Also

$$(5.11)\qquad \int_E \frac{dx}{C(x)} = \sum_i \int_{a_i} \frac{dx}{C(x)} \leqslant \sum_i \frac{\beta_i - a_i}{C(a_i)} = \frac{1}{K} \sum_i \frac{\beta_i - a_i}{a_i}.$$

From (5.10) we see that $\beta_i/a_i \to 1$ as $i \to \infty$, and hence for all large i

$$(\beta_i - a_i)/a_i \leqslant 2\log(\beta_i/a_i).$$

From (5.10) and (5.11) we now deduce that $\int_E \frac{dx}{C(x)} < \infty$. Hence $\int_F \frac{dx}{C(x)} = \infty$, and therefore

$$\int_F \frac{dx}{C(x)+Kx} \geqslant \frac{1}{2} \int_F \frac{dx}{C(x)} = \infty,$$

so that (5.9) holds. This completes the proof of the lemma and of Theorem 4.

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Spherical convergence and integrability of multiple trigonometric series on hypersurfaces

by

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Abstract. Let T be a trigonometric series in k variables and let Γ be a subset of E^k of $k-1$ dimensional character. We investigate conditions on the coefficients of T and on the structure of Γ so that T converges spherically a.e. on Γ and can be formally termwise integrated over Γ. We apply our results to improve a k-dimensional version of Riemann's Theorem for formally integrated series.

1. Introduction. Let

$$(1.1)\qquad \sum c_n e^{in\cdot x}$$

be a trigonometric series in k variables, and let Γ be a subset of E^k of Hausdorff dimension $k-1$ (e.g., a hypersurface). We investigate conditions on Γ and on the coefficients c_n of (1.1) so that the series converges spherically almost everywhere on Γ and can be integrated formally over Γ. We apply our results to improve a theorem of Shapiro, [5], on Riemann summability of multiple trigonometric series.

The results of this paper form part of the author's Doctoral dissertation. The author wishes to record his debt to Professor A. Zygmund, under whose direction the dissertation was written.

2. Notation. We denote points of Euclidean space E^k, $k \geqslant 2$, by $x = (x_1, \ldots, x_k)$ and integral lattice points by $n = (n_1, \ldots, n_k)$. We write $n\cdot x = n_1x_1 + \ldots + n_kx_k$ and $|x| = (x_1^2 + \ldots + x_k^2)^{\frac{1}{2}}$. We put $T^k = \{x \in E^k: |x_i| \leqslant \pi, i = 1, \ldots, k\}$.

For the series (1.1) we write

$$S_R(x) = \sum_{|n|<R} c_n e^{in\cdot x}$$

and we say the series converges spherically at x to sum s if

$$\lim_{R\to\infty} S_R(x) = s.$$

We denote by $H_\beta(\Gamma)$ the Hausdorff measure of order β of a set Γ. We are concerned with sets of Hausdorff dimension $k-1$, that is, sets Γ

such that $k-1 = \inf\{\beta: H_\beta(\Gamma) = 0\}$. We will say a property is true almost everywhere on such a set Γ if the set of points of Γ at which the property does not hold has Hausdorff $k-1$ measure zero.

3. Statement of results.

THEOREM 1. *Let Γ be a subset of E^k of Hausdorff dimension $k-1$ and let the coefficients c_n of* (1.1) *satisfy*

(3.1) $$\sum_n |n|^a |c_n|^2 < \infty$$

for some $a > k-1$. Then (1.1) *converges spherically almost everywhere on Γ.*

Theorem 1 was proved for the special case when Γ is a circle in the plane, under stronger conditions on c_n, by E. M. Stein in a letter to A. Zygmund.

Now let Γ be a subset of E^k with $H_{k-1}(\Gamma) < \infty$. For $x \in E^k$ we define

$$D^*(x, \Gamma) = \limsup_{r\to 0} \frac{H_{k-1}(\Gamma \cap S(x, r))}{(2r)^{k-1}},$$

where $S(x, r)$ is the ball in E^k with center x and radius r. $D^*(x, r)$ is called the *upper Hausdorff-$(k-1)$ density of x in Γ.*

This notation, for $k = 2$, is due to Besicovitch [1], who proved that if Γ is any linearly (i.e., Hausdorff- —1) measurable set in the plane, then almost all points of Γ have upper Hausdorff- —1 density equal to one.

However, our results limit us to surfaces for which $D^*(x, \Gamma)$ exists as an uniform lim sup for all $x \in \Gamma$. That is, the ratio defining $D^*(x, \Gamma)$ is uniformly bounded in x as r tends to zero. If Γ is reasonably smooth, for example if

$$\Gamma = \{x \in E^k: x_k = f(x_1, \ldots, x_{k-1})\},$$

where

$$\left(\frac{\partial f}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_k}\right)^2 \leqslant M,$$

then this hypothesis is satisfied.

Given Γ which has finite, positive Hausdorff-$(k-1)$ dimeusional measure we define a mass distribution on E^k by

$$\mu(S) = H_{k-1}(S \cap \Gamma).$$

We then define

$$\int_\Gamma f(x)\, ds(x) = \int_{E^k} f(x)\, d\mu(x).$$

THEOREM 2. *Let $1 \leqslant p < \infty$, and assume*

$$\sum |n|^a |c_n|^2 < \infty$$

where $a > \frac{(p-1)k+1}{p}$ and $a > k-1$. Let Γ be a subset of E^k with $H_{k-1}(\Gamma) < \infty$ and such that there is a number M with $H_{k-1}(\Gamma \cap (S(x, r))) \leqslant Mr^{k-1}$ for all $x \in \Gamma$ and $r > 0$. Let

$$S^*(x) = \sup_R \Big|\sum_{|n|<R} c_n e^{in\cdot x}\Big|.$$

Then,

$$\int_\Gamma S^*(x)^p\, ds(x) < \infty.$$

4. We first establish the following lemma, whose proof is similar to the proof of similar results in [6], Chapter 1.

LEMMA. *Let $k-1 < \gamma < k$. Then there is a number C_γ such that for $|x| < 1$, and for all $R > 0$,*

(4.1) $$\Big|\sum_{|n|<R} |n|^{-\gamma} e^{in\cdot x}\Big| < C_\gamma |x|^{\gamma-k}.$$

Proof. We define C^∞ functions $\psi(t)$ and $b(t)$ satisfying

$$\psi(t) = \begin{cases} 0 & \text{for } t \leqslant 1/2, \\ 1 & \text{for } t \geqslant 1, \end{cases}$$

$$b(t) = \begin{cases} 1 & \text{for } t \leqslant 0, \\ 0 & \text{for } t \geqslant 1, \end{cases}$$

let $b_N(t) = \psi(t) b(t-N)$, and define

$$F_N(x) = \{|x|^{-\gamma} b_N(|x|)\}^{\check{}}.$$

We assert

(4.2) $$|F_N(x)| \leqslant C_\gamma |x|^{\gamma-k} \quad \text{for all } N \text{ and all } |x| < 1,$$

(4.3) $$|F_N(x)| \leqslant C_\gamma |x|^{-k-1} \quad \text{for all } N \text{ and all } |x| > 1.$$

Put $\mu = \frac{1}{2}(k-2)$ and denote by $J_\mu(t)$ the Bessel function of order μ. We recall the following estimates:

(4.4) $$|J_\mu(t)| \leqslant C_\mu \quad \text{for all } t > 0,$$

(4.5) $$|J_\mu(t)| \leqslant C_\mu t^\mu \quad \text{for } 0 \leqslant t \leqslant 2.$$

We first prove

$$(4.6)\qquad \Big|2\pi R^{-\mu}\int_0^\infty \psi(t)t^{-\gamma+\frac12 k}J_\mu(2\pi Rt)\,dt\Big| \leqslant C_\gamma R^{\gamma-k}$$

for $0<R\leqslant 1$.

$$2\pi R^{-\mu}\int_0^\infty \psi(t)t^{-\gamma+\frac12 k}J_\mu(2\pi Rt)\,dt = 2\pi R^{-\mu}\Big[\int_0^{1/R}+\int_{1/R}^\infty\Big] = A+B.$$

We use (4.5) to evaluate A.

$$|A|\leqslant 2\pi R^{-\mu}\int_0^{1/R} t^{-\gamma+\frac12 k}(2\pi Rt)^\mu\,dt = C_\mu\int_0^{1/R} t^{-\gamma+k-1}\,dt \leqslant C_\mu R^{\gamma-k}.$$

To estimate B we integrate by parts, using the fact that $t^{\mu+1}J_\mu(t) = \frac{d}{dt}\{t^{\mu+1}J_{\mu+1}(t)\}$,

$$B = 2\pi R^{-\mu}\int_{1/R}^\infty \{t^{-\gamma+\frac12 k}(2\pi Rt)^{-\mu-1}(2\pi R)^{-1}\}\cdot\{(2\pi Rt)^{\mu+1}(2\pi R)J_\mu(2\pi Rt)\}\,dt$$

$$= 2\pi R^{-\mu}\int_{1/R}^\infty \{t^{-\gamma+\frac12 k}(2\pi R)^{-\mu-1}(2\pi R)^{-1}\}\cdot\frac{d}{dt}\{(2\pi Rt)^{\mu+1}J_{\mu+1}(2\pi Rt)\}\,dt$$

$$= C_\gamma R^{-\mu-1}t^{-\gamma+\frac12 k}J_{\mu+1}(2\pi Rt)\,\Big|_{1/R}^\infty + C_\gamma R^{-\mu-1}\int_{1/R}^\infty t^{-\gamma+\frac12 k-1}J_{\mu+1}(2\pi Rt)\,dt.$$

Using (4.5),

$$|B|\leqslant C_\gamma R^{\gamma-k}+C_\gamma R^{-\mu-1}\int_{1/R}^\infty t^{-\gamma+\frac12 k-1}\,dt = C_\gamma R^{\gamma-k}.$$

Thus (4.6) is proved.

We now establish (4.2). By Lemma 3, Chapter 1 of [6],

$$F_N(x) = 2\pi R^{-\mu}\int_0^\infty t^{-\gamma+\frac12 k}b_N(t)J_\mu(2\pi Rt)\,dt.$$

We treat separately the cases when $R(N+1)<1$ and when $R(N+1)\geqslant 1$.

If $R(N+1)<1$, using (4.5)

$$|F_N(x)|\leqslant 2\pi R^{-\mu}\int_0^{N+1} t^{-\gamma+\frac12 k}(2\pi Rt)^\mu\,dt = C_\gamma\int_0^{N+1} t^{-\gamma+k-1}\,dt\leqslant C_\gamma.$$

If $R(N+1)\geqslant 1$,

$$|F_N(x)| = 2\pi R^{-\mu}\int_0^N \psi(t)t^{-\gamma+\frac12 k}J_\mu(2\pi Rt)\,dt + 2\pi R^{-\mu}\int_N^{N+1} t^{-\gamma+\frac12 k}b_N(t)J_\mu(2\pi Rt)\,dt = P+Q.$$

$$|Q|\leqslant C_\gamma R^{-\mu}\int_N^{N+1} t^{-\gamma+\frac12 k}\,dt \leqslant C_\gamma R^{-\mu}\int_0^{1/R} t^{-\gamma+\frac12 k}\,dt = C_\gamma R^{\gamma-k}.$$

To estimate P we consider first

$$2\pi R^{-\mu}\int_N^\infty t^{-\gamma+\frac12 k}J_\mu(2\pi Rt)\,dt.$$

Integrating by parts as before,

$$(4.7)\qquad \Big|2\pi R^{-\mu}\int_N^\infty t^{-\gamma+\frac12 k}J_\mu(2\pi Rt)\,dt\Big|$$
$$= \Big|C_\gamma R^{\gamma-k}+C_\mu R^{-\mu-1}\int_N^\infty t^{-\gamma+\frac12 k-1}J_{\mu+1}(2\pi Rt)\,dt\Big|$$
$$\leqslant C_\gamma R^{\gamma-k}+C_\gamma R^{-\mu-1}N^{-\gamma+\frac12 k}$$
$$= C_\gamma R^{\gamma-k}.$$

Hence,

$$|P| = \Big|2\pi R^{-\mu}\int_0^N \psi(t)t^{-\gamma+\frac12 k}J_\mu(2\pi Rt)\,dt\Big|$$
$$\leqslant \Big|2\pi R^{-\mu}\int_0^\infty\cdot\Big|+\Big|2\pi R^{-\mu}\int_N^\infty\Big|\leqslant C_\gamma R^{\gamma-k}$$

by (4.6) and (4.7).

This completes the proof of (4.2). The proof of (4.3) follows from equations 1.38 to 1.41 of [6]. We comment only that for fixed j,t he derivatives of order j of $b_N(x)$ are bounded in x independently of N.

Having established (4.2) and (4.3), the Lemma follows immediately upon application of the Poisson summation formula to $F_N(x)$.

5. Proof of Theorem 2. Fix N. We will prove

$$\int_\Gamma S_N^*(x)^p\,ds(x) < M,$$

where

$$S_N^*(x) = \sup_{R\leqslant N}\Big|\sum_{|n|<R} c_n e^{in\cdot x}\Big|.$$

Since $S_N^*(x)$ increases to $S^*(x)$ for all x, Theorem 2 will follow upon application of Fatou's lemma.

Write

$$(5.1)\qquad I=\int_{\Gamma} S_N^*(x)^p\,ds(x)=\int_{\Gamma}|S_{n(x)}(x)|^p\,ds(x)=\int_{E^k}|S_{n(x)}(x)|^p\,d\mu(x),$$

where $d\mu$ is the mass distribution on E^k with $\mu(S)=H_{k-1}(S\cap\Gamma)$, for $S\subset E^k$.

By the converse to Hölder's inequality there is a function $\Psi(x)$ with

$$(5.2)\qquad \int_{E^k}|\Psi(x)|^{p'}\,d\mu(x)=1$$

such that

$$I^{1/p}=\int_{E^k}S_{n(x)}(x)\Psi(x)\,d\mu(x).$$

Since $\sum|n|^\alpha|c_n|^2<\infty$, $\sum|n|^{\alpha/2}c_ne^{in\cdot x}$ is the Fourier series of a function $F(x)\in L^2(T^k)$. Write

$$G_R(x)=\sum_{|n|<R}|n|^{-\alpha/2}e^{in\cdot x}.$$

Then

$$I^{1/p}=\int_{E^k}\Psi(x)\Big\{(2\pi)^{-k}\int_{T^k}F(z)G_{n(x)}(x-z)\,dz\Big\}d\mu(x)$$
$$=(2\pi)^{-k}\int_{T^k}F(z)\Big\{\int_{E^k}\Psi(x)G_{n(x)}(x-z)\,d\mu(x)\Big\}dz$$
$$=(2\pi)^{-k}\int F(z)J(z)\,dz,$$

where $J(z)=\int_{E^k}\Psi(x)G_{n(x)}(x-z)\,d\mu(x)$.

Applying Hölder's inequality,

$$I^{1/p}=(2\pi)^{-k}\Big[\int_{T^k}|F(z)|^2dz\Big]^{1/2}\Big[\int_{T^k}J(z)^2\,dz\Big]^{1/2}$$
$$=(2\pi)^{-k}\Big[\sum|n|^\alpha|c_n|^2\Big]^{1/2}\Big[\int_{T^k}J(z)^2dz\Big]^{1/2}$$
$$=C\Big[\int_{T^k}J(z)^2\,dz\Big]^{1/2}.$$

The proof of Theorem 2 will be complete when we show

$$\int_{T^k}J(z)^2dz<\infty,$$

$$(5.3)\qquad \int J(z)^2dz$$
$$=\int_{T^k}\Big\{\int_{E^k}\Psi(x)G_{n(x)}(x-z)\,d\mu(x)\Big\}\Big\{\int_{E^k}\Psi(y)G_{n(y)}(y-z)\,d\mu(y)\Big\}dz$$
$$=\int_{E^k}\int_{E^k}\Psi(x)\Psi(y)\Big\{\int_{T^k}G_{n(x)}(x-z)G_{n(y)}(y-z)\,dz\Big\}d\mu(x)\,d\mu(y)$$
$$=(2\pi)^{-k}\int_{E^k}\Psi(y)\Big[\int_{E^k}\Psi(x)H_{n(x,y)}(x-y)\,d\mu(x)\Big]d\mu(y),$$

where

$$H_R(z)=\sum_{|n|<R}|n|^{-\alpha}e^{in\cdot z}$$

and

$$n(x,y)=\min\big(n(x),n(y)\big).$$

Using (5.2), the integral enclosed in brackets in (5.3) is majorized by

$$\Big[\int_{E^k}|H_{n(x,y)}(x-y)|^p\,d\mu(x)\Big]^{1/p}.$$

By (4.1), this is majorized by

$$\Big[\int_{E^k}C_\alpha|x-y|^{p(\alpha-k)}\,d\mu(x)\Big]^{1/p}.$$

We will prove this last integral is bounded independently of y. Then substituting back into (5.3), the proof of Theorem 2 will be complete.

By hypothesis, $p(\alpha-k)>1-k$, say

$$p(\alpha-k)=1-k+\varepsilon.$$

We may assume $0<\varepsilon<1$.

$$\int_{E^k}|x-y|^{p(\alpha-k)}\,d\mu(x)$$
$$=\int_{E^k}|x-y|^{1-k+\varepsilon}\,d\mu(x)=\int_0^\infty r^{1-k+\varepsilon}\,d\mu\big(S(y,r)\big)=\int_0^1+\int_1^\infty=A+B.$$

$$B=\int_1^\infty r^{1-k+\varepsilon}\,d\mu\big(S(y,r)\big)\leqslant\int_1^\infty d\mu\big(S(y,r)\big)\leqslant H_{k-1}(\Gamma)<\infty.$$

$$A=\int_0^1 r^{1-k+\varepsilon}\,d\mu\big(S(y,r)\big)=r^{1-k+\varepsilon}\mu\big(S(y,r)\big)\Big|_0^1-\int_0^1 r^{-k+\varepsilon}\mu\big(S(y,r)\big)\,dr.$$

The smoothness condition on Γ gives $\mu\big(S(y,r)\big)\leqslant Mr^{k-1}$, so

$$A=r^{1-k+\varepsilon}O(r^{k-1})\Big|_0^1-\int_0^1 r^{-k+\varepsilon}O(r^{k-1})\,dr=O(1).$$

This completes the proof of Theorem 2.

We remark that if in equation (5.1), $n(x)$ is replaced by any Borel-measurable function and $d\mu$ is any mass distribution with

$$\int_{E^k}|x-y|^{\alpha-k}\,d\mu(x)\leqslant M,$$

where M is independent of y, then the proof of Theorem 2 shows that

$$\int_{E^k}|S_{n(x)}(x)|\,d\mu(x)\leqslant C_\alpha M.$$

We will use this fact in the proof of Theorem 1.

6. Proof of Theorem 1. If U is a Borel set in E^k the β-capacity of U, $C_\beta(U)$, is defined by

$$\frac{1}{C_\beta(U)} = \inf_\mu \sup_{x\in E^k} \int_{E^k} |x-y|^{-\beta} d\mu(y),$$

the infinum being taken over all (non-negative) mass distributions concentrated in U with total mass one,

$$\int_{E^k} d\mu = \int_U d\mu = 1.$$

It is well known, see for example [2], [3], that if U is a Borel set, then

(6.1) $$\inf\{\beta\colon H_\beta(U) = 0\} = \inf\{\beta\colon C_\beta(U) = 0\}.$$

We may assume $\Gamma \subset T^k$. Denote by E the set of points of T^k at which (1.1) does not spherically converge. We will prove that E has outer $(k-\alpha)$-capacity zero. That is,

(6.2) $$\inf\{C_{k-\alpha}(U)\colon U \text{ is open and } E \subset U\} = 0.$$

Thus, there is a G_δ set, W, which contains E, such that $C_{k-\alpha}(W) = 0$. Hence by (6.1) $H_{k-1}(W) = 0$. Therefore, $H_{k-1}(\Gamma \cap E) = 0$.

We now establish (6.2). Its proof is analogous to the proof of corresponding one variable results due to Beurling, and Salem and Zygmund, see [4]. We may assume $S_R(x)$ is unbounded as, R tends to infinity, at each point x of E. For if we let $T_R(x) = \sum_{|n|<R} \omega(|n|) c_n e^{in\cdot x}$, where $\omega(t)$ is a positive, differentiable function increasing monotonically to infinity with t such that $\sum \omega(|n|)^2 |n|^\alpha |c_n|^2 < \infty$, then

$$S_R(x) = \sum_{|n|<R} \frac{1}{\omega(|n|)} \omega(|n|) c_n e^{in\cdot x}$$
$$= -\int_0^R T_u(x) \frac{d}{du}\left(\frac{1}{\omega(u)}\right) du + T_R(x)\frac{1}{\omega(R)}$$

Hence $\lim_{R\to\infty} S_R(x)$ exists at each point x, where $T_R(x) = O(1)$ as R tends to infinity.

Fix a number N. For each x in E there is a number $n(x)$ such that $|S_{n(x)}(x)| > N$. Since, for fixed x, $S_{n(x)}(z)$ is a trigonometric polynomial and therefore continuous, $|S_{n(x)}(y)| > N$ for y in a neighborhood U_x of x. Cover E by a countable number of U_x, and denote by O_N the union of these U_x.

E is contained in O_N, O_N is open, and $|S_{n(x)}(x)| > N$ for each x in O_N and some Borel measurable function $n(x)$.

If (6.2) is false, then there is a number M such that for every open set O containing E and for some $d\mu$ of total mass one concentrated in O

$$\int_{E^k} |x-y|^{\alpha-k} d\mu(x) < M$$

for all $x \in E^k$. Then, by the remark made at the end of the proof of Theorem 2,

(6.3) $$\int_{E^k} |S_{n(x)}(x)|\, d\mu(x) \leqslant C_\alpha M.$$

But $|S_{n(x)}| > N$ in O_N. Hence,

$$\int_{E^k} |S_{n(x)}(x)|\, d\mu(x) = \int_{O_N} |S_{n(n)}(x)|\, d\mu(x) \geqslant \int_{O_N} N d\mu(x) \geqslant N.$$

This contradicts (6.3) if N is chosen large enough. This completes the proof of Theorem 1.

7. Let $F(x)$ be a function defined in a neighborhood of $x_0 \in E^k$. We say $F(x)$ has a generalized Laplacian at x_0 equal to s if $F(x)$ is integrable on each $(k-1)$ dimensional sphere $|x-x_0| = t$ for t small, and if

$$(2\pi)^{-\frac{1}{2}k} \int_{|x|=1} F(x_0+tx)\, ds(x) = a_0 + \frac{s}{2^{1+k/2}\Gamma(1+k/2)} t^2 + o(t^2)$$

as t tends to 0.

Shapiro [5] proved the following: *Let the coefficients c_n of* (1.1) *satisfy $c_n = O(|n|^\delta)$ for some $\delta < 2-k$. If the series* (1.1) *converges spherically at a point x_0 to a finite sum s, then*

(7.1) $$F(x) = \frac{c_0(x_1+\ldots+x_n)^2}{2k} - \sum_{n\neq 0} \frac{c_n}{|n|^2} e^{in\cdot x}$$

has a generalized Laplacian at x_0 equal to s.

The condition on the coefficients c_n imposed by Shapiro is needed to make the series of (7.1) converge uniformly and, therefore, insure that $F(x)$ be continuous and, so, integrable over every $(k-1)$ dimensional sphere $|x-x_0| = t$.

We are able to improve Shapiro's result by applying Theorem 1 and 2 to the series in (7.1). We then have the following result:

THEOREM 3. *Let the coefficients of* (1.1) *satisfy $\sum |n|^\alpha |c_n|^2 < \infty$ for some $\alpha > k-5$. If at some point x_0 the series* (1.1) *converges spherically to a finite sum s, then $F(x)$ defined by* (7.1) *has a generalized Laplacian at x_0 equal to s.*

In particular we are able to prove the theorem for series which do not converge uniformly, and so for which the function $F(x)$ need not be continuous.

In addition somewhat weaker conditions than spherical convergence of (1.1) are sufficient to prove Shapiro's Theorem. These will be the subject of a later note.

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Boundary values of bounded holomorphic functions

by

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Abstract. Suppose f is a bounded measurable function on the n-torus $\boldsymbol{T}^n$ which is the (distinguished-) boundary value function of a bounded holomorphic function in the polydisc. If, for some point $x^0 \in \boldsymbol{T}^n$, $f(x) \to \lambda$ as x tends to x^0 through points of a set E which, in a specified measure-theoretic sense, is "thick" at x^0, then $f(x) \to \lambda$ "on the average" as $x \to x^0$; here, the "average" can be any one of a very broad class of summability methods applied to the Fourier series of f.

1. Let U denote the open upper half-plane, R its boundary. By $H^\infty(U^n)$ we denote the bounded holomorphic functions in U^n, and by $H^\infty(R^n)$ the Fatou boundary values of these functions on the distinguished boundary R^n of $U^n \cdot H^\infty(R^n)$ is naturally identified as that subspace of $L^\infty(R^n)$ consisting of the functions whose (distributional) Fourier transforms are supported in the "first quadrant", i.e. the set $\{\xi_1 \geqslant 0, \ldots, \xi^n \geqslant 0\}$. For an elaboration of these matters (in the slightly different context where U is the open unit disc) see [1].

The purpose of this note is the proof of the following theorem, a refinement of Theorem 1 of [2]. By $B(x, a)$ we denote the closed ball in R^n with center x and radius a, and $k_{(a)}(x)$ denotes the "dilated" function $a^{-n} k(a^{-1}x)$; dx denotes Haar measure in R^n, and $|E|$ denotes the measure of $E \subset R^n$.

THEOREM. *Let $\varphi \in H^\infty(R^n)$, $x^0 \in R^n$, and suppose there exist $b > 0$, a complex number λ, and a function $s(a)$ tending to zero as $a \to 0+$, with the following properties. For each sufficiently small a, $B(x^0, a)$ contains a set K_a of measure at least $b|B(x^0, a)|$ such that*

$$\{\text{ess sup}\,|\varphi(x) - \lambda|,\ x \in K_a\} \leqslant s(a).$$

Then, for every $k \in L^1(R^n)$,

$$(\varphi * k_{(a)})(x^0) = \int \varphi(x^0 - ax) k(x)\, dx \to \lambda \int k(x)\, dx \tag{1}$$

as $a \to 0$.

Let us say that a point x^0 is a *point of density* $\geqslant b$ (*where* $0 < b \leqslant 1$) *relative to* $E \subset R^n$, if $|E \cap B(x^0, a)| \geqslant b\,|B(x^0, a)|$ for all sufficiently small a. Then the above theorem may be reformulated as follows: