for each a, the function $x \to \lambda(x)$ is, under the stated assumptions, necessarily continuous on J.)

An interesting consequence is:

COROLLARY 3. *Let J be a compact subset of R^n. Suppose for a certain positive number b each point of J is a point of density $\geqslant b$ relative to J. Let $\varphi \in H^\infty(R^n)$, and suppose the restriction of f to J coincides a.e. with a continuous function λ on J. Then, for each $k \in L^1(R^n)$ satisfying $\int k\,dx = 1$,*

$$\lim_{a\to 0+} \max_{\xi\in J} \left|\lambda(\xi) - \int \varphi(\xi - ax)\,k(x)\,dx\right| = 0.$$

This result (which answers question (a) posed in [2], p. 116) can be viewed as a very strong "localization principle" for compact sets. In particular, on a compact subset J of the circle satisfying the density condition of Corollary 3, the Fejér means of any $f \in H^\infty(T)$ whose restriction to J is (after correction on a set of measure zero) continuous converge uniformly. Even for J an arc we have not found this result in the literature, although its direct deduction by means of known Tauberian theorems would not be difficult.

In conclusion we remark that the proof of the lemma has a "function algebras" flavor, and an analogously formulated proposition is valid for certain function algebras on T^m (in particular, for the polydisc algebra itself); what is decisive is certain properties of the "Jensen measures" that would be easy to formulate explicitly. For the weak* closure in $L^\infty(T^m)$ of such an algebra the analogs of Corollaries 2 and 3 would then be valid.

Also, it is possible to estimate how rapidly $\Delta(k, a)$ tends to zero, enabling the rate of convergence in (1) to be estimated in terms of $s(a)$ and properties of the kernels k. This will be discussed in [3].

I wish to express my thanks to Charles Fefferman for a valuable remark converning the subject of this paper.

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On the derivation and covering properties of a differentiation basis

by

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Dedicated to Professor Zygmund with gratitude

Abstract. A theorem is presented that determines the type of covering properties of a differentiation basis which differentiates integrals of functions satisfying a rather general integrability condition.

We define a *differentiation basis* $\boldsymbol{R}$ in n-dimensional Euclidean space R^n as a collection of open bounded sets of R^n such that for every $x \in R^n$ there is at least one sequence $\{R_k\} \subset \boldsymbol{R}$ so that $R_k \to x$ as $k \to \infty$ (i.e. for all $k = 1, 2, \ldots, x \in R_k$ and for every neighborhood U of x there is a $k_0 = k_0(U)$ such that $R_k \subset U$ for $k \geqslant k_0$). An example of a differentiation basis in R^n is the collection $\boldsymbol{R}_1$ of all open cubic intervals. Another one is the collection $\boldsymbol{R}_2$ of all open bounded intervals. A third one is the collection $\boldsymbol{R}_3$ of all open bounded rectangles of R^n.

We consider a locally (Lebesgue) integrable function $f\colon R^n \to R$ and define the upper derivative $\overline{D}(\int f, x)$ of $\int f$ with respect to $\boldsymbol{R}$ at the point x in the following way

$$\overline{D}\left(\int f, x\right) = \sup \overline{\lim_{k\to\infty}} \frac{1}{|R_k|} \int_{R_k} f(y)\,dy,$$

where the sup is taken over all sequences $\{R_k\} \subset \boldsymbol{R}$ such that $R_k \to x$ as $k \to \infty$. The *lower derivative* is defined setting $\inf \underline{\lim}$ above. We shall say that $\boldsymbol{R}$ *differentiates* $\int f$ if $\overline{D}(\int f, x) = \underline{D}(\int f, x) = f(x)$ at almost all points $x \in R^n$.

The type of covering properties considered in this paper find its motivation in the following. Let $E \subset R^n$ be bounded and measurable. We say that $\boldsymbol{T} \subset \boldsymbol{R}$ is an $\boldsymbol{R}$-fine cover of E if for every $x \in E$ there is a sequence $\{T_k\} \subset \boldsymbol{T}$ such that $T_k \to x$ as $k \to \infty$. There are differentiation

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bases $\boldsymbol{R}$ such that, given any E bounded and measurable, any $\boldsymbol{R}$-fine cover of $E, \boldsymbol{T}$, and any $\varepsilon > 0$, one can select a finite collection $\{S_k\} \subset \boldsymbol{T}$ such that $|E - \bigcup S_k| < \varepsilon$ and the $\{S_k\}$ are disjoint (i.e. $\sum \chi_{S_k} - \chi_{\cup S_k} \equiv 0$, χ_A representing the characteristic function of the set A). Such is, for example, the basis $\boldsymbol{R}_1$, by virtue of the classical lemma of Vitali, but not $\boldsymbol{R}_2$ nor $\boldsymbol{R}_3$. However, with an appropriate extension of the notion of disjointness, one obtains a positive result for $\boldsymbol{R}_2$.

We shall consider functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying: (i) $\varphi(0) = 0$, (ii) φ is continuous, (iii) φ is strictly increasing, (iv) $\varphi(u) \to \infty$ as $u \to \infty$. Such functions will be called *strength functions*. For such a function we consider its inverse. It is easy to see that φ^{-1} is a also a strength function. Furthermore, if ψ and σ are strength functions, so is ψ_1 defined by $\psi_1(u) = u\sigma^{-1}(\psi(u))$ for $u \geqslant 0$.

We say that a differentiation basis $\boldsymbol{R}$ *has* (covering) *strength* φ whenever for every measurable bounded set E, any $\boldsymbol{R}$-fine cover $\boldsymbol{T}$ of E and any $\varepsilon > 0$ one can select a finite collection $\boldsymbol{T}^* = \{S_k\} \subset \boldsymbol{T}$ such that $|E - \bigcup S_k| < \varepsilon$ and $\int \varphi\big(\sum \chi_{S_k}(x) - \chi_{\cup S_k}(x)\big)\,dx < \varepsilon$. (Observe that these two conditions vaguely mean that E is nearly covered by the S_k's and that the overlap $\sum \chi_{S_k} - \chi_{\cup S_k}$ is "φ-small". In general, for $x \in R^n$ and for a finite sequence of measurable sets $\{A_k\}$ we shall denote $\nu(\{A_k\}, x) = \sum \chi_{A_k}(x) - \chi_{\cup A_k}(x)$ and we call $\nu(\{A_k\}, \cdot)$ the *overlapping function of* $\{A_k\}$).

A theorem of de Possel [7] affirms that for a differentiation basis $\boldsymbol{R}$ the two following statements are equivalent:

(a) $\boldsymbol{R}$ *differentiates* $\int f$, *for all* $f \in L^\infty_{\mathrm{loc}}(R^n)$ (i.e., the space of all measurable functions which are essentially bounded on every compact set).

(b) $\boldsymbol{R}$ *has strength* θ, *for* $\theta(u) = u$.

For a simple proof of this theorem we refer to [2]. In virtue of this theorem, $\boldsymbol{R}_2$ has strength θ, for $\theta(u) = u$, but not $\boldsymbol{R}_3$. For these facts we refer to [1] or [6]. One can also see them in [5].

The problem we deal with in this note is to establish connections between differentiation properties of the type

(A) $\boldsymbol{R}$ *differentiates* $\int f$, *for all* f *in some space of locally integrable functions*

and properties of the type

(B) $\boldsymbol{R}$ *has strength* φ, *for some strength function* φ.

It is rather easy in some cases to obtain theorems of the type $(B) \Rightarrow (A)$. Implications in the other direction seem to be deeper. In 1955, Hayes and Pauc [4] obtained a theorem stating that, if we call, for $1 < p < \infty$, $\varphi_p(u) = u^p$, $u \geqslant 0$, and $q = p/(p-1)$, and if (a), (b), (b_1) are the following statements:

(a) $\boldsymbol{R}$ differentiates $\int f$, for all $f \in L^p(\boldsymbol{R}^n)$,

(b) $\boldsymbol{R}$ has strength φ_q

(b_1) $\boldsymbol{R}$ has strength φ_{q_1}, for all $q_1 < q$,

then one has (b) $\Rightarrow$ (a) and (a) $\Rightarrow$ (b_1).

More recently, other implications of the type (B) $\Rightarrow$ (A) have been obtained, φ being in (B) a function defining an Orlicz space, and the corresponding space in (A) being then the conjugate Orlicz space. For these results one can see [3] and [2].

The following theorem, an extension of the above mentioned theorem of Hayes and Pauc, gives a general result in the direction (A) $\Rightarrow$ (B). The idea of the proof is essentially that of their theorem, which appears then as an easy consequence of this result.

Let σ be a strength function. We define $\sigma(L_{\mathrm{loc}})$ as the space of functions $f\colon R^n \to R$ which are measurable and such that $\int_K \sigma(|f(x)|)\,dx$ for every compact set K. (The spaces $L^p_{\mathrm{loc}}(R^n)$ are particular cases of this type of space. Observe that $\sigma(L_{\mathrm{loc}})$ is not necessarily linear).

THEOREM. *Let ψ and σ be two strength functions and assume $\psi(u) \geqslant u$ for $u \geqslant 1$. Assume also that $\boldsymbol{R}$ is a differentiation basis which differentiates $\int f$ for all $f \in \sigma(L_{\mathrm{loc}})$ and has strength ψ. Then $\boldsymbol{R}$ has also strength ψ_1, where $\psi_1(u) = u\sigma^{-1}(\psi(u))$.*

The theorem will be proved by means of the following lemma.

LEMMA. *Let θ be the identity function, $\theta(u) = u$, $u \geqslant 0$ and φ a strength function such that $\varphi(u)/u$ is non-decreasing. Assume $\boldsymbol{R}$ is a differentiation basis having strength θ but not strength φ. Then there exist a set $E \subset R^n$, bounded and measurable with $|E| > 0$, two numbers $a > 0$, $b > 0$, and an $\boldsymbol{R}$-fine cover $\boldsymbol{T}$ of E such that for every finite subcollection of $\boldsymbol{T}$, $\boldsymbol{T}^* = \{T_k\} \subset \boldsymbol{T}$ with $|E - \bigcup T_k| < b$ and $\int \nu(\{T_k\}, x)\,dx < b$ (such $\boldsymbol{T}^*$ exist, since $\boldsymbol{R}$ has strength θ) one has $|\bigcup\{T\colon T \in \boldsymbol{T}^{**}\}| > b$ where $\boldsymbol{T}^{**}$ is defined by $\boldsymbol{T}^{**} = \{T \in \boldsymbol{T}^*\colon \int_T [\nu(\boldsymbol{T}^*, x)]^{-1}\varphi(\nu(\boldsymbol{T}^*, x))\,dx > a|T|\}$ (the function inside the integral is defined to be 0 whenever $\nu(\boldsymbol{T}^*, x) = 0$).*

P r o o f o f t h e l e m m a. Assume the lemma is not true. This means that for some differentiation basis $\boldsymbol{R}$, having strength θ but not strength φ one has that for every $E \subset R^n$ bounded and measurable with $|E| > 0$, for every $a > 0$, $b > 0$, and for every $\boldsymbol{R}$-fine cover $\boldsymbol{T}$ of E, it happens that for some finite sequence $\boldsymbol{T}^* = \{T_k\} \subset \boldsymbol{T}$, satisfying $|E - \bigcup T_k| < b$ and also $\int \nu(\{T_k\}, x)\,dx < b$, one has $|\bigcup\{T\colon T \in \boldsymbol{T}^{**}\}| \leqslant b$. We shall prove that under these circumstances $\boldsymbol{R}$ necessarily has strength φ, reaching so a contradiction. This will prove the lemma.

Let E, bounded and measurable with $|E| > 0$, $\varepsilon > 0$ and $\boldsymbol{T}$, an $\boldsymbol{R}$-fine cover of E be given. Take G open, $G \supset E$, $|G| < 2|E|$ and two numbers $a, b > 0$, to be fixed later. According to our assumptions, we can choose $\boldsymbol{T}^* = \{T_k\} \subset \boldsymbol{T}$ satisfying $|E - \bigcup T_k| < b$, $\int \nu(\{T_k\}, x)\,dx < b$, $|\bigcup\{T\colon T \in \boldsymbol{T}^{**}\}| \leqslant b$, and also $T_k \subset G$. We consider $\boldsymbol{S} = \boldsymbol{T}^* - \boldsymbol{T}^{**} = \{S_k\}$. Then, obviously, $|E - \bigcup S_k| \leqslant |E - \bigcup\{T_k\colon T_k \in \boldsymbol{T}^*\}| + |\bigcup\{T_k\colon T_k \in \boldsymbol{T}^{**}\}| \leqslant 2b$. Furthermore, defining again the integrand as 0 when $\nu(S, x) = 0$,

$$\int \varphi\big(\nu(\boldsymbol{S}, x)\big)\,dx = \int [\nu(\boldsymbol{S}, x)]^{-1}\varphi\big(\nu(\boldsymbol{S}, x)\big)\Big(\sum \chi_{S_k}(x)\Big)\,dx$$
$$\leqslant \sum_{S_k \in \boldsymbol{S}} \int_{S_k} [\nu(\boldsymbol{T}^*, x)]^{-1}\varphi\big(\nu(\boldsymbol{T}^*, x)\big)\,dx$$
$$\leqslant \sum_{S_k \in \boldsymbol{S}} a|S_k| \leqslant a\int \nu(\boldsymbol{S}, x)\,dx + a\Big|\bigcup_{S_k \in \boldsymbol{S}} S_k\Big|$$
$$\leqslant a\int \nu(\boldsymbol{T}^*, x)\,dx + a|G| \leqslant ab + 2a|E|$$

having made use, for the first inequality, of the condition that $\varphi(u)/u$ is non-decreasing.

If we first fix b so that $2b < \varepsilon$, and then a so that $ab + 2a|E| < \varepsilon$, we see that $\boldsymbol{R}$ has strength φ. So the lemma is established.

Proof of the theorem. Assume $\boldsymbol{R}$ has not strength ψ_1. Since $L^\infty_{\mathrm{loc}}(R^n) \subset \sigma(L_{\mathrm{loc}})$ we have, by the above mentioned result of de Possel, that $\boldsymbol{R}$ has strength θ. We also see that $\psi_1(u)/u$ is non-decreasing. Hence we can apply the lemma for $\varphi = \psi_1$ and affirm that there is an E, bounded and measurable with $|E| > 0$, that there are $a, b > 0$ and an $\boldsymbol{R}$-fine cover of $E, \boldsymbol{T}$, so that for every finite subcollection $\boldsymbol{T}^* = \{T_k\} \subset \boldsymbol{T}$, with $|E - \bigcup T_k| < b$, $\int \nu(\boldsymbol{T}, x)\,dx < b$ we have $|\bigcup\{T\colon T \in \boldsymbol{T}^{**}| > b$, $\boldsymbol{T}^{**}$ being defined as above.

We shall try to construct a function $f \in \sigma(L_{\mathrm{loc}})$ such that $\boldsymbol{R}$ does not differentiate $\int f$, reaching a contradiction. This will prove the theorem. First we take a sequence $\{b_m\}$, $b_m > 0$, $b_m \downarrow 0$, which will be fixed later conveniently. For each $m = 1, 2, \ldots$ we take a finite sequence $\boldsymbol{T}^*_m = \{T^m_k\} \subset \boldsymbol{T}$ with $\delta(T^m_k) < \dfrac{1}{m}$, $\delta(A)$ meaning the diameter of the set A, such that $|E - \bigcup T^m_k| < b_m$, $\int \psi\big(\nu(\boldsymbol{T}^*_m, x)\big)\,dx < b_m$ (observe that we also have $\int \nu(\boldsymbol{T}^*_m, x)\,dx < b$, since $\psi(u) \geqslant u$). We consider then the corresponding $\boldsymbol{T}^{**}_m$ and define

$$\omega_m(x) = \sigma^{-1}\Big(\psi\big(\nu(\boldsymbol{T}^*_m, x)\big)\Big).$$

We now set

$$f(x) = \sup_m \omega_m(x).$$

So we have,

$$\int \sigma\big(|f(x)|\big)\,dx \leqslant \int \sum_m \sigma\big(\omega_m(x)\big)\,dx \leqslant \sum_m \int \psi\big(\nu(\boldsymbol{T}^*_m, x)\big)\,dx \leqslant \sum_m b_m.$$

If we take b_m sufficiently small as $m \to \infty$, then $\sum b_m < \infty$ and so $f \in \sigma(L_{\mathrm{loc}})$.

Consider now the set $O_m = \bigcup_m \{T^m_k\colon T^m_k \in \boldsymbol{T}^{**}_m\}$ and $A = \limsup_m O_m$. Then we have $|A| \geqslant b$. Every $x \in A$ is in some $T^m_k \in \boldsymbol{T}^{**}_m$ for an infinite number of m's and so, if $\boldsymbol{R}$ differentiates $\int f$ for $f \in \sigma(L_{\mathrm{loc}})$ we have, since

$$\int_{T^m_k} f(x)\,dx > a|T^m_k| \text{ for } T^m_k \in \boldsymbol{T}^{**}_m,$$

that $f(x) \geqslant a$ for almost all points x of A. Hence $\int \sigma\big(|f(x)|\big)\,dx \geqslant \sigma(a)|A| \geqslant \sigma(a)b$. If we choose b_m small enough as $m \to \infty$, we can make $\sum b_m < \sigma(a)b$ and this is a contradiction. So the theorem is proved.

COROLLARY. *With the notation of the theorem, define $\psi_2(u) = u\sigma^{-1}\big(\psi_1(u)\big)$, $\psi_{k+1}(u) = u\sigma^{-1}\big(\psi_k(u)\big)$ and assume $\psi_k(u) \geqslant u$ for all k. Then $\boldsymbol{R}$ has strength ψ_k for all k.*

It is enough to apply the theorem k times. The theorem of Hayes and Pauc is a particular case of this corollary. In fact, if $\boldsymbol{R}$ differentiates $\int f$, $f \in L^p(R^n)$ for some p, $1 < p < \infty$, then, by the de Possel theorem, $\boldsymbol{R}$ has strength θ, with $\theta(u) = u$. Hence, by the corollary, $\boldsymbol{R}$ has strength φ_{h_k} with $\varphi_{h_k}(u) = u^{h_k}$, $h_k = \sum_0^k \dfrac{1}{p^k}$. Therefore $\boldsymbol{R}$ has strength φ_{q_1} for all $q_1 < q = p/(p-1)$.

Remarks. (1) The setting of this note is a rather concrete one. It is possible to consider more general measure spaces and more general types of differentiation bases. By the same methods one would obtain similar results.

(2) It is interesting to observe that the spaces $\sigma(L_{\mathrm{loc}})$ considered in the theorem are not necessarily linear. There are other situations in differentiation theory where linearity is irrelevant. Compare the problems recently treated by Rubio [8] on the relationships between the Hardy–Littlewood maximal operator with respect to $\boldsymbol{R}$ and the differentiation properties of $\boldsymbol{R}$.

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The integrability class of the sine transform of a monotonic function

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Abstract. This note introduces a method for discussing the weighted Lebesgue class of the Fourier transform of a monotonic function, a method that is rather more direct than those that have been used for similar problems about Fourier series. The method depends on Steffensen's version of Jensen's inequality (see [6], p. 109 ff.). and a theorem of S. M. Edmonds [4] on Parseval's theorem for monotonic functions.

Two classical theorems of Hardy and Littlewood (see [8], vol. 2, p. 129–130) state that if φ is an integrable function, λ_n are its Fourier sine or cosine coefficients, and $1 < p < \infty$, then

(1) *If* $\lambda_n \downarrow 0$, *then* $\varphi \in L^p$ *if and only if* $n^{1-2/p}\lambda_n \in l^p$.

(2) *If* $\varphi(x) \geqslant 0$ *and* φ *decreases on* $(0, \pi)$, *then* $\lambda_n \in l^p$ *if and only if* $x^{1-2/p}\varphi(x) \in L^p$.

These theorems have been extended (see [1], p. 35) to weighted L^p and l^p spaces. Here and subsequently, $p' = p/(p-1)$.

(3) (Generalization of (1) by Y.-M. Chen) *If* $\lambda_n \downarrow 0$, *then* $x^{-\gamma}\varphi(x) \in L^p$, $-1/p' < \gamma < 1/p$, *if and only if* $n^{\gamma+1-2/p}\lambda_n \in l^p$. *Alternatively (replace* γ *by* $(2/p)-1-\gamma$) *if* $\lambda_n \downarrow 0$, *then* $n^{-\gamma}\lambda_n \in l^p$ *if and only if* $x^{\gamma+1-2/p}\varphi(x) \in L^p$, $-1/p' < \gamma < 1/p$.

The corresponding generalization (4) of (2) has the same conclusions but the hypothesis $\lambda_n \downarrow 0$ is replaced by $\varphi(x) \geqslant 0, \varphi$ decreasing, and φ integrable (all on $(0, \pi)$).

Theorems (3) and (4) reduce to (1) and (2) if in each we take $\gamma = 0$ in the first statement or $\gamma = (2/p)-1$ in the second statement.

Theorems of this kind naturally have analogues for Fourier transforms. A partial analogue of (1) is given by Titchmarsh ([7], p. 113). This is:

(5) *If* $\varphi(x) \downarrow 0$ *on* $(0, \infty)$, Φ *is its cosine transform, and* $\varphi(x)x^{1-2/p} \in L^p$ $(1 < p < 2)$, *then* $\Phi \in L^p$.

Disregarding the distinction between a function's having a Fourier transform and its being a Fourier transform, we may loosely formulate a hypothetical integral analogue of (3) and (4) as follows: