In particular we are able to prove the theorem for series which do not converge uniformly, and so for which the function $F(x)$ need not be continuous.

In addition somewhat weaker conditions than spherical convergence of (1.1) are sufficient to prove Shapiro's Theorem. These will be the subject of a later note.

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Boundary values of bounded holomorphic functions

by

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Abstract. Suppose f is a bounded measurable function on the n-torus $\boldsymbol{T}^n$ which is the (distinguished-) boundary value function of a bounded holomorphic function in the polydisc. If, for some point $x^0 \in \boldsymbol{T}^n, f(x) \to \lambda$ as x tends to x^0 through points of a set E which, in a specified measure-theoretic sense, is "thick" at x^0, then $f(x) \to \lambda$ "on the average" as $x \to x^0$; here, the "average" can be any one of a very broad class of summability methods applied to the Fourier series of f.

1. Let U denote the open upper half-plane, R its boundary. By $H^\infty(U^n)$ we denote the bounded holomorphic functions in U^n, and by $H^\infty(R^n)$ the Fatou boundary values of these functions on the distinguished boundary R^n of $U^n \cdot H^\infty(R^n)$ is naturally identified as that subspace of $L^\infty(R^n)$ consisting of the functions whose (distributional) Fourier transforms are supported in the "first quadrant", i.e. the set $\{\xi_1 \geqslant 0, \ldots, \xi^n \geqslant 0\}$. For an elaboration of these matters (in the slightly different context where U is the open unit disc) see [1].

The purpose of this note is the proof of the following theorem, a refinement of Theorem 1 of [2]. By $B(x, a)$ we denote the closed ball in R^n with center x and radius a, and $k_{(a)}(x)$ denotes the "dilated" function $a^{-n} k(a^{-1}x)$; dx denotes Haar measure in R^n, and $|E|$ denotes the measure of $E \subset R^n$.

THEOREM. *Let $\varphi \in H^\infty(R^n)$, $x^0 \in R^n$, and suppose there exist $b > 0$, a complex number λ, and a function $s(a)$ tending to zero as $a \to 0+$, with the following properties. For each sufficiently small a, $B(x^0, a)$ contains a set K_a of measure at least $b|B(x^0, a)|$ such that*

$$\{\text{ess sup}\, |\varphi(x) - \lambda|,\ x \in K_a\} \leqslant s(a).$$

Then, for every $k \in L^1(R^n)$,

$$(\varphi * k_{(a)})(x^0) = \int \varphi(x^0 - ax) k(x)\, dx \to \lambda \int k(x)\, dx \tag{1}$$

as $a \to 0$.

Let us say that a point x^0 is a *point of density $\geqslant b$ (where $0 < b \leqslant 1$) relative to $E \subset R^n$*, if $|E \cap B(x^0, a)| \geqslant b\,|B(x^0, a)|$ for all sufficiently small a. Then the above theorem may be reformulated as follows:

Suppose $E \subset R^n$, and x^0 is a point of density $\geqslant b > 0$ relative to E. If $\varphi \in H^\infty(R^n)$, and $\varphi(x) \to \lambda$ as x tends to x^0 through points in E, then, for every $k \in L^1(R^n)$, (1) holds.

For the proof, we shall require some notation. If ϱ is a probability measure on a measure space X, the *geometric mean* (relative to ϱ) of a non-negative measurable function F on X is

$$G(F;\varrho) = \exp \int (\log F)\, d\varrho.$$

For a point $z = (x_1+iy_1, \ldots, x_n+iy_n) \in U^n$ we denote by P_z the Poisson kernel,

$$P_z(\xi) = \pi^{-n} \prod_{\nu=1}^{n} y_\nu\big((x_\nu-\xi_\nu)^2+y_\nu^2\big)^{-1}, \qquad \xi = (\xi_1, \ldots, \xi_n) \in R^n.$$

Thus, $P_z\, d\xi$ is a "representing measure" for evaluation at z:

$$\Phi(z) = \int \varphi(\xi) P_z(\xi)\, d\xi, \qquad \text{all } \varphi \in H^\infty(R^n)$$

whereby Φ is that element of $H^\infty(U^n)$ whose Fatou boundary function is φ. For $z^0 = (0+i, \ldots, 0+i)$, the measure $P_{z^0}\, d\xi$ is denoted by σ, i.e. $d\sigma = \prod_{\nu=1}^{n} \big(\pi(1+\xi_\nu^2)\big)^{-1} d\xi$.

LEMMA. *For $k \in L^1(R^n)$, and $a > 0$, let*

$$\Delta(k;a) = \sup \Big| \int_{R^n} k\varphi\, dx \Big|,$$

where

$$\varphi \in H^\infty(R^n), \qquad \|\varphi\|_\infty = 1, \qquad G(|\varphi|;\sigma) \leqslant a.$$

Then, $\lim\limits_{a\to 0} \Delta(k;a) = 0$.

Proof of lemma. Let S denote the set of $k \in L^1$ for which $\Delta(k;a) \to 0$. It is easy to check that S is a closed subspace of L^1. We claim that, for every $z \in U^n$, $P_z \in S$. This will prove the lemma, since the closed linear span of $\{P_z, z \in U^n\}$ is L^1.

Let now $\varphi \in H^\infty(R^n)$, $\|\varphi\|_\infty = 1$, $G(|\varphi|;\sigma) \leqslant a$. Then, if Φ is the element of $H^\infty(U^n)$ with boundary function φ, we have by subharmonicity,

$$\Big|\int \varphi P_z\, d\xi\Big| = |\Phi(z)| \leqslant G(|\varphi|;\, P_z\, d\xi) \leqslant G(|\varphi|;\sigma)^{\delta(z)} \leqslant a^{\delta(z)},$$

where

$$\delta(z) = \inf\big(P_z(\xi)/P_{z^0}(\xi)\big), \qquad \xi \in R^n$$

is positive. This proves the lemma.

Proof of theorem. Without loss of generality we may assume x^0 is the origin, $\lambda = 0$, and $\|\varphi\|_\infty = 1$. By virtue of the lemma, the theorem will be proved if the geometric mean of the function $x \to |\varphi(-ax)|$, relative to σ, tends to zero as $a \to 0$. But, the hypotheses imply that on the subset $E_a = -a^{-1}K_a$ of the unit ball, whose measure is $\geqslant b\,|B(0,1)|$, we have $|\varphi(-ax)| \leqslant s(a)$, therefore the geometric mean in question does not exceed $\exp\big(\sigma(E_a)\log s(a)\big) = s(a)^{\sigma(E_a)}$; this tends to zero as $a \to 0$ since $\sigma(E_a)$ is larger than a positive constant (depending only on n) times $|E_a|$, hence bounded away from zero. The theorem is proved.

COROLLARY 1. *Under the hypotheses of Theorem 1, if K denotes an open cone with vertex at x^0, we have*

$$\lim_{a\to 0} |K \cap B(x^0,a)|^{-1} \int_{K\cap B(x^0,a)} \varphi(x)\, dx = \lambda.$$

Proof. Apply the theorem, with $k(-x)$ equal to the characteristic function of $K \cap B(x^0, 1)$.

We can specialize the theorem for φ which have period 2π in each variable, and thereby obtain an analogous theorem for the polydisc. Letting D denote the open unit disc, and T its boundary, each function $f \in H^\infty(T^n)$ can be "lifted" into $H^\infty(R^n)$ by the formula $\varphi(x_1, \ldots, x_n) = f\big(\exp(ix_1), \ldots, \exp(ix_n)\big)$. Then, defining the notion "point of density $\geqslant b$" relative to a set in T^n, in the obvious way, we have:

COROLLARY 2. *Let $f \in H^\infty(T^n)$ have the Fourier expansion $\sum \hat{f}(\nu) e^{i\nu x}$ (here ν runs through lattice points in the "first quadrant", and νx denotes $\nu_1 x_1 + \ldots + \nu_n x_n$). Let $R \subset T^n$, and suppose $t^0 = (e^{ix_1^0}, \ldots, e^{ix_n^0})$ is a point of T^n, which is of density $\geqslant b > 0$ relative to E. If $f(t) \to \lambda$ as t tends to t^0 through points in E, then for every $k \in L^1(R^n)$ satisfying $\int k\, dx = 1$,*

$$\lim_{a\to 0+} \sum \hat{k}(a\nu) \hat{f}(\nu) e^{i\nu x^0} = \lambda. \tag{2}$$

Clearly (2) follows from (1) by inserting for φ its Fourier expansion; it is a reinterpretation of (1) in the framework of summability of Fourier series. For instance, taking $n = 1$, $\hat{k}(x) = (1-|x|)^+$, (2) states that the Fejér means of the Fourier series of f converge to λ at x^0. The point is that this is valid under a much weaker hypothesis than the classical one (continuity at x^0), precisely because f is here assumed to belong to $H^\infty(T)$, not merely $L^\infty(T)$.

Theorem 1 can also be stated in a "uniform" version: if J is a compact subset of R^n, each point x^0 of which satisfies the hypotheses of Theorem 1 for suitable $\lambda(x^0)$, $s(a;x^0)$ and b, *with $b > 0$ independent of x^0*, then $(\varphi * k_{(a)})(x^0) \to \lambda(x^0)\int k(x)\,dx$ *uniformly for* $x^0 \in J$. Indeed, given $x^0 \in J$ and $\varepsilon > 0$, for all $a \leqslant a_0(\varepsilon, x^0)$, $B(x^0,a)$ contains a set $K_a(x^0)$ of measure $\geqslant b\,|B(x^0,a)|$ on which $\operatorname{ess\,sup}|\varphi(x)-\lambda(x^0)| \leqslant \varepsilon$. The compactness of J guarantees that $a_0(\varepsilon, x^0)$ can be chosen independently of x^0, and now it is easy to verify that the previous convergence proof yields the desired uniformity with respect to $x^0 \in J$. (Observe that since $\varphi * k_{(a)}$ is continuous for each a, the function $x \to \lambda(x)$ is, under the stated assumptions, necessarily continuous on J.)

An interesting consequence is:

COROLLARY 3. *Let J be a compact subset of R^n. Suppose for a certain positive number b each point of J is a point of density $\geqslant b$ relative to J. Let $\varphi \in H^\infty(R^n)$, and suppose the restriction of f to J coincides a.e. with a continuous function λ on J. Then, for each $k \in L^1(R^n)$ satisfying $\int k\,dx = 1$,*

$$\lim_{a\to 0+} \max_{\xi \in J} \left| \lambda(\xi) - \int \varphi(\xi - ax) k(x)\,dx \right| = 0.$$

This result (which answers question (a) posed in [2], p. 116) can be viewed as a very strong "localization principle" for compact sets. In particular, on a compact subset J of the circle satisfying the density condition of Corollary 3, the Fejér means of any $f \in H^\infty(T)$ whose restriction to J is (after correction on a set of measure zero) continuous converge uniformly. Even for J an arc we have not found this result in the literature, although its direct deduction by means of known Tauberian theorems would not be difficult.

In conclusion we remark that the proof of the lemma has a "function algebras" flavor, and an analogously formulated proposition is valid for certain function algebras on T^m (in particular, for the polydisc algebra itself); what is decisive is certain properties of the "Jensen measures" that would be easy to formulate explicitly. For the weak* closure in $L^\infty(T^m)$ of such an algebra the analogs of Corollaries 2 and 3 would then be valid.

Also, it is possible to estimate how rapidly $\Delta(k, a)$ tends to zero, enabling the rate of convergence in (1) to be estimated in terms of $s(a)$ and properties of the kernels k. This will be discussed in [3].

I wish to express my thanks to Charles Fefferman for a valuable remark converning the subject of this paper.

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On the derivation and covering properties of a differentiation basis

by

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Dedicated to Professor Zygmund with gratitude

Abstract. A theorem is presented that determines the type of covering properties of a differentiation basis which differentiates integrals of functions satisfying a rather general integrability condition.

We define a *differentiation basis* $\boldsymbol{R}$ in n-dimensional Euclidean space R^n as a collection of open bounded sets of R^n such that for every $x \in R^n$ there is at least one sequence $\{R_k\} \subset \boldsymbol{R}$ so that $R_k \to x$ as $k \to \infty$ (i.e. for all $k = 1, 2, \ldots, x \in R_k$ and for every neighborhood U of x there is a $k_0 = k_0(U)$ such that $R_k \subset U$ for $k \geqslant k_0$). An example of a differentiation basis in R^n is the collection $\boldsymbol{R}_1$ of all open cubic intervals. Another one is the collection $\boldsymbol{R}_2$ of all open bounded intervals. A third one is the collection $\boldsymbol{R}_3$ of all open bounded rectangles of R^n.

We consider a locally (Lebesgue) integrable function $f\colon R^n \to R$ and define the upper derivative $\bar{D}(\int f, x)$ of $\int f$ with respect to $\boldsymbol{R}$ at the point x in the following way

$$\bar{D}\left(\int f, x\right) = \sup \overline{\lim_{k\to\infty}} \frac{1}{|R_k|} \int_{R_k} f(y)\,dy,$$

where the sup is taken over all sequences $\{R_k\} \subset \boldsymbol{R}$ such that $R_k \to x$ as $k \to \infty$. The *lower derivative* is defined setting $\inf \underline{\lim}$ above. We shall say that $\boldsymbol{R}$ *differentiates* $\int f$ if $\bar{D}(\int f, x) = \underline{D}(\int f, x) = f(x)$ at almost all points $x \in R^n$.

The type of covering properties considered in this paper find its motivation in the following. Let $E \subset R^n$ be bounded and measurable. We say that $\boldsymbol{T} \subset \boldsymbol{R}$ is an $\boldsymbol{R}$-fine cover of E if for every $x \in E$ there is a sequence $\{T_k\} \subset \boldsymbol{T}$ such that $T_k \to x$ as $k \to \infty$. There are differentiation

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