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Received September 23, 1971 (392)

STUDIA MATHEMATICA T. XLIV. (1972)

The integrability class of the sine transform of a monotonic function

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Abstract. This note introduces a method for discussing the weighted Lebesgue class of the Fourier transform of a monotonic function, a method that is rather more direct than those that have been used for similar problems about Fourier series. The method depends on Steffensen's version of Jensen's inequality (see [6], p. 109 ff.). and a theorem of S. M. Edmonds [4] on Parseval's theorem for monotonic functions.

Two classical theorems of Hardy and Littlewood (see [8], vol. 2, p. 129–130) state that if φ is an integrable function, λ_n are its Fourier sine or cosine coefficients, and $1 < p < \infty$, then

(1) *If* $\lambda_n \downarrow 0$, *then* $\varphi \in L^p$ *if and only if* $n^{1-2/p}\lambda_n \in l^p$.

(2) *If* $\varphi(x) \geqslant 0$ *and* φ *decreases on* $(0, \pi)$, *then* $\lambda_n \in l^p$ *if and only if* $x^{1-2/p}\varphi(x) \in L^p$.

These theorems have been extended (see [1], p. 35) to weighted L^p and l^p spaces. Here and subsequently, $p' = p/(p-1)$.

(3) (Generalization of (1) by Y.-M. Chen) *If* $\lambda_n \downarrow 0$, *then* $x^{-\gamma}\varphi(x) \in L^p$, $-1/p' < \gamma < 1/p$, *if and only if* $n^{\gamma+1-2/p}\lambda_n \in l^p$. *Alternatively* (*replace* γ *by* $(2/p)-1-\gamma$) *if* $\lambda_n \downarrow 0$, *then* $n^{-\gamma}\lambda_n \in l^p$ *if and only if* $x^{\gamma+1-2/p}\varphi(x) \in L^p$, $-1/p' < \gamma < 1/p$.

The corresponding generalization (4) of (2) has the same conclusions but the hypothesis $\lambda_n \downarrow 0$ is replaced by $\varphi(x) \geqslant 0, \varphi$ decreasing, and φ integrable (all on $(0, \pi)$).

Theorems (3) and (4) reduce to (1) and (2) if in each we take $\gamma = 0$ in the first statement or $\gamma = (2/p)-1$ in the second statement.

Theorems of this kind naturally have analogues for Fourier transforms. A partial analogue of (1) is given by Titchmarsh ([7], p. 113). This is:

(5) *If* $\varphi(x) \downarrow 0$ *on* $(0, \infty)$, Φ *is its cosine transform, and* $\varphi(x)x^{1-2/p} \in L^p$ $(1 < p < 2)$, *then* $\Phi \in L^p$.

Disregarding the distinction between a function's having a Fourier transform and its being a Fourier transform, we may loosely formulate a hypothetical integral analogue of (3) and (4) as follows:

(6) *If Φ and φ are a pair of Fourier sine or cosine transforms, and one of them is positive and decreasing on $(0,\infty)$, then $x^{-\gamma}\varphi(x)\in L^p$ if and only if $x^{1+\gamma-2/p}\Phi(x)\in L^p$ provided that $-1/p'<\gamma<1/p$.*

Of course φ and Φ can be interchanged in (6).

As far as I know, nobody has written out a proof of any version of (6). Here I shall establish a partial result for sine transforms.

THEOREM. *If $\varphi(x)\downarrow 0$, $x^{1/p}\varphi(x)\in L^p(0,1)$, and Φ is the sine transform of φ, then $x^{\gamma+1-2/p}\Phi(x)\in L^p$ provided that $x^{-\gamma}\varphi(x)\in L^p$, where $p>1$ and $-1/p<\gamma<1/p$.*

Note that the range for γ is not the same as in (6); it is wider when $p<2$ but narrower when $p>2$.

We need the following version of the Jensen–Steffensen inequality ([2]; see also [3]).

LEMMA. *Let λ be a function of bounded variation on every finite subinterval of $(0,\infty)$; $\lambda(0)\leqslant\lambda(x)$ for all $x>0$; and $\lambda(0)<\Lambda=\sup\lambda(x)$. Let $f(x)$ decrease and $f(x)\geqslant 0$. If ψ is continuous and convex over $(0,f(0))$, $\psi(0)\leqslant 0$, and $\int_0^\infty d\mu(x)\geqslant\Lambda-\lambda(0)$, then*

$$\psi\left\{\frac{\int_0^\infty f(x)\,d\lambda(x)}{\int_0^\infty d\mu(x)}\right\}\leqslant\frac{\int_0^\infty \psi\big(f(x)\big)\,d\lambda(x)}{\int_0^\infty d\mu(x)}.$$

We use this when $\psi(u)=u^p$, $p>1$; in this case it says in particular that

$$\left\{\int_0^\infty f(x)\,d\lambda(x)\right\}^p\leqslant\Lambda^{p-1}\int_0^\infty f(x)^p\,d\lambda(x). \tag{7}$$

We take $\lambda(x)=1-\cos x$, $\Lambda=2$, and apply (7) to $f(x)x^{a-1}$, where $-1<a<1$ and $f(x)x^{a-1}$ decreases. We then have

$$\left\{\int_0^\infty f(t)t^{a-1}\sin t\,dt\right\}^p\leqslant 2^{p-1}\int_0^\infty f(t)^p t^{p(a-1)}\sin t\,dt, \tag{8}$$

where the right-hand side is positive, and finite if $f(t)t^{a-1+1/p}\in L^p(0,1)$. Now replace $f(t)$ by $f(xt)$, multiply (8) by $x^{-\delta p}$, and integrate over $(0,\infty)$. We obtain

$$\int_0^\infty x^{-\delta p}\,dx\left\{\int_0^\infty f(xt)t^{a-1}\sin t\,dt\right\}^p\leqslant 2^{p-1}\int_0^\infty x^{-\delta p}\,dx\int_0^\infty f(xt)^p t^{p(a-1)}\sin t\,dt. \tag{9}$$

Put $x=1/y$, $t=yu$; (9) now reads

$$\int_0^\infty y^{\delta p-2+ap}\,dy\left\{\int_0^\infty f(u)u^{a+1}\sin yu\,du\right\}^p \leqslant 2^{p-1}\int_0^\infty y^{\delta p-1+p(a-1)}\,dy\int_0^\infty f(u)^p u^{p(a-1)}\sin yu\,du. \tag{10}$$

We now want to change the order of integration on the right-hand side of (10). The following argument lets us do this under less restrictive conditions than would be required for applying Fubini's theorem.

The integral on the right of (10) is of the form

$$(\pi/2)^{\frac12}\int_0^\infty B(y)b(y)\,dy,$$

where $B(y)=y^{p(\delta+a-1)-1}$ and b is the sine transform of $g(u)=f(u)^p u^{p(a-1)}$. It is a theorem of Edmonds ([4]; see also [1], p. 58) that if g and B decrease (hence if $p(\delta+a-1)<1$) and $ug(u)$, $yB(y)$ are integrable on $(0,1)$ (hence if $p(\delta+a-1)>-1$), then $g(u)G(u)$ is integrable (where G is the sine transform of B), and Parseval's theorem holds, i.e. the right-hand side of (10) is equal to

$$\begin{aligned}
2^{p-1}(\pi/2)^{\frac12}\int_0^\infty G(u)g(u)\,du
&=2^{p-1}\int_0^\infty f(u)^p u^{p(a-1)}\,du\int_0^\infty y^{p(\delta+a-1)-1}\sin yu\,dy\\
&=2^{p-1}\int_0^\infty f(u)^p u^{-\delta p}\,du\int_0^\infty t^{p(\delta+a-1)-1}\sin t\,dt\\
&=2^{p-1}\Gamma\big(p(\delta+a-1)\big)\big(\sin\tfrac12\pi p(\delta+a-1)\big)\int_0^\infty f(u)^p u^{-\delta p}\,du.
\end{aligned}$$

We have therefore justified the change in the order of integration on the right of (10) under the conditions $-1<p(\delta+a-1)<1$. Under these conditions, then, we have

$$\int_0^\infty y^{p(\delta+a)-2}\,dy\left\{\int_0^\infty f(u)u^{a-1}\sin yu\,du\right\}^p\leqslant C_{a,\delta,p}\int_0^\infty f(u)^p u^{-\delta p}\,du, \tag{11}$$

where

$$C_{a,\delta,p}=2^{p-1}\Gamma\big(p(\delta+a-1)\big)\big((\sin\tfrac12\pi p(\delta+a-1)\big) \tag{12}$$

(when $\delta+a=1$, the right-hand side of (12) is to be interpreted as $2^{p-1}\pi/2$).

Now take $f(u) = \varphi(u)u^{1-\alpha}$, where $\varphi(u)\downarrow 0$, and write $\delta = 1-\alpha+\gamma$. Then

$$\int_0^\infty y^{p+p\gamma-2}dy\Big\{\int_0^\infty \varphi(u)\sin yu\,du\Big\}^p \leqslant 2^{p-1}\Gamma(p\gamma)(\sin\tfrac{1}{2}\pi p\gamma)\int_0^\infty \varphi(u)^p u^{-\gamma p}du,$$

with $|\gamma| < 1/p$. This is the conclusion of the theorem.

The analogy with trigonometric series suggests that the condition $\gamma > -1/p$ may be an artifact of the proof. The condition $\gamma < 1/p$, on the other hand, is essential, since when $\lim\limits_{x\to 0+}\varphi(x) > 0$ we cannot have $x^{-\gamma}\varphi(x) \in L^p(0,1)$ with $\gamma \geqslant 1/p$. It is less obvious that we also cannot have $x^{\gamma+1-2/p}\Phi(x)\in L^p$ when $\gamma > 1/p$, at least as long as $\gamma < 1+1/p$. To see this, suppose that $\gamma p > 1$ and $x^{\gamma+1-2/p}\Phi(x)\in L^p$. Then we would have, since $\Phi(x)\geqslant 0$,

$$\begin{aligned}\int_0^\infty x^{-\gamma p}\varphi(x)^p dx &= (2/\pi)^{\frac12}\int_0^\infty x^{-\gamma p}dx\Big\{\int_0^\infty \Phi(t)\sin xt\,dt\Big\}^p\\ &\leqslant 2^{p-1}(2/\pi)^{\frac12}\Big\{\int_0^\infty x^{-\gamma p}dx\Big(\int_0^{1/x}\Phi(t)\,xt\,dt\Big)^p+\\ &\quad+\int_0^\infty x^{-\gamma p}dx\Big(\int_{1/x}^\infty \Phi(t)\,dt\Big)^p\Big\}\\ &= 2^{p-1}(2/\pi)^{\frac12}\Big\{\int_0^\infty y^{\gamma p-p+2}dy\Big(\int_0^y \Phi(t)\,t\,dt\Big)^p+\\ &\quad+\int_0^\infty y^{\gamma p-2}dy\Big(\int_y^\infty \Phi(t)\,dt\Big)^p\Big\}\\ &\leqslant 2^{p-1}(2/\pi)^{\frac12}\Big\{\Big(\frac{p}{p+1-p\gamma}\Big)^p\int_0^\infty \Phi(t)^p t^{p\gamma+p-2}dt+\\ &\quad+\Big(\frac{p}{p\gamma-1)}\Big)^p\int_0^\infty t^{p\gamma+p-2}\Phi(t)^p dt\Big\},\end{aligned}$$

by a variant of Hardy's inequality ([5], Theorem 330), provided that $1<\gamma p<p+1$. We have assumed the right-hand side to be finite; then the left-hand side would be finite also, but this is impossible when $\gamma p > 1$ since $\lim\limits_{x\to 0+}\varphi(x) > 0$.

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Received September 25, 1971 (401)