Some applications of Zygmund's lemma to non-linear differential equations in Banach and Hilbert spaces

by

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Abstract. This paper deals with existence theorems for the differential equation

\[ y' = f(t, y), \]

where the solution \( y \) takes its values in a Banach or Hilbert space \( Y \). A new proof is given of the theorem of Ważewski that the conditions of Kamke's uniqueness criterion imply local existence of solutions. The argument covers also monotonicity conditions, and a generalization of a theorem of Browder of this type is given.

1. In this paper we consider existence theorems for the differential equation

(1.1)

\[ y' = f(t, y), \]

where the solution \( y \) takes its values in a Banach or Hilbert space \( Y \). If \( Y \) is finite-dimensional, then the continuity of \( f \) alone implies the local existence of solutions, but this is no longer so when the dimension of \( Y \) is infinite (see, for example, Dieudonné [3], p. 287, Exercise 5). It has been proved by Ważewski [12] that in the infinite-dimensional case the conditions of Kamke's well-known uniqueness theorem imply local existence, and we give a new proof of this result. We also prove a result involving a monotonicity condition that is the 'one-sided' analogue of Ważewski's theorem.

As in Kamke's theorem, we compare the differential equation (1.1) with a scalar equation

(1.2)

\[ x' = g(t, x), \]

and in our first two theorems we suppose that

(A) \( g \) is a continuous function from the rectangle \( [t_0, t_0 + a] \times [0, b] \)

in \( \mathbb{R}^2 \) into \( [0, \infty] \) with the properties that

(i) \( g(t, 0) = 0 \) for all \( t \in [t_0, t_0 + a] \),

(ii) for each \( t_0 \in [t_0, t_0 + a] \), \( x = 0 \) is the only solution of equation (1.2) on \( [t_0, t_1] \) satisfying the conditions that

\[ x(t_0 +) = 0 \quad \text{and that} \quad \lim_{t \to t_0^+} x(t) = 0. \]
For completeness, we state both the uniqueness and existence results.

**Theorem 1.** Let $Y$ be a complex Banach space, let $(t_0, y_0) \in X \times Y$ be the closed ball in $Y$ with center $y_0$ and radius $r > 0$, and let $f: [t_0, t_0 + a] \times \times B \rightarrow Y$ be a continuous function such that for all $t \in [t_0, t_0 + a] \times B$

\[
\|f(t, y) - f(t, z)\| \leq g(t, \|y - z\|),
\]

where $g$ satisfies condition (A) with $\beta = 2g$. Then equation (1.1) has at most one solution $u$ on $[t_0, t_0 + a]$. If in addition $f$ is bounded, then $\|f(t, y)\|$ and $\eta = \min\{\alpha, \|U(t, y)\|\}$, then equation (1.1) has exactly one solution on $[t_0, t_0 + a]$ taking the value $y_0$ at $t_0$.

**Theorem 2.** If $Y$ is a complex Hilbert space, condition (1.3) in Theorem 1 can be replaced by the condition that for all $(t, y), (t, z) \in [t_0, t_0 + a] \times B$

\[
2\text{re}\langle f(t, y) - f(t, z), (y - z) \rangle \leq g(t, \|y - z\|),
\]

where $g$ satisfies condition (A) with $\beta = 4g$.

Here the uniqueness part of Theorem 1 is Kamke's theorem (see [5], p. 31), while the existence part is the result of Weiswafski [12] mentioned above.

The uniqueness part of Theorem 2 is essentially known (for a similar argument, see [6], p. 15; Exercise 68), while the existence part appears to be new. We also obtain global theorems of similar type (Section 5, Theorems 3 and 4), which include results of Murakami and Browder.

2. Following Gál [4], we say that a property $P(t)$ holds nearly everywhere in a set $E \subset R$, or for nearly all $t \in E$, if there is a countable set $E' \subset E$ such that $P(t)$ holds for all $t \in E \setminus E'$. We also use $\forall t \in E$ here and later to denote the closed interval $[t_0, t_0 + a]$.

The following well-known lemma of Zygmund (see [6], p. 300) plays a fundamental role in our arguments.

**Lemma 1.** Let $M \in \text{Re}$, and let $\varphi: J \rightarrow R$ be a continuous function whose lower right-hand Dini derivative $D_\varphi$ satisfies the inequality $D_\varphi(t) \leq M$ for nearly all $t \in J$. Then

\[
\varphi(t_0 + a) - \varphi(t_0) \leq Ma.
\]

We require also some further lemmas.

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(1) It has been shown by Oloch [8] (see also [6], p. 30) that the results of Theorem 1 are implied by the corresponding results in which $\varphi$ satisfies the simpler conditions of Perron's uniqueness theorem. However, this simplification is not available for Theorem 2, and since our proofs of Theorems 1 and 2 are closely similar, we prove Theorem 1 directly.

We remark also that Oloch [8] has given an extension of Theorem 1 involving conditions of 'Carathéodory type'.

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**Lemma 2.** Let $(\varphi_n(n = 1, 2, \ldots))$ be a sequence of continuous functions from $J$ into $R$ converging uniformly on $J$ to a function $\varphi$. Let also $E$ be a set in $R^2$ containing the graphs of $\varphi_n(n = 1, 2, \ldots)$ and of $\varphi$, let $h: E \rightarrow R$ be continuous, and suppose that for each $n$

\[
D_\varphi \varphi_n(t) \leq h(t, \varphi(t))
\]

nearly everywhere in $J$. Then for all $t \in J$

\[
D_\varphi \varphi(t) \leq h(t, \varphi(t)).
\]

Let $\varepsilon > 0$ and let $t \in J$. Since $h$ is continuous at $(t, \varphi(t))$, $\varphi$ is continuous at $t$, and $\varphi_n \rightarrow \varphi$ uniformly on $J$, we can find a positive number $\eta$ and a positive integer $n$ such that $\varphi_n \in J$ and $|h(t, \varphi_n(t)) - h(t, \varphi(t))| \leq \varepsilon$ whenever $|s - t| \leq \eta$ and $n \geq q$. Hence for each $n \geq q$

\[
D_\varphi \varphi_n(t) \leq h(t, \varphi(t)) + \varepsilon
\]

nearly everywhere in $(t, t + \eta)$, and therefore, by Lemma 1,

\[
\varphi_n(t) - \varphi(t) \leq (s - t)|h|_t \varphi(t) + \varepsilon
\]

whenever $s \in (t, t + \eta)$. Hence also

\[
\varphi(s) - \varphi(t) \leq (s - t)|h|_t \varphi(t) + \varepsilon,
\]

so that $D_\varphi \varphi(t) \leq h(t, \varphi(t))^{(3)}$.

**Lemma 3.** Let $M > 0$, let $\lambda$ be a class of uniformly bounded continuous functions $\psi: J \rightarrow R$ with the property that for all $s \in J$

\[
|\psi(s) - \psi(t)| \leq M |s - t|
\]

and let $\mathcal{V} = \sup \lambda$. Let also $E$ be a set in $R^2$ containing the graphs of each $\varphi \in \lambda$ and of $\varphi$, let $h: E \rightarrow R$ be continuous, and suppose that for each $\varphi \in \lambda$

\[
D_\varphi \varphi(t) \leq h(t, \varphi(t))
\]

nearly everywhere in $J$. Then for all $s \in J$

\[
|\mathcal{V}(s) - \mathcal{V}(t)| \leq M |s - t|
\]

(3) that $\mathcal{V}$ is continuous, and for all $t \in J$

\[
D_\varphi \mathcal{V}(t) \leq h(t, \mathcal{V}(t)).
\]

We remark first that if $\varphi_1, \ldots, \varphi_L$ and $\psi = \max\{\varphi_1, \ldots, \varphi_L\}$, then $\psi$ satisfies (2.2) nearly everywhere in $J$. Here the case $k = 2$ is almost immediate (consider separately the cases where $\varphi_1(t) > \varphi_2(t)$, $\varphi_1(t) < \varphi_2(t)$, and $\varphi_1(t) = \varphi_2(t)$), and the general case follows from this by induction.
Next, from (2.1) we obtain (2.3), and this, together with (2.1), implies that for all $w \in \mathcal{D}$ and all $s \in \mathcal{T}$

$$0 \leq \psi(t) - \psi(s) \leq \psi(t) - \psi(s) + 2M|s-t|.$$ 

From this it follows easily that for each positive integer $n$ we can find a positive integer $k$, a partition of $J$ into $k$ subintervals of equal lengths, and $k$ functions $\psi_1, \ldots, \psi_k \in \mathcal{D}$ such that in the $j$th subinterval

$$0 \leq \psi(t) - \psi_j(t) \leq 1/n.$$ 

Let $\psi(t) = \max\{\psi_j(t), \ldots, \psi_k(t)\}$. Then

$$0 \leq \psi(t) - \psi(k)(t) \leq 1/n$$

for all $t \in J$, so that the sequence $(\psi(k))_n$ converges uniformly to $\psi$ on $J$. Also $D_s \psi(k)(t) \leq |t - \psi(k)(t)|$ nearly everywhere in $J$ (by the remark above), and the required result therefore follows from Lemma 2.

**Lemma 4.** Let $g$ satisfy condition (A) of Section 1, and let $w : J \to [0, \beta]$ be a continuous function such that $w(t_0) = w(t_0) = 0$ and that

$$D_s w(t) \leq g(t, w(t))$$

for nearly all $t \in J$. Then $w = 0$.

This is the differential inequality that underlies the proof of Kamke's uniqueness theorem (see, for example, [3], p. 51, or [11], p. 45).

**Lemma 5.** Let $Y$ be a complex Banach space, let $(t_0, y_0) \in B \times Y$, and let $B$ be the closed ball in $Y$ with centre $y_0$ and radius $\delta > 0$. Let $f : J \times B \to Y$ be continuous and bounded, let $M = \sup |f(t, y)|$, and let $I = [t_0, t_0 + \eta]$, where $\eta = \min(\eta, y/M)$. Then for each $\epsilon > 0$ the equation $y' = f(t, y)$ has an $\epsilon$-approximate solution on $I$ such that $y(t_0) = y_0$. Moreover, $\eta$ can be chosen so that for all $s \in I$

$$\|y(s) - y(t)\| \leq M|s-t|.$$ 

This is proved, for instance, by Cartan [2, Theorem 1.3.1].

3. Consider now the proof of the existence part of Theorem 1. Let $f, g$ satisfy the hypotheses of Theorem 1, let $I = [t_0, t_0 + \eta]$, and let $\sigma_n$ be a decreasing sequence of positive numbers with the limit 0. By Lemma 5, for each positive integer $n$ we can find an $\epsilon_n$-approximate solution $y_n$ of the equation $y' = f(t, y)$ on $I$, satisfying $y(t_0) = y_0$, with the property that for all $s \in I$

$$\|y_n(s) - y_n(t)\| \leq M|s-t|.$$ 

Let $\sigma_n(t) = \|y_n(t) - y_n(s)\|$, where $t \in I$ and $m > n \geq 1$. Obviously $\sigma_n(t_0) = 0$, and for all $s \in I$

$$\|\sigma_n(s) - \sigma_n(s)\| \leq 2M|s-t|.$$ 

Further, for all except a finite number of points of $I$

$$D_s \sigma_m(t) \leq \|\psi(t) - \psi(s)\|$$

$$\leq \|f(t, \psi(t)) - f(t, \psi(s))\| + c_{m+1} + c_m$$

$$\leq g(t, \psi_m(t)) + 2c_m + c_m.$$ 

For each positive integer $n$ we let $\sigma_n = \sup_{m>n} \sigma_m$. Then $\sigma_n(t_0) = 0$, and, by (3.2), (3.3), and Lemma 3 (applied to each compact subinterval of $[t_0, t_0 + \eta]$),

$$|\sigma_n(t) - \sigma_n(t_0)| \leq 2M|s-t|$$

for all $t \in I$ and

$$D_s \sigma_n(t) \leq g(t, \sigma_n(t)) + 2c_n$$

for all $t \in I$. The sequence $(\sigma_n)$ is therefore equicontinuous and uniformly bounded, and hence it has a subsequence $(\sigma_{n_k})$ converging uniformly on $I$ to a function $\sigma$, and clearly $\sigma(t_0) = 0$. By (3.4) and Lemma 2,

$$D_s \sigma(t) \leq g(t, \sigma(t)) + 2c_n$$

for all $t \in I$, and therefore also

$$D_s \sigma(t) \leq g(t, \sigma(t)).$$

We show next that $\sigma(t_0) = 0$. Since $f$ is continuous at $(t_0, y_0)$, given $\epsilon > 0$ we can find $\delta > 0$ such that $\|f(t, y) - f(t_0, y_0)\| < \epsilon$ whenever $t_0 - \delta \leq t \leq t_0 + \delta$ and $|y - y_0| < \delta$. Let $\delta = \min(\delta, 1, |M|)$. By (3.1), $|y_n(t) - y_0| < \delta$ for all $n$ and all $t \in [t_0, t_0 + \delta]$, and therefore

$$\|f(t, y_n(t)) - f(t, y_0(t))\| < 2\epsilon$$

whenever $m > n > 1$ and $t \in [t_0, t_0 + \delta]$. By the penultimate inequality in (3.3),

$$D_s \sigma_n(t) \leq 2\epsilon + 2c_n$$

for all but a finite number of points $t \in [t_0, t_0 + \delta]$, and hence, by Lemma 1,

$$0 \leq \sigma_n(t) - \sigma_n(t_0) \leq (2\epsilon + 2c_n)(t - t_0)$$

whenever $t \in [t_0, t_0 + \delta]$. This implies in turn that

$$0 \leq \sigma_n(t) \leq (2\epsilon + 2c_n)(t - t_0)$$

whence $\sigma(t_0) = 0$. From Lemma 4, we deduce now that $\sigma = 0$, and this implies that the sequence $(\sigma_n)$ is uniformly convergent on $I$. The limit of this sequence is then the required solution.
4. The proof of the existence part of Theorem 2 is similar. We choose the sequence \( (\varphi_n) \) as before, and we define \( \sigma_{\alpha n}(t) = (\varphi_n(t) - \varphi_n(t))^2 \) (so that \( \sigma_{\alpha n}(t) \leq 4 M_p^2 \)). Then for all but a finite number of \( t \in I \)

\[
\sigma_{\alpha n}(t) = 2 \Re \langle \varphi_n(t) - \varphi_n(t), \varphi_n(t) - \varphi_n(t) \rangle \\
= 2 \Re \langle f(t), \varphi_n(t) - f(t), \varphi_n(t) - \varphi_n(t) \rangle + \\
+ 2 \Re \langle \varphi_n(t) - f(t), \varphi_n(t) - f(t), \varphi_n(t) - \varphi_n(t) \rangle + 4 \langle \sigma_{\alpha n} + \epsilon_n \rangle M_p^2 \\
\leq 2 \Re \langle f(t), \varphi_n(t) - f(t), \varphi_n(t) - \varphi_n(t) \rangle + 4 \langle \sigma_{\alpha n} + \epsilon_n \rangle M_p^2 \\
\leq g(t, \sigma_{\alpha n}(t)) + 8 \epsilon_n M_p^2.
\]

If now \( \omega_0 = \sup_n \omega_{\alpha n} \), then exactly as before we see that there exists a subsequence \( (\omega_{\alpha n}) \) of \( (\omega_n) \) converging uniformly on \( I \) to a function \( \omega \), and that \( \omega(t_0) = 0 \) and

\[
D_{\alpha n} u(t) \leq g(t, o(t))
\]

for all \( t \in I \). The first inequality in (4.1) shows also that \( \omega'(t_0) = 0 \), and the proof is then completed as above.

5. We consider next global analogues of Theorems 1 and 2, and here we suppose that

(i) \( g(t, 0) = 0 \) for all \( t > t_0 \),

(ii) for each \( t_1 > t_0, x = 0 \) is the only solution of the equation

\[
x' = g(t, x)
\]

on \( [t_0, t_1] \) satisfying the conditions that

\[
x(t_0) = 0 \quad \text{and} \quad \lim_{t \to t_1} x(t) = 0,
\]

(iii) for each compact subinterval \( [t_2, t_3] \) of \( [t_0, \infty) \), the function \( C \) given by

\[
C(x) = \sup_{t \in [t_2, t_3]} \int_0^x g(t, u) \, du
\]

satisfies \( \int_0^\infty \frac{dx}{C(x)} = \infty \).

We remark that condition (B) implies that, on each compact subinterval \([t_1, t_2]\) of \([0, \infty)\), the zero function is the only solution of (5.1) on \([t_1, t_2]\) taking the value \( 0 \) at \( t_0 \) (for if there is a non-zero solution \( x \) on \([t_1, t_2]\) with \( x(t_0) = 0 \), then this, together with the zero function on \([t_0, t_1]\), provides a non-zero solution \( x \) on \([t_0, t_2]\) satisfying (5.2)).

It is known, for instance, that if

\[
g(t, x) = g(t, t_0) x,
\]

where \( g \) is a continuous function from \([0, \infty)\) into \([0, \infty)\), then \( g \) satisfies condition (B) if and only if

\[
\lim_{t \to \infty} \int_0^t g(s) \, ds + \log |t| < \infty
\]

(see [5], p. 33, Exercise 6.3). In particular, the case \( g(t) = 1/t \) corresponds to Nagumo’s uniqueness criterion.

Theorem 3. Let \( X \) be a complex Banach space, let \( (t_n, y_n) \in X \times X \), and let \( f: [t_n, \infty) \times X \to X \) be a continuous function such that for all \( t \geq t_0 \) and \( y, z \in X \)

\[
\|f(t, y) - f(t, z)\| \leq g(t, \|y - z\|),
\]

where \( g \) satisfies condition (B). Then the equation

\[
y' = f(t, y)
\]

has a unique solution on \([t_n, \infty] \) taking the value \( y_n \) at \( t_n \).

Theorem 4. Let \( X \) be a complex Hilbert space, let \( (t_n, y_n) \in X \times X \), and let \( f: (t_n, \infty) \times X \to X \) be a continuous function mapping bounded sets onto bounded sets such that for all \( t \geq t_0 \) and \( y, z \in X \)

\[
2 \Re \langle f(t, y) - f(t, z), y - z \rangle \leq g(t, \|y - z\|^2),
\]

where \( g \) satisfies condition (B). Then the result of Theorem 3 holds.

Here the case of Theorem 4 in which \( g(t, x) = x(t-t_0) \) is a result of Murakami ([7], p. 155; see also [6], ii, p. 240), while the case where \( g(t, x) = \beta(t,x) \), where \( \beta \) is continuous on \([0, \infty) \), is a result of Browder ([1], Theorem 3). We mention also that Ważewski [11] has given a global extension of Theorem 1 different from Theorem 3.

To prove Theorem 3, we note first that, by Theorem 1, there exists a unique solution of (5.3) on some non-degenerate interval \([t_0, t_1]\) with \( \varphi(t_0) = y_n \). We prove that this solution can be extended to \([t_0, t_2]\) for every \( t_2 > t_1 \).

Let \( t_2 > t_1 \), let \( y_1 = \varphi(t_0) \), and let \( A \) be the supremum of \( \|f(t, y_1)\| \) for \( t_1 \leq t \leq t_2 \). If \( A = 0 \), then \( f(t, y_1) = 0 \) for all \( t \in [t_1, t_2] \), and hence \( t \mapsto y_1 \) is a solution of (5.3) on \([t_1, t_2]\) which continues \( \varphi \) to \( t_0 \). We may therefore suppose that \( A > 0 \).

If \( y \) is a solution of (5.3) on some interval \([t_0, t_2] \subseteq [t_1, t_2]\) satisfying \( \varphi(t_0) = y_1 \), and \( \chi(t) = |y(t) - y_1| \), then \( \chi(t) = 0 \), and for all \( t \in [t_1, t_2] \)

\[
D_{x^2} x(t) \leq |y'(t)| \leq |f(t, y(t))| + |f(t, y_1)|
\]

\[
\leq |f(t, y(t)) - f(t, y_1)| + |f(t, y_1)|
\]

\[
\leq g(t, \chi(t)) + A \leq C(\chi(t)) + A,
\]
where \( C \) is defined as in (B) (iii). Since \( A > 0 \), the differential equation
\[(5.5)\]
\[w' = C(w^2) + A\]
has a unique solution \( \Phi \) on \([t_1, \infty[\) taking the value 0 at \( t_1 \). Indeed, this solution is given by
\[(5.6)\]
\[
\int_0^t \frac{ds}{C(s^2) + A} = \int_{t_1}^t ds = t - t_1.
\]
By (B) (iii), \( \int (C(x))^{-1} dx \to \infty \), and since \( C \) is increasing, this trivially implies that \( \int C(x)^{-1} dx = \infty \). Hence for each \( t \geq t_1 \), equation (5.6) has a unique solution \( z = \Phi(t) \), and \( \Phi \) is the required solution of (5.5). It therefore follows from (5.4) and a familiar differential equality (see [53], p. 26, or [61], (i), p. 15) that for all \( t \in [t_1, t_2] \)
\[z(t) \leq \Phi(t) \leq \Phi(t_2) = \varphi, \quad \text{say},
\]
and here \( \varphi \) is independent of \( t_1 \).

Now let \( B \) be the closed ball in \( Y \) with centre \( y_1 \) and radius \( 2\varphi \). Then for \( t \in [t_1, t_2] \), \( y \in B \),
\[(5.7)\]
\[
\|f(t, y)\| = \|f(t, y) - f(t, y_1) + f(t, y_1)\| \\
\leq g(t, |y - y_1| + A) \leq C(2\varphi^2) + A = M, \quad \text{say}.
\]
By repeated applications of Theorem 1, we can thus continue the solution \( \varphi \) of (5.3) through successive intervals of length \( \eta = \min\{t_2 - t_1, \varphi / M\} \), and hence we can continue \( \varphi \) to \( t_2 \).

In the case of Theorem 4, we again reduce the result to the case where \( A > 0 \), but here we take \( z(t) = |y(t) - y_1|^2 \). Then \( z(t_1) = 0 \), and for all \( t \in [t_1, t_2] \)
\[z'(t) = 2v^2 \langle f(t, y), v(t) \rangle - 2v \langle f(t, y), y(t) - y_1 \rangle
\]
\[= 2v \langle f(t, y), y(t) - y_1 \rangle - 2v \langle f(t, y_1), y(t) - y_1 \rangle + 2v \langle f(t, y_1), y(t) - y_1 \rangle
\]
\[\leq g(t, z(t)) + 2\|f(t, y_1)\|^2 z(t)^{\frac{1}{2}}
\]
\[\leq C(z(t)) + 2A(z(t))^{\frac{1}{2}}.
\]

We now have to consider the maximal solution of
\[(5.8)\]
\[w' = C(w^3) + 2Aw^2 + A,
\]
where \( x > 0 \), and \( x = 0 \) when \( t = t_1 \). Any solution of this equation (5.8) is increasing, and hence either is identically zero, or is zero on \([t_1, t^*]\) for some \( t^* > t_1 \) and non-zero to the right of \( t^* \). For a solution of the latter type, we can set \( x = u^3 \) when \( t > t^* \), so that \( u = 0 \) when \( t = t^* \), and for \( t > t^* \)
\[w' = C(u^3) + 2Au^2 + A,
\]

It is now obvious that we obtain the maximal solution of (5.8) by taking \( t^* = t_1 \), and that this maximal solution is given by
\[
\int_{t_1}^t \frac{ds}{C(u^3) + 2Au^2 + A} = \int_{t_1}^t ds = t - t_1.
\]
We can therefore complete the argument as before (\textit{c}), provided that we can show that the integral
\[
\int \frac{dx}{C(x) + 2Ax^2}
\]
is divergent. This is less trivial than the corresponding result for
\[
\int \frac{1}{C(x) + A}
\]
and we give the proof in the following lemma.

**Lemma 6.** Let \( C, D : ]0, \infty[ \to ]0, \infty[ \) be continuous functions such that \( C \) is increasing and that \( D(x) = O(x) \) as \( x \to \infty \). Then
\[
\int \frac{dx}{C(x)} \quad \text{and} \quad \int \frac{dx}{C(x) + D(x)}
\]
converge or diverge together.

It is obvious that if the second integral diverges, so does the first. Suppose then that the first integral converges; we have to show that the second integral diverges. Further, we can find \( X, k \) such that \( D(x) < Kx \) for all \( x > k \); and since then \( C(x) + D(x) < C(x) + Kx \) for all \( x > k \), it is enough to prove that
\[(5.9)\]
\[
\int \frac{dx}{C(x) + Kx} = \infty.
\]
Let \( E = \{x : C(x) < Kx\} \), \( F = \{x : C(x) > Kx\} \). It is easily verified that if either \( E \) or \( F \) contains the interval \( [1, \infty[ \) for some \( \lambda \), then (5.9) holds. On the other hand, if both \( E \) and \( F \) meet every interval \( [\lambda, \infty[ \), then \( E \) is the union of a sequence of disjoint open intervals \( ]a_i, \beta_i[ \), with \( a_i \) increasing to \( \infty \), such that \( C(x) = Kx \) for all \( x > \beta_i \). If now the integral of \( 1/C(x + Kx) \) over \( E \) is finite, then
\[(5.10)\]
\[
\infty > \frac{1}{K} \int \frac{dx}{C(x) + Kx} \geq \frac{1}{2K} \int \frac{dx}{x} = \frac{1}{2K} \sum_{i=1}^\infty \beta_i - a_i,
\]
\[= \frac{1}{2K} \sum_{i=1}^\infty \log \frac{\beta_i}{a_i}.
\]
(\textit{c}) In place of (5.7) we use the fact that \( f \) maps bounded sets into bounded sets.
Also
\[
\int \frac{dx}{C(x)} = \sum_i \int \frac{dx}{C_i(x)} = \sum_i \frac{\beta_i - a_i}{a_i} = \frac{1}{K} \sum_i \frac{\beta_i - a_i}{a_i}.
\]
From (5.10) we see that \(\beta_i / a_i \rightarrow 1\) as \(i \rightarrow \infty\), and hence for all large \(i\)
\[(\beta_i - a_i)/a_i \leq 2 \log(\beta_i/a_i).
\]
From (5.10) and (5.11) we now deduce that \(\int \frac{dx}{C(x)} < \infty\). Hence
\[
\int \frac{dx}{C(x)} = \infty, \quad \text{and therefore}
\]
\[
\int \frac{dx}{C(x) + Kx} = -\frac{1}{2} \int \frac{dx}{C(x)} = \infty,
\]
so that (5.9) holds. This completes the proof of the lemma and of Theorem 4.

References


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