $\bar{f}(x) > y$ for all x.) For $x \in P$, we have

$$|\tilde{b}(x)| \leqslant Cy\,\delta(x),$$

where

$$\delta(x) = \sum \int_{I_j} \frac{|I_j|}{(x-t)^2+|I_j|^2}\,dt.$$

Hence, it is sufficient to show

$$m\{x \in (-\pi,\pi):\ \delta(x) > \lambda\} \leqslant Ce^{-C\lambda}.$$

Following Carleson, we let $E = \{x \in (-\pi, \pi): \ \delta(x) > \lambda\}$ and set $\varphi(x) = \chi_E(x)/(mE\cdot\log(C/mE))$. Then

$$\lambda(\log(C/mE))^{-1} \leqslant \int_{-\pi}^{\pi} \varphi(x)\,\delta(x)\,dx$$
$$= \int_{-\pi}^{\pi} \varphi(x)\Big[\sum \int_{I_j} \frac{|I_j|}{(x-t)^2+|I_j|^2}\,dt\Big]dx$$
$$\leqslant \sum \int_{I_j}\Big[\int_{-\pi}^{\pi} \varphi(x)\frac{|I_j|}{(x-t)^2+|I_j|^2}\,dx\Big]dt$$
$$\leqslant \sum \int_{I_j}\Big[\frac{1}{|I_j|}\int_{|x-t|\leqslant|I_j|}\varphi(x)\,dx+$$
$$+\sum_k \frac{|I_j|}{(2^k|I_j|)^2}\int_{2^k|I_j|<|x-t|\leqslant 2^{k+1}|I_j|}\varphi(x)\,dx\Big]dt$$
$$\leqslant C\int_{-\pi}^{\pi}\bar{\varphi}(t)\,dt \leqslant C\int_{-\pi}^{\pi}\varphi(t)\log^+\varphi(t)\,dt + C \leqslant C$$

(see Zygmund [5; I, p. 33]). Hence, $mE \leqslant Ce^{-C\lambda}$ and this completes the proof of Theorem 1.

Given a lacunary sequence $\{n_k\}_{k\geqslant 1}$ we will show that

$$\sup_k \frac{|S_{n_k}(f,x)|}{\log\log n_k} < \infty$$

a.e. for all $f \in L^1(-\pi, \pi)$. A theorem of E. M. Stein then yields the weak type (1, 1) estimate of Theorem 2 (see Stein [4]) and the "o" result follows.

Set $f_{n,c}(x) = f(x)\cos nx$, $f_{n,s}(x) = f(x)\sin nx$, and let f_n denote either $f_{n,c}$ or $f_{n,s}$. Since

$$S_n(f,x) = \tilde{f}_{n,c}(x)\sin nx - \tilde{f}_{n,s}(x)\cos nx + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos n(t-x)\,dt,$$

we see that it is sufficient to show that

$$\sup_k \frac{|\tilde{f}_{n_k}(x)|}{\log\log n_k} < \infty \text{ a.e.}$$

Using the intervals I_j and the sets S^* and P as previously defined, we set

$$g_n(x) = \begin{cases} \dfrac{1}{|I_j|}\displaystyle\int_{I_j} f_n(t)\,dt, & x \in I_j, \\ f_n(x), & x \notin \bigcup I_j \end{cases}$$

and $b_n = f_n - g_n$. g_n and b_n enjoy the properties of the previously defined functions g and b. It follows that

$$m\{x \in P: \ |\tilde{f}_n(x)| > y^2\log\log n\} \leqslant Ce^{-Cy\log\log n} = C(\log n)^{-C\lambda}.$$

Hence,

$$m\Big\{x \in (-\pi, \pi): \ \sup_k \frac{|\tilde{f}_{n_k}(x)|}{\log\log n_k} > y^2\Big\}$$
$$\leqslant mS^* + \sum_k \{x \in P: \ |\tilde{f}_{n_k}(x)| > y^2\log\log n_k\}$$
$$\leqslant \frac{C\|f\|_1}{y} + C\sum_k (\log n_k)^{-Cy}.$$

We are done if the above sum approaches zero as y approaches infinity. In particular, if $n_{k+1} > an_k$, $a > 1$, and $\log n_1 > 1$, we have $n_{k+1} > a^k n_1$, so

$$\sum_{k=1}^{\infty}(\log n_k)^{-Cy} \leqslant (\log n_1)^{-Cy} + \sum_{k=1}^{\infty}(k\log a + \log n_1)^{-Cy},$$

which approaches zero as y approaches infinity.

References

[1] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88 (1952), pp. 85–139.

[2] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), pp. 135–157.

[3] Y. M. Chen, *A remarkable divergent Fourier series*, Proc. Japan Acad. 38 (1962), pp. 239–244.

[4] E. M. Stein, *On limits of sequences of operators*, Annals of Math. 74 (1961), pp. 140–170.

[5] A. Zygmund, *Trigonometric series*, 2nd. ed., New York 1959.

PURDUE UNIVERSITY
LAFAYETTE, INDIANA

Received September 27, 1971 (402)