Since w decreases as $[z]$ increases, this is at most

$$\sum_{k=0}^{\infty}\Big(\int_{2^k\delta}^{\infty}\frac{dt}{t^{2a+1}\nu^2(B_t)}\Big)^{1/2}\nu(B_{2^{k+1}\delta})\leqslant c\sum_{0}^{\infty}\nu(B_{2^k\delta})^{-1}(2^k\delta)^{-a}\nu(B_{2\cdot 2^k\delta})$$

$$\leqslant c\delta^{-a}\sum_{0}^{\infty}2^{-ak}=O(\delta^{a}).$$

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A characterization of commutators with Hilbert transforms

by

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Dedicated to Professor Antoni Zygmund

Abstract. There are given necessary and sufficient conditions for the commutator of the Hilbert transform with an operator bounded in L^2 to be compact. Similar results are obtained for the cotangent Hilbert transform and for the Cauchy singular integral operator on a closed arc. These conditions follow from a property of commutators of linear operators with an involution.

The purpose of the present note is to give necessary and sufficient conditions for the commutator of the Hilbert transform with an operator bounded in L^2 to be compact. Similar results will be obtained for the cotangent Hilbert transform and for the Cauchy singular integral operator on a closed arc, and so on. The proofs are based on a simple property of commutators with an involution, which are presented at the begining.

1. Let $\mathfrak{X}$ be an algebra (a linear ring) with unit e over the field of complex scalars. An element $a\in\mathfrak{X}$ is said to be an *involution* if $a\neq e$ and $a^2=e$. An element $a\in\mathfrak{X}$ is said to be an *almost involution*, with respect to a proper two-sided ideal $J\subset\mathfrak{X}$, if the coset $[a]$ is an involution in the quotient algebra $\mathfrak{X}/J$, i.e. if there is a $b\in J$ such that $a^2=e+b$ (see [3] and [4]). Let us denote the commutator and the anticommutator of two elements $a,b\in\mathfrak{X}$ as follows:

$$[a,b]=ab-ba,\qquad (a,b)=ab+ba.$$

PROPOSITION 1.1. *Let a be an involution in an algebra $\mathfrak{X}$ with unit. An element $b\in\mathfrak{X}$ commutes with a if and only if there is a $b_0\in\mathfrak{X}$ such that $b=(a,b_0)$.*

Proof. Let us suppose that $b=(a,b_0)$ for an element $b_0\in\mathfrak{X}$. Then

$$[a,b]=ab-ba=a(ab_0+b_0a)-(ab_0+b_0a)a$$
$$=a^2b_0+ab_0a-ab_0a-b_0a^2=eb_0-b_0e=0.$$

Conversely, let $[a, b] = 0$. We put $b_0 = \frac{1}{2}ab$. Then

$$(a, b_0) = ab_0 + b_0 a = \tfrac{1}{2}a^2 b + \tfrac{1}{2}aba$$
$$= \tfrac{1}{2}(eb + aba - ba^2 + ba^2) = \tfrac{1}{2}[b + (ab - ba)a + be] = b$$

what had to be proved.

We also have a dual statement:

PROPOSITION 1.2. *Let a be an involution in an algebra $\mathfrak{X}$ with unit. An element $b \in \mathfrak{X}$ anticommutes with a if and only if there is a $b_0 \in \mathfrak{X}$ such that $b = [a, b_0]$.*

The proof is just the same as the preceding one, if we change the roles of signes "+" and "−".

COROLLARY 1.1. *Let a be an involution in an algebra $\mathfrak{X}$ with unit. Let J be a proper two-sided ideal in $\mathfrak{X}$. Then $[a, b] \in J$ for $b \in \mathfrak{X}$ if and only if $b = da + ad + g$, where $d \in \mathfrak{X}$ and $g \in J$.*

The proof is immediate if we apply Proposition 1.1 to the quotient algebra $\mathfrak{X}/J$.

COROLLARY 1.2. *Let J be a proper two-sided ideal in an algebra $\mathfrak{X}$ with unit. Let $a \in \mathfrak{X}$ be an almost involution with respect to J. Then $[a, b] \in J$ for $b \in \mathfrak{X}$ if and only if $b = da + ad + g$, where $d \in \mathfrak{X}$ and $g \in J$.*

The proof follows immediately from Proposition 1.1 applied to the quotient algebra $\mathfrak{X}/J$.

It is obvious that we can obtain results dual (in the sense of Proposition 1.2) to Corollaries 1.1 and 1.2.

2. Let $\mathfrak{X} = B(L^2(R))$ be the algebra of all bounded operators transforming $L^2(R)$ into itself(¹). In this algebra there is only one closed proper two-sided ideal, namely the ideal $T(L^2(R))$ of compact operators (see Gohberg, Markus, Feldman [1]). Let us consider the Hilbert transform:

$$(Hx)(t) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{x(s)}{s-t}\,ds \qquad (x \in L^2(R))$$

(here the integral is understood in the sense of the Cauchy principal value). It is well-known that $H \in B(L^2(R))$ and that H is an involution on $L^2(R)$: $H^2 = I$, where I is the identity operator (Zygmund [5]). From Proposition 1.1 we obtain immediately

PROPOSITION 2.1. *If $A \in B(L^2(R))$, then $HA - AH = 0$ if and only if $A = HA_0 + A_0 H$, where $A_0 \in B(L^2(R))$.*

From Corollary 1.1 we obtain

PROPOSITION 2.2. *If $A \in B(L^2(R))$, then $HA - AH$ is compact if and only if $A = HA_0 + A_0 H + T$, where $A_0 \in B(L^2(R))$, $T \in T(L^2(R))$.*

(¹) Here and in the sequel all considered functions are complex-valued.

3. Let $\mathfrak{X} = B(L^2(0, 2\pi))$ be the algebra of all bounded operators transforming $L^2(0, 2\pi)$ into iself. Also in this algebra the unique closed proper two-sided ideal is the ideal $T(L^2(0, 2\pi))$ of compact operators (see [1]). Let us consider the cotangent Hilbert transform:

$$(H_0 x)(t) = \frac{1}{2\pi i}\int_0^{2\pi} x(s)\cot\frac{s-t}{2}\,ds \qquad (x \in L^2(0, 2\pi))$$

(the integral is understood in the sense of the Cauchy principal value). It is known that $H_0 \in B(L^2(0, 2\pi))$ and that

(1) $$H_0^2 = I - K, \quad \text{where } (Kx)(t) = \frac{1}{2\pi}\int_0^{2\pi} x(s)\,ds.$$

Since K is one-dimensional, it is compact. Hence H_0 is an almost involution with respect to the ideal $T(L^2(0, 2\pi))$. Corollary 1.2 implies the following

PROPOSITION 3.1. *If $A \in B(L^2(0, 2\pi))$, then $H_0 A - AH_0$ is compact if and only if $A = H_0 A_0 + A_0 H_0 + T$, where $A_0 \in B(L^2(0, 2\pi))$ and $T \in T(L^2(0, 2\pi))$.*

By a simple calculation we obtain

COROLLARY 3.1. *If $A \in B(L^2(0, 2\pi))$, then $H_0 A - AH_0 = 0$ if and only if $A = H_0 A_0 + A_0 H_0 + T$, where $A_0 \in B(L^2(0, 2\pi))$, $T \in T(L^2(0, 2\pi))$ and*

$$H_0 T - TH_0 = A_0 K - KA_0$$

(K being defined by formula (1)).

4. Let $H^\mu(L)$ be the space of all functions satysfying the Hölder condition with exponent μ, $0 < \mu < 1$, on a closed regular arc L with the norm

$$\|x\| = \sup_{t \in L}|x(t)| + \sup_{t, t' \in L}\frac{|x(t) - x(t')|}{|t - t'|^\mu}.$$

Let $B(H^\mu(L))$ be the algebra of all bounded operators transforming $H^\mu(L)$ into itself. Let us denote by $T(H^\mu(L))$ the ideal of compact operators over the space $H^\mu(L)$.(²)

Let us consider the singular integral operator with the Cauchy kernel:

(2) $$(Sx)(t) = \frac{1}{\pi i}\int_L \frac{x(s)}{s-t}\,dt \qquad (x \in H^\mu(L))$$

(²) Theorem 2.5 of [3] (see also [4], Theorem 3.1 on p. 197) shows the following property: *if T is a continuous map from $C(L)$ into $H^\mu(L)$ then the operator $\tilde{T} = T|_{H^\mu(L)}$ is compact in the $H^\mu(L)$-topology.*

(the integral is understood in the sense of the Cauchy principal value). The Plemelj–Privalov theorem (see [2]) asserts that $S \in B(H^{\mu}(L))$. Moreover it is an involution on $H^{\mu}(L)$ ([2]). Therefore from Proposition 1.1 and from Corollary 1.1 we conclude

PROPOSITION 4.1. *If* $A \in B(H^{\mu}(L))$, *then* $SA - AS = 0$ *if and only if* $A = SA_0 + A_0S$, *where* $A_0 \in B(H(L))$.

PROPOSITION 4.2. *If* $A \in B(H^{\mu}(L))$ *then* $SA - AS$ *is compact if and only if* $A = SA_0 + A_0S + T$, *where* $A_0 \in B(H^{\mu}(L))$, $T \in T(H^{\mu}(L))$.

The list of similar results can be easily prolongated.

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Hardy's inequality with weights

by

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Abstract. This paper is concerned with conditions on measures μ and ν that are both necessary and sufficient for the existence of a finite C such that

$$\Big[\int_0^\infty \Big|\int_0^x f(t)\,dt\Big|^p d\mu\Big]^{1/p} \leqslant C\Big[\int_0^\infty |f(x)|^p d\nu\Big]^{1/p},$$

where p is a fixed number satisfying $1 \leqslant p \leqslant \infty$. For absolutely continuous measures a new proof is given for a known condition, and a new condition is given that arises from an interpolation with change of measures. The case when $\int_0^x f(t)\,dt$ is replaced by $\int_x^\infty f(t)\,dt$ is sketched. For Borel measures a condition like the first one for absolutely continuous measures is proved. Estimates for C in terms of the constants of the conditions are also given.

1. Introduction. Hardy's inequality, [5], p. 20, states that if p and b satisfy $1 \leqslant p \leqslant \infty$ and $bp < -1$, then

$$\Big[\int_0^\infty \Big|x^b \int_0^x f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant \frac{-p}{bp+1}\Big[\int_0^\infty |x^{b+1} f(x)|^p dx\Big]^{1/p}, \tag{1.1}$$

and the indicated constant is the best possible. Several authors, Tomaselli [4], Talenti [3], and Artola [1], have recently investigated the problem of for what functions, $U(x)$ and $V(x)$, there is a finite constant, C, such that

$$\Big[\int_0^\infty \Big|U(x)\int_0^x f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant C\Big[\int_0^\infty |V(x)f(x)|^p dx\Big]^{1/p}; \tag{1.2}$$

this is, of course, just the inequality (1.1) with x^b and x^{b+1} replaced by the weight functions $U(x)$ and $V(x)$. Their principal result is the following theorem.

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