An estimate of the conjugate function

by

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Abstract. It is shown that the conjugate function of a function \( f \in L^1(-\pi, \pi) \) is of exponential type off any set where the Hardy-Littlewood maximal function of \( f \) is uniformly bounded. Thus, there is considerable overlap between the set where the conjugate function is large and the set where the Hardy-Littlewood maximal function is large. This general principle is used to show that \( S_{n_k}(f, x) = o(\log \log n_k) \) a.e., where \( S_{n_k}(f, x) \) is the \( n_k \)th partial sum of the Fourier series of \( f \in L^1(-\pi, \pi) \) and \( (n_k) \) is any lacunary sequence.

We will show that the conjugate function of a function \( f \in L^1(-\pi, \pi) \) is of exponential type off any set where the Hardy-Littlewood maximal function of \( f \) is uniformly bounded. Thus, there is considerable overlap between the set where the conjugate function is large and the set where the Hardy-Littlewood maximal function is large. This general principle will be used to obtain a result on the rate of growth of certain partial sums of the Fourier series of \( f \).

\[
\hat{f}(\omega) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{2 \tan \frac{1}{2}(t-\omega)} \, dt
\]

will denote the conjugate function of \( f \in L^1(-\pi, \pi) \),

\[
f(\omega) = \sup \left\{ \frac{1}{|I|} \int_I |f(t)| \, dt : I \text{ an interval with center } \omega \right\}
\]

will denote the Hardy-Littlewood maximal function of \( f \), and

\[
S_n(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin (\pi + \frac{1}{2})(t-\omega)}{2 \sin \frac{1}{2}(t-\omega)} \, dt
\]

will denote the \( n \)th partial sum of the Fourier series of \( f \). \( m_E \) will denote the Lebesgue measure of the set \( E \) and \( C \) will denote an absolute constant, although not always the same constant.

* This work was partially supported by the National Science Foundation GP-18491 and the Alfred P. Sloan Foundation.
The idea of the proof of our main result is to take a double look at the Calderón-Zygmund proof that \( \tilde{f} \) satisfies a weak type \((1, 1)\) estimate. For \( f \in L^1 \) we use the Calderón-Zygmund decomposition \( f = g + b \), where \( g \) is bounded and \( b \) is supported on a set of small measure. (See Calderón and Zygmund [1]). The weak type \((1, 1)\) estimate for \( \tilde{f} \) is obtained by applying the strong type \((2, 2)\) result to \( \tilde{g} \) and noting that \( \tilde{b} \) is integrable off a set of small measure. The weak type \((1, 1)\) estimate for \( \tilde{f} \) when \( f \in L^1 \) leads to an exponential type estimate for \( \tilde{f} \) when \( f \in L^\infty \). With this in mind we take a second look at the \( L^1 \) proof. Since \( g \) is bounded, \( \tilde{g} \) is of exponential type. Using a lemma of L. Carleson (see Carleson [2, p. 140]) we show that \( \tilde{b} \) is also of exponential type when restricted to a set where \( f \) is bounded. This yields

**Theorem 1.** \( m \{ x \in (-\pi, \pi); \tilde{f}(x) \leq y, |\tilde{f}(x)| > \lambda y \} \leq C e^{-\lambda t}, \) where \( y > 0, \lambda > 0 \) and \( f \in L^1 (-\pi, \pi). \)

The application of our main result to Fourier series is based on the fact that \( S_\alpha(f, \cdot) \) is essentially \( f_{\alpha} \), where \( f_{\alpha}(x) = e^{i\alpha x} \). Hence, \( S_\alpha(f, \cdot) \) is of exponential type if we avoid the set where \( f_{\alpha} \) is large. Since \( \tilde{f}_{\alpha} = \tilde{f} \), we see there is considerable overlap of the sets where \( S_\alpha(f, \cdot) \) is large as \( \alpha \) varies. This leads easily to

**Theorem 2.** Given \( (n_k) \in N \) and \( a > 1 \) with \( n_{k+1} > an_k \), we have \( S_n(f, x) = o(\log \log \log n) \) a.e. and

\[
\sup_k \frac{|S_n(f, x)|}{\log \log n} > y \leq C \frac{|f(x)|}{y},
\]

\( f \in L^1 (-\pi, \pi). \)

Theorem 2 is best possible in the sense there are functions \( f \in L^1 \) with \( S_n(f, \cdot) \) growing a.e. at a rate arbitrarily close to \( \log \log n \). (See Chen [3]).

Note Theorem 3 shows that in order to find a function \( f \in L^1 \) with \( S_n(f, \cdot) \) growing on the order of \( \log \log n \), it would be necessary to consider more than a lacunary sequence of partial sums. (See Zygmund [5; I, p. 308].)

The proofs of Theorems 1 and 2 are based on a collection of intervals which is related to the size of \( f \).

**Given**

\[
\begin{align*}
f & \in L^1 (-\pi, \pi) \quad \text{and} \\
y & \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| dt
\end{align*}
\]

we define \( \mathcal{J} \) to be the collection of all dyadic subintervals \( I \) of \((-\pi, \pi)\) such that

\[
\frac{1}{|I|} \int_{I} |f(t)| dt > 4y, \quad \text{but} \quad \frac{1}{|I|} \int_{I} |f(t)| dt \leq 4y
\]

for every dyadic subinterval \( I \in \mathcal{J} \). (Dyadic subintervals are those obtained by dividing \((-\pi, \pi)\) into 2\( n \) intervals each of length \( 2\pi \cdot 2^{-n} \), \( n = 0, 1, \ldots \).)

Note that the intervals \( I \in \mathcal{J} \) are pairwise disjoint and

\[
4y \leq \frac{1}{|I|} \int_{I} |f(t)| dt \leq 8y.
\]

Also, if \( x \in \bigcup I \), then

\[
\frac{1}{|I|} \int_{I} |f(t)| dt \leq 4y
\]

for every dyadic subinterval \( I \) which contains \( x \). This implies \( |f(x)| \leq 4y \) for a.e. \( x \) not in \( \bigcup I \).

For each \( I \in \mathcal{J} \), consider \( I \) as a set on the unit circle, and let \( I^* \) denote the interval on the unit circle which has the same center as \( I \) and \( |I^*| = 3|I| \). Set \( S^* = \bigcup I^* \) and let \( P \) denote the complement of \( S^* \). Note

\[
\mu S^* \leq \sum |I^*| = 3 \sum |I| \leq 3 \frac{1}{4y} \int_{-\pi}^{\pi} |f(t)| dt.
\]

Also, if \( x \in S^* \), say in \( I^* \), let \( I \) be the smallest interval with center \( x \) and \( I \supseteq I^* \). Then \( |I| \leq 4|I^*| \). Hence

\[
\tilde{f}(x) \geq \frac{1}{|I|} \int_{I} |f(t)| dt \geq \frac{1}{4|I^*|} \int_{I^*} |f(t)| dt > y.
\]

It follows that \( S^* \cap \{ \tilde{f}(x) > y \} = \emptyset \) or \( |f(x)| \leq 8y \) in \( P \).

In order to prove Theorem 1 we set

\[
g(x) = \begin{cases} 
\frac{1}{|I|} \int_{I} f(t) dt, & x \in I, \\
f(x), & x \notin \bigcup I
\end{cases}
\]
and \( b(x) = f(x) - g(x) \). Note that \( \|g\|_\infty \leq 8y \), \( \|g\|_1 \leq \|f\|_1 \), \( b \circ \mathbb{1}_I \), \( b(x) = 0 \) for \( x \notin \bigcup_I I_j \), \( \int I_j b(t) \, dt = 0 \), and \( \int I_j b(t) \, dt \leq C y \|f\|_1 \). We have

\[
\hat{b}(x) = -\frac{1}{\pi} \sum_I \int_{I_j} b(t) \frac{1}{2 \tan \frac{1}{2} (t - x)} \, dt \\
= -\frac{1}{\pi} \sum_I \int_{I_j} b(t) \left[ \frac{1}{2 \tan \frac{1}{2} (t - x)} - \frac{1}{2 \tan \frac{1}{2} (t_j - x)} \right] \, dt \\
= -\frac{1}{\pi} \sum_I \int_{I_j} b(t) \left[ \frac{\sin \frac{1}{2} (t_j - t)}{2 \sin \frac{1}{2} (t - x) \sin \frac{1}{2} (t_j - x)} \right] \, dt,
\]

where \( t_j \) denotes the center of \( I_j \). Note that \( x \notin \mathbb{1}_I I_j \) implies \( |x - t_j| \gg |I_j| \) and \( (2/3) |x - t| \leq |x - t_j| \leq |I_j| \) for \( t \in I_j \).

It follows that \( x \notin P \) implies

\[
|\hat{b}(x)| \leq C \sum_I \int_{I_j} |b(t)| \left[ \frac{|I_j|}{(t_j - x)^2 + |I_j|^3} \right] \, dt \\
\leq C y \sum_I \int_{I_j} \frac{|I_j|^3}{(t_j - x)^2 + |I_j|^3} \, dt.
\]

At this point we could complete the proof of Theorem 1 by applying the exponential result to \( \hat{g} \) and using Carlsson's exponential estimate of the function

\[
A(x) = \sum \frac{|I_j|^3}{(t_j - x)^2 + |I_j|^3}.
\]

Instead, we assume the strong type (2, 2) result for the conjugate function (see Zygmund [5; I, p. 233]) and recall how the Calderón–Zygmund decomposition is used to obtain the weak type \((1, 1)\) result. This allows an interesting comparison between the weak type \((1, 1)\) proof and the proof of Theorem 1. Also, it is interesting that the weak type \((1, 1)\) result implies the exponential result. This illustrates a double use of the Calderón–Zygmund decomposition in the proof of Theorem 1.

We wish to prove the weak type \((1, 1)\) result,

\[
m(x \circ \cdot - \eta, \mathbb{1}_I; \mathbb{1}_I) \sim C \|f\|_1.
\]

Applying the strong type \((2, 2)\) result to \( g \) we have

\[
m(g(x)) > y \leq \frac{1}{y^2} \int \hat{g}^2 \leq C \int |g|^2 \leq C \int |g| \leq C \int |f|.
\]

Note \( m S^* \leq \frac{C}{y} \int |f| \) and

\[
m(x \circ P; \hat{b}(x)) > y \leq \frac{1}{y^2} \int \hat{b}(x) \, dx \\
\leq C \sum_I \int \left( \frac{|I_j|^3}{(t_j - x)^2 + |I_j|^3} \right) \, dx \\
\leq C \sum_I |I_j| \leq \frac{C}{y} \int |f|.
\]

The weak type \((1, 1)\) result follows.

We will need the exponential result,

\[
m(x \circ \cdot - \eta, \mathbb{1}_I; \mathbb{1}_I) \sim C \exp(-Cy\|f\|_1), \quad f \circ \mathbb{1}_I(\cdot - \eta)
\]

(see Zygmund [5; I, p. 234]). To obtain this from the weak type \((1, 1)\) and the \(L^2\) results, we first apply the Marcinkiewicz Interpolation Theorem to obtain \(|\mathbb{1}_x f\|_p \leq C_p \|f\|_{L^p}, \quad 1 \leq p < 2\), where \( C_p = O(1/(p-1)) \) (\( p \to 1 \)).

A duality argument and an extrapolation theorem then yield the exponential result (see Zygmund [5; II, p. 112 and p. 119]).

We now return to the proof of Theorem 1.

We have \( f = g + b \). Since \( \|g\|_\infty \leq 8y \) we can use the exponential result in place of the \( L^2 \) result to obtain

\[
m(x \circ \cdot - \eta, \mathbb{1}_I; \mathbb{1}_I) \sim C \exp(-Cy\|g\|_\infty) \leq C e^{-Ct}.
\]

Since \( \{f(x) \leq y\} \subset P \), it is now enough to show

\[
m(x \circ P; \hat{b}(x)) > \lambda y \leq C e^{-Ct}.
\]

(Note there is nothing to prove if \( \frac{1}{2\pi} \int |f(t)| \, dt > y \), since this implies \( f(x) > y \) for all \( x \).) For \( x \circ P \), we have

\[
|\hat{b}(x)| \leq C y \delta(x),
\]

where

\[
\delta(x) = \sum_{I_j} \int \frac{|I_j|^3}{(t_j - x)^2 + |I_j|^3} \, dx.
\]

Hence, it is sufficient to show

\[
m(x \circ \cdot - \eta, \mathbb{1}_I; \delta(x) > \lambda) \leq C e^{-Ct}.
\]
Following Carleson, we let $E = \{x \in (-\pi, \pi): \delta(x) > 1\}$ and set $\psi(x) = \mu_{E}(x|mE, \log(C|mE))$. Then

$$
\lambda \log(C|mE)^{-1} \leq \int_{-\pi}^{\pi} \psi(x) \delta(x) \, dx
$$

$$
= \int_{-\pi}^{\pi} \psi(x) \left[ \sum_{I_{j}} \int_{I_{j}} \frac{|I_{j}|}{|x-t|^{2} + |I_{j}|^{2}} \, dt \right] \, dx
$$

$$
\leq \sum_{I_{j}} \left[ \int_{-\pi}^{\pi} \psi(x) \frac{|I_{j}|}{|x-t|^{2} + |I_{j}|^{2}} \, dx \right] \, dt
$$

$$
\leq \sum_{I_{j}} \left[ \frac{|I_{j}|}{|x-t|^{2}} + \int_{-\pi}^{\pi} \psi(x) \, dx \right] \, dt
$$

$$
+ \sum_{j} \frac{|I_{j}|}{2(2|I_{j}|)^{2}} \int_{y \in I_{j} \cap \log(2)} \int_{y \in I_{j} \cap \log(1/2)} \psi(x) \, dx \right] \, dt
$$

$$
\leq C \int_{-\pi}^{\pi} \psi(t) \, dt \leq C \int_{-\pi}^{\pi} \psi(t) \log^{2} \psi(t) \, dt + C \leq C
$$

(see Zygmund [5; I, p. 33]). Hence, $mE \leq C \delta^{-C}$ and this completes the proof of Theorem 1.

Given a lacunary sequence $\{n_{k}\}_{k>1}$, we will show that

$$
\sup_{k} \frac{|S_{n_{k}}(f; \omega)|}{\log \log n_{k}} < \infty
$$

a.e. for all $f \in L^{1}(-\pi, \pi)$. A theorem of E. M. Stein then yields the weak type $(1, 1)$ estimate of Theorem 2 (see Stein [4]) and the "$o$" result follows.

Set $f_{n_{k}}(x) = f(x) \sin n_{k}x$, $f_{n_{k}}(x) = f(x) \sin n_{k}x$, and let $f_{n_{k}}$ denote either $f_{n_{k}}$ or $f_{n_{k}}$. Since

$$
S_{n}(f, \omega) = f_{n_{k}}(x) \sin n_{k}x - f_{n_{k}}(x) \sin n_{k}x + \frac{1}{2n_{k}} \int_{-\pi}^{\pi} f(t) \cos n_{k}(t-x) \, dt,
$$

we see that it is sufficient to show that

$$
\sup_{k} \frac{|f_{n_{k}}(x)|}{\log \log n_{k}} < \infty \text{ a.e.}
$$

References


