STUDIA MATHEMATICA T. XLIV. (1972)

Some applications of Zygmund's lemma to non-linear differential equations in Banach and Hilbert spaces

by

T. M. FLETT (Sheffield)

Abstract. This paper deals with existence theorems for the differential equation $y' = f(t, y)$ where the solution y takes its values in a Banach or Hilbert space Y. A new proof is given of the theorem of Ważewski that the conditions of Kamke's uniqueness criterion imply local existence of solutions. The argument covers also monotonicity conditions, and a generalization of a theorem of Browder of this type is given.

1. In this paper we consider existence theorems for the differential equation

$$y' = f(t, y), \tag{1.1}$$

where the solution y takes its values in a Banach or Hilbert space Y. If Y is finite-dimensional, then the continuity of f alone implies the local existence of solutions, but this is no longer so when the dimension of Y is infinite (see, for example, Dieudonné [3], p. 287, Exercise 5). It has been proved by Ważewski [12] that in the infinite-dimensional case the conditions of Kamke's well-known uniqueness theorem imply local existence, and we give a new proof of this result. We also prove a result involving a monotonicity condition that is the 'one-sided' analogue of Ważewski's theorem.

As in Kamke's theorem, we compare the differential equation (1.1) with a scalar equation

$$x' = g(t, x), \tag{1.2}$$

and in our first two theorems we suppose that

(A) *g is a continuous function from the rectangle $]t_0, t_0+a] \times [0, \beta]$ in $\boldsymbol{R}^2$ into $[0, \infty[$ with the properties that*

(i) *$g(t, 0) = 0$ for all $t \in]t_0, t_0+a]$,*

(ii) *for each $t_1 \in]t_0, t_0+a]$, $\chi = 0$ is the only solution of equation (1.2) on $]t_0, t_1]$ satisfying the conditions that*

$$\chi(t_0+) = 0 \quad \textit{and that} \quad \lim_{t\to t_0+} \frac{\chi(t)}{t-t_0} = 0.$$

For completeness, we state both the uniqueness and existence results.

THEOREM 1. *Let Y be a complex Banach space, let $(t_0, y_0) \in R \times Y$, let B be the closed ball in Y with centre y_0 and radius $\varrho > 0$, and let $f\colon [t_0, t_0+a] \times B \to Y$ be a continuous function such that for all $(t, y), (t, z) \in]t_0, t_0+a] \times B$*

$$\|f(t, y) - f(t, z)\| \leqslant g(t, \|y - z\|), \tag{1.3}$$

where g satisfies condition (A) *with $\beta = 2\varrho$. Then equation* (1.1) *has at most one solution φ on $[t_0, t_0+a]$ satisfying the condition $\varphi(t_0) = y_0$. If in addition f is bounded, $M = \sup\|f(t, y)\|$, and $\eta = \min\{a, \varrho/M\}$, then equation* (1.1) *has exactly one solution on $[t_0, t_0+\eta]$ taking the value y_0 at t_0.*

THEOREM 2. *If Y is a complex Hilbert space, condition* (1.3) *in Theorem* 1 *can be replaced by the condition that for all $(t, y), (t, z) \in]t_0, t_0+a] \times B$*

$$2\,\mathrm{re}\langle f(t, y) - f(t, z), y - z\rangle \leqslant g(t, \|y - z\|^2), \tag{1.4}$$

where g satisfies condition (A) *with $\beta = 4\varrho^2$.*

Here the uniqueness part of Theorem 1 is Kamke's theorem (see [5], p. 31), while the existence part is the result of Ważewski [12] mentioned above(1).

The uniqueness part of Theorem 2 is essentially known (for a similar argument, see [5], p. 35, Exercise 6.8), while the existence part appears to be new. We also obtain global theorems of similar type (Section 5. Theorems 3 and 4), which include results of Murakami and Browder,

2. Following Gál [4], we say that a property $P(t)$ holds *nearly everywhere* in a set $E \subset \mathbf{R}$, or *for nearly all* $t \in E$, if there is a countable set $H \subset E$ such that $P(t)$ holds for all $t \in E \setminus H$. We also use J, here and later, to denote the closed interval $[t_0, t_0+a]$.

The following well-known lemma of Zygmund (see [10], p. 204) plays a fundamental role in our arguments.

LEMMA 1. *Let $M \in \mathbf{R}$, and let $\varphi\colon J \to \mathbf{R}$ be a continuous function whose lower right-hand Dini derivative $D_+\varphi$ satisfies the inequality $D_+\varphi(t) \leqslant M$ for nearly all $t \in J$. Then*

$$\varphi(t_0+a) - \varphi(t_0) \leqslant Ma.$$

We require also some further lemmas.

(1) It has been shown by Olech [8] (see also [6], i, p. 50) that the results of Theorem 1 are implied by the corresponding results in which g satisfies the simpler conditions of Perron's uniqueness theorem. However, this simplification is not available for Theorem 2, and since our proofs of Theorems 1 and 2 are closely similar, we prove Theorem 1 directly.

We remark also that Olech [9] has given an extension of Theorem 1 involving conditions of 'Carathéodory type'.

LEMMA 2. *Let (φ_n) be a sequence of continuous functions from J into $\mathbf{R}$ converging uniformly on J to a function φ. Let also E be a set in $\mathbf{R}^2$ containing the graphs of φ_n $(n = 1, 2, \ldots)$ and of φ, let $h\colon E \to \mathbf{R}$ be continuous, and suppose that for each n*

$$D_+\varphi_n(t) \leqslant h(t, \varphi_n(t))$$

nearly everywhere in J. Then for all $t \in J^\circ$

$$D_+\varphi(t) \leqslant h(t, \varphi(t)).$$

Let $\varepsilon > 0$ and let $t \in J^\circ$. Since h is continuous at $(t, \varphi(t))$, φ is continuous at t, and $\varphi_n \to \varphi$ uniformly on J, we can find a positive number η and an integer q such that $s \in J^\circ$ and $|h(s, \varphi_n(s)) - h(t, \varphi(t))| \leqslant \varepsilon$ whenever $|s - t| \leqslant \eta$ and $n \geqslant q$. Hence for each $n \geqslant q$

$$D_+\varphi_n(s) \leqslant h(t, \varphi(t)) + \varepsilon$$

nearly everywhere in $[t, t+\eta]$, and therefore, by Lemma 1,

$$\varphi_n(s) - \varphi_n(t) \leqslant (s - t)\big(h(t, \varphi(t)) + \varepsilon\big)$$

whenever $s \in [t, t+\eta]$. Hence also

$$\varphi(s) - \varphi(t) \leqslant (s - t)\big(h(t, \varphi(t)) + \varepsilon\big),$$

so that $D_+\varphi(t) \leqslant h(t, \varphi(t))$(2).

LEMMA 3. *Let $M \geqslant 0$, let Δ be a class of uniformly bounded continuous functions $\psi\colon J \to \mathbf{R}$ with the property that for all $s, t \in J$*

$$|\psi(s) - \psi(t)| \leqslant M|s - t|, \tag{2.1}$$

and let $\Psi = \sup_\Delta \psi$. Let also E be a set in $\mathbf{R}^2$ containing the graphs of each $\psi \in \Delta$ and of Ψ, let $h\colon E \to \mathbf{R}$ be continuous, and suppose that for each $\psi \in \Delta$

$$D_+\psi(t) \leqslant h(t, \psi(t)) \tag{2.2}$$

nearly everywhere in J. Then for all $s, t \in J$

$$|\Psi(s) - \Psi(t)| \leqslant M|s - t| \tag{2.3}$$

(so that Ψ is continuous), and for all $t \in J^\circ$

$$D_+\Psi(t) \leqslant h(t, \Psi(t)).$$

We remark first that if $\psi_1, \ldots, \psi_k \in \Delta$ and $\psi = \max\{\psi_1, \ldots, \psi_k\}$, then ψ satisfies (2.2) nearly everywhere in J. Here the case $k = 2$ is almost immediate (consider separately the cases where $\psi_1(t) > \psi_2(t)$, $\psi_1(t) < \psi_2(t)$, $\psi_1(t) = \psi_2(t)$), and the general case follows from this by induction.

(2) Indeed, $D^+\varphi(t) \leqslant h(t, \varphi(t))$.

Next, from (2.1) we obtain (2.3), and this, together with (2.1), implies that for all $\psi \in \Delta$ and all $s, t \in J$

$$0 \leqslant \Psi(t) - \psi(t) \leqslant \Psi(s) - \psi(s) + 2M|s-t|.$$

From this it follows easily that for each positive integer n we can find a positive integer k, a partition of J into k subintervals of equal lengths, and k functions $\psi_1, \ldots, \psi_k \in \Delta$ such that in the jth subinterval

$$0 \leqslant \Psi(t) - \psi_j(t) \leqslant 1/n.$$

Let $\psi^{(n)} = \max\{\psi_1, \ldots, \psi_k\}$. Then $0 \leqslant \Psi(t) - \psi^{(n)}(t) \leqslant 1/n$ for all $t \in J$, so that the sequence $(\psi^{(n)})$ converges uniformly to Ψ on J. Also $D_+\psi^{(n)}(t) \leqslant h(t, \psi^{(n)}(t))$ nearly everywhere in J (by the remark above), and the required result therefore follows from Lemma 2.

LEMMA 4. *Let g satisfy condition* (A) *of Section* 1, *and let $\omega\colon J \to [0, \beta]$ be a continuous function such that $\omega(t_0) = \omega'(t_0) = 0$ and that*

$$D_+\omega(t) \leqslant g(t, \omega(t))$$

for nearly all $t \in J$. Then $\omega = 0$.

This is the differential inequality that underlies the proof of Kamke's uniqueness theorem (see, for example, [5], p. 31, or [11], p. 45).

LEMMA 5. *Let Y be a complex Banach space, let $(t_0, y_0) \in \mathbf{R} \times Y$, and let B be the closed ball in Y with centre y_0 and radius $\varrho > 0$. Let also $f\colon J \times B \to Y$ be continuous and bounded, let $M = \sup\|f(t, y)\|$, and let $I = [t_0, t_0+\eta]$, where $\eta = \min\{\alpha, \varrho/M\}$. Then for each $\varepsilon > 0$ the equation $y' = f(t, y)$ has an ε-approximate solution ψ on I such that $\psi(t_0) = y_0$. Moreover, ψ can be chosen so that for all $s, t \in I$*

$$\|\psi(s) - \psi(t)\| \leqslant M|s-t|.$$

This is proved, for instance, by Cartan [2], Theorem 1.3.1.

3. Consider now the proof of the existence part of Theorem 1. Let f, g satisfy the hypotheses of Theorem 1, let $I = [t_0, t_0+\eta]$, and let (ε_n) be a decreasing sequence of positive numbers with the limit 0. By Lemma 5, for each positive integer n we can find an ε_n-approximate solution ψ_n of the equation $y' = f(t, y)$ on I, satisfying $\psi_n(t_0) = y_0$, with the property that for all $s, t \in I$

$$\|\psi_n(s) - \psi_n(t)\| \leqslant M|s-t|. \tag{3.1}$$

Let $\sigma_{m,n}(t) = \|\psi_m(t) - \psi_n(t)\|$, where $t \in I$ and $m > n \geqslant 1$. Obviously $\sigma_{m,n}(t_0) = 0$, and for all $s, t \in I$

$$\|\sigma_{m,n}(s) - \sigma_{m,n}(t)\| \leqslant 2M|s-t|. \tag{3.2}$$

Further, for all except a finite number of points of I

$$\begin{aligned} D_+\sigma_{m,n}(t) &\leqslant \|\psi_m'(t) - \psi_n'(t)\| \\ &\leqslant \|f(t, \psi_m(t)) - f(t, \psi_n(t))\| + \varepsilon_m + \varepsilon_n \\ &\leqslant g(t, \sigma_{m,n}(t)) + 2\varepsilon_n. \end{aligned} \tag{3.3}$$

For each positive integer n let $\omega_n = \sup_{m>n} \sigma_{m,n}$. Then $\omega_n(t_0) = 0$, and, by (3.2), (3.3), and Lemma 3 (applied to each compact subinterval of $]t_0, t_0+\eta]$),

$$|\omega_n(s) - \omega_n(t)| \leqslant 2M|s-t|$$

for all $s, t \in I$ and

$$D_+\omega_n(t) \leqslant g(t, \omega_n(t)) + 2\varepsilon_n \tag{3.4}$$

for all $t \in I^\circ$. The sequence (ω_n) is therefore equicontinuous and uniformly bounded, and hence it has a subsequence (ω_{n_r}) converging uniformly on I to a function ω, and clearly $\omega(t_0) = 0$. By (3.4) and Lemma 2,

$$D_+\omega(t) \leqslant g(t, \omega(t)) + 2\varepsilon_n$$

for all $t \in I^\circ$, and therefore also

$$D_+\omega(t) \leqslant g(t, \omega(t)).$$

We show next that $\omega'(t_0) = 0$. Since f is continuous at (t_0, y_0), given $\varepsilon > 0$ we can find $\delta > 0$ such that $\|f(t, y) - f(t_0, y_0)\| \leqslant \varepsilon$ whenever $t_0 \leqslant t \leqslant t_0 + \delta$ and $\|y - y_0\| \leqslant \delta$. Let $\lambda = \min\{\delta, \delta/M\}$. By (3.1), $\|\psi_n(t) - y_0\| \leqslant \delta$ for all n and all $t \in [t_0, t_0+\lambda]$, and therefore

$$\|f(t, \psi_m(t)) - f(t, \psi_n(t))\| \leqslant 2\varepsilon$$

whenever $m > n \geqslant 1$ and $t \in [t_0, t_0+\lambda]$. By the penultimate inequality in (3.3),

$$D_+\sigma_{m,n}(t) \leqslant 2\varepsilon + 2\varepsilon_n$$

for all but a finite number of points $t \in\,]t_0, t_0+\lambda[$, and hence, by Lemma 1,

$$0 \leqslant \sigma_{m,n}(t) = \sigma_{m,n}(t) - \sigma_{m,n}(t_0) \leqslant (2\varepsilon + 2\varepsilon_n)(t - t_0)$$

whenever $t \in [t_0, t_0+\lambda]$. This implies in turn that

$$0 \leqslant \omega_n(t) \leqslant (2\varepsilon + 2\varepsilon_n)(t - t_0),$$

and so also

$$0 \leqslant \omega(t) \leqslant (2\varepsilon + 2\varepsilon_n)(t - t_0),$$

whence $\omega'(t_0) = 0$.

From Lemma 4, we deduce now that $\omega = 0$, and this implies that the sequence (ψ_n) is uniformly convergent on I. The limit of this sequence is then the required solution.

4. The proof of the existence part of Theorem 2 is similar. We choose the sequence (ψ_n) as before, and we define $\sigma_{m,n}(t) = \|\psi_m(t)-\psi_n(t)\|^2$ (so that $\sigma_{m,n}(t) \leqslant 4M^2\eta^2$). Then for all but a finite number of $t \in I$

$$\begin{aligned}(4.1)\qquad \sigma'_{m,n}(t) &= 2\operatorname{re}\langle\psi'_m(t)-\psi'_n(t),\, \psi_m(t)-\psi_n(t)\rangle \\ &= 2\operatorname{re}\langle f(t,\psi_m(t))-f(t,\psi_n(t)),\, \psi_m(t)-\psi_n(t)\rangle + \\ &\quad + 2\operatorname{re}\langle \psi'_m(t)-f(t,\psi_m(t))-\psi'_n(t)+f(t,\psi_n(t)),\, \psi_m(t)-\psi_n(t)\rangle \\ &\leqslant 2\operatorname{re}\langle f(t,\psi_m(t))-f(t,\psi_n(t)),\, \psi_m(t)-\psi_n(t)\rangle + 4(\varepsilon_m+\varepsilon_n)M\eta \\ &\leqslant g(t,\sigma_{m,n}(t)) + 8\varepsilon_n M\eta .\end{aligned}$$

If now $\omega_n = \sup_{m>n} \sigma_{m,n}$, then exactly as before we see that there exists a subsequence (ω_{n_r}) of (ω_n) converging uniformly on I to a function ω, and that $\omega(t_0) = 0$ and

$$D_+\omega(t) \leqslant g(t, \omega(t))$$

for all $t \in J^\circ$. The first inequality in (4.1) shows also that $\omega'(t_0) = 0$, and the proof is then completed as above.

5. We consider next global analogues of Theorems 1 and 2, and here we suppose that

(B) *g is a continuous function from $]t_0, \infty[\times[0,\infty[$ into $[0,\infty[$ with the properties that*

(i) *$g(t, 0) = 0$ for all $t > t_0$,*

(ii) *for each $t_1 > t_0$, $\chi = 0$ is the only solution of the equation*

$$(5.1)\qquad x' = g(t, x)$$

on $]t_0, t_1]$ satisfying the conditions that

$$(5.2)\qquad \chi(t_0+) = 0 \quad \text{and that} \quad \lim_{t\to t_0+} \frac{\chi(t)}{t-t_0} = 0,$$

(iii) *for each compact subinterval $[t_1, t_2]$ of $]t_0, \infty[$ the function C given by $C(x) = \sup\{g(t,u)\colon t_1 \leqslant t \leqslant t_2,\ 0 \leqslant u \leqslant x\}$ satisfies $\int_1^\infty \frac{dx}{C(x)} = \infty$.*

We remark that condition (B) implies that, on each compact subinterval $[t_1, t_2]$ of $]t_0, \infty[$, the zero function is the only solution of (5.1) on $[t_1, t_2]$ taking the value 0 at t_1 (for if there is a non-zero solution χ_1 on $[t_1, t_2]$ with $\chi_1(t_1) = 0$, then this, together with the zero function on $]t_0, t_1[$, provides a non-zero solution χ on $]t_0, t_2]$ satisfying (5.2)).

It is known, for instance, that if

$$g(t, x) = \gamma(t-t_0)x,$$

where γ is a continuous function from $]0,\infty[$ into $[0,\infty[$, then g satisfies condition (B) if and only if

$$\liminf_{t\to 0+}\Big\{\int_t^1 \gamma(s)\,ds + \log t\Big\} < \infty$$

(see [5], p. 33, Exercise 6.3). In particular, the case $\gamma(t) = 1/t$ corresponds to Nagumo's uniqueness criterion.

THEOREM 3. *Let Y be a complex Banach space, let $(t_0, y_0) \in \boldsymbol{R}\times Y$, and let $f\colon [t_0, \infty[\times Y \to Y$ be a continuous function such that for all $t > t_0$ and $y, z \in Y$*

$$\|f(t,y)-f(t,z)\| \leqslant g(t, \|y-z\|),$$

where g satisfies condition (B). *Then the equation*

$$(5.3)\qquad y' = f(t, y)$$

has a unique solution on $[t_0, \infty[$ taking the value y_0 at t_0.

THEOREM 4. *Let Y be a complex Hilbert space, let $(t_0, y_0) \in \boldsymbol{R}\times Y$, and let $f\colon [t_0,\infty[\times Y \to Y$ be a continuous function mapping bounded sets onto bounded sets such that for all $t > t_0$ and $y, z \in Y$*

$$2\operatorname{re}\langle f(t,y)-f(t,z),\, y-z\rangle \leqslant g(t, \|y-z\|^2),$$

where g satisfies condition (B). *Then the result of Theorem 3 holds.*

Here the case of Theorem 4 in which $g(t,x) = x/(t-t_0)$ is a result of Murakami ([7], p. 155; see also [6], ii, p. 246), while the case where $g(t,x) = \beta(t)x$, where β is continuous on $[0,\infty[$, is a result of Browder ([1], Theorem 3). We mention also that Ważewski [11] has given a global extension of Theorem 1 different from Theorem 3.

To prove Theorem 3, we note first that, by Theorem 1, there exists a unique solution of (5.3) on some non-degenerate interval $[t_0, t_1]$ with $\varphi(t_0) = y_0$. We prove that this solution can be extended to $[t_0, t_2]$ for every $t_2 > t_1$.

Let $t_2 > t_1$, let $y_1 = \varphi(t_1)$, and let A be the supremum of $\|f(t, y_1)\|$ for $t_1 \leqslant t \leqslant t_2$. If $A = 0$, then $f(t, y_1) = 0$ for all $t \in [t_1, t_2]$, and hence $t \mapsto y_1$ is a solution of (5.3) on $[t_1, t_2]$ which continues φ to t_2. We may therefore suppose that $A > 0$.

If ψ is a solution of (5.3) on some interval $[t_1, t_3] \subset [t_1, t_2]$ satisfying $\psi(t_1) = y_1$, and $\chi(t) = \|\psi(t)-y_1\|$, then $\chi(t_1) = 0$, and for all $t \in]t_1, t_3[$

$$\begin{aligned}(5.4)\qquad D_+\chi(t) &\leqslant \|\psi'(t)\| = \|f(t,\psi(t))\| \\ &\leqslant \|f(t,\psi(t))-f(t,y_1)\| + \|f(t,y_1)\| \\ &\leqslant g(t,\chi(t)) + A \leqslant C(\chi(t)) + A,\end{aligned}$$

where C is defined as in (B) (iii). Since $A > 0$, the differential equation

(5.5) $$x' = C(x)+A$$

has a unique solution Φ on $[t_1, \infty[$ taking the value 0 at t_1. Indeed, this solution is given by

(5.6) $$\int_0^x \frac{du}{C(u)+A} = \int_{t_1}^t ds = t-t_1.$$

By (B) (iii), $\int^\infty (C(x))^{-1}dx = \infty$, and since C is increasing, this trivially implies that $\int^\infty (C(x)+A)^{-1}dx = \infty$. Hence for each $t \geqslant t_1$ equation (5.6) has a unique solution $x = \Phi(t)$, and Φ is the required solution of (5.5). It therefore follows from (5.4) and a familiar differential equality (see [5], p. 26, or [6], (i), p. 15) that for all $t \in [t_1, t_3]$

$$\chi(t) \leqslant \Phi(t) \leqslant \Phi(t_2) = \varrho, \quad \text{say},$$

and here ϱ is independent of t_3.

Now let B be the closed ball in Y with centre y_1 and radius 2ϱ. Then for $t \in [t_1, t_2]$, $y \in B$,

(5.7) $$\begin{aligned}\|f(t, y)\| &\leqslant \|f(t, y)-f(t, y_1)\| + \|f(t, y_1)\| \\ &\leqslant g(t, \|y-y_1\|)+A \leqslant C(2\varrho)+A = M, \quad \text{say}.\end{aligned}$$

By repeated applications of Theorem 1, we can thus continue the solution φ of (5.3) through successive intervals of length $\eta = \min\{t_2-t_1, \varrho/M\}$, and hence we can continue φ to t_2.

In the case of Theorem 4, we again reduce the result to the case where $A > 0$, but here we take $\chi(t) = \|\psi(t)-y_1\|^2$. Then $\chi(t_1) = 0$, and for all $t \in]t_1, t_3[$

$$\begin{aligned}\chi'(t) &= 2\mathrm{re}\langle\psi'(t), \psi(t)-y_1\rangle = 2\mathrm{re}\langle f(t, \psi(t)), \psi(t)-y_1\rangle \\ &= 2\mathrm{re}\langle f(t, \psi(t))-f(t, y_1), \psi(t)-y_1\rangle + 2\mathrm{re}\langle f(t, y_1), \psi(t)-y_1\rangle \\ &\leqslant g(t, \chi(t))+2\|f(t, y_1)\|(\chi(t))^{\frac{1}{2}} \\ &\leqslant C(\chi(t))+2A(\chi(t))^{\frac{1}{2}}.\end{aligned}$$

We now have to consider the maximal solution of

(5.8) $$x' = C(x)+2Ax^{\frac{1}{2}},$$

where $x \geqslant 0$, and $x = 0$ when $t = t_1$. Any solution of this equation (5.8) is increasing, and hence either is identically zero, or is zero on $[t_1, t^*]$ for some $t^* \geqslant t_1$ and non-zero to the right of t^*. For a solution of the latter type, we can set $x = u^2$ when $t \geqslant t^*$, so that $u = 0$ when $t = t^*$, and for $t > t^*$

$$u' = C(u^2)/(2u)+A.$$

It is now obvious that we obtain the maximal solution of (5.8) by taking $t^* = t_1$, and that this maximal solution is given by

$$\int_0^u \frac{dv}{C(v^2)/(2v)+A} = \int_{t_1}^t ds = t-t_1.$$

We can therefore complete the argument as before[3], provided that we can show that the integral

$$\int^\infty \frac{dv}{C(v^2)/(2v)+A} = \int^\infty \frac{dx}{C(x)+2A\sqrt{x}}$$

is divergent. This is less trivial than the corresponding result for

$$\frac{1}{C(x)+A}$$

and we give the proof in the following lemma.

Lemma 6. *Let C, D: $]0, \infty[\to]0, \infty[$ be continuous functions such that C is increasing and that $D(x) = O(x)$ as $x \to \infty$. Then*

$$\int^\infty \frac{dx}{C(x)} \quad \text{and} \quad \int^\infty \frac{dx}{C(x)+D(x)}$$

converge or diverge together.

It is obvious that if the second integral diverges, so does the first. Suppose then that the first integral diverges; we have to show that the second integral diverges. Further, we can find K, k such that $D(x) \leqslant Kx$ for all $x \geqslant k$, and since then $C(x)+D(x) \leqslant C(x)+Kx$ for all $x \geqslant k$, it is enough to prove that

(5.9) $$\int^\infty \frac{dx}{C(x)+Kx} = \infty.$$

Let $E = \{x\colon C(x) < Kx\}$, $F = \{x\colon C(x) \geqslant Kx\}$. It is easily verified that if either E or F contains the interval $[\lambda, \infty[$ for some λ, then (5.9) holds. On the other hand, if both E and F meet every interval $[\lambda, \infty[$, then E is the union of a sequence of disjoint open intervals $]\alpha_i, \beta_i[$, with α_i increasing to ∞, such that $C(x) = Kx$ for $x = \alpha_i, \beta_i$. If now the integral of $1/(C(x)+Kx)$ over E is finite, then

(5.10) $$\begin{aligned}\infty > \int_E \frac{dx}{C(x)+Kx} &\geqslant \frac{1}{2K}\int_E \frac{dx}{x} = \frac{1}{2K}\sum_i \int_{\alpha_i}^{\beta_i} \frac{dx}{x} \\ &= \frac{1}{2K}\sum_i \log\left(\frac{\beta_i}{\alpha_i}\right).\end{aligned}$$

[3] In place of (5.7) we use the fact that f maps bounded sets onto bounded sets.

Also

$$(5.11)\qquad \int_E \frac{dx}{C(x)} = \sum_i \int_{a_i} \frac{dx}{C(x)} \leqslant \sum_i \frac{\beta_i - a_i}{C(a_i)} = \frac{1}{K}\sum_i \frac{\beta_i - a_i}{a_i}.$$

From (5.10) we see that $\beta_i/a_i \to 1$ as $i \to \infty$, and hence for all large i

$$(\beta_i - a_i)/a_i \leqslant 2\log(\beta_i/a_i).$$

From (5.10) and (5.11) we now deduce that $\int_E \frac{dx}{C(x)} < \infty$. Hence $\int_F \frac{dx}{C(x)} = \infty$, and therefore

$$\int_F \frac{dx}{C(x)+Kx} \geqslant \frac{1}{2}\int_F \frac{dx}{C(x)} = \infty,$$

so that (5.9) holds. This completes the proof of the lemma and of Theorem 4.

References

[1] F. E. Browder, *Non-linear equations of evolution*, Ann. of Math. 80 (1964), pp. 485–523.

[2] H. Cartan, *Calcul différentiel*, Paris 1967.

[3] J. Dieudonné, *Foundations of modern analysis*, New York 1960.

[4] I. S. Gál, *On the fundamental theorems of the calculus*, Trans. Amer. Math. Soc. 86 (1957), pp. 309–320.

[5] P. Hartman, *Ordinary differential equations*, New York 1964.

[6] V. Lakshmikantham and S. Leela, *Differential and integral inequalities*, 2 vols, New York 1969.

[7] H. Murakami, *On non-linear ordinary and evolution equations*, Funkcial. Ekvac. 9 (1966), pp. 151–162.

[8] C. Olech, *Remarks concerning criteria for uniqueness of solutions of ordinary differential equations*, Bull. Acad. Polon. Sci. Série des Sci. Math. Astr. Phys. 8 (1960), pp. 661–666.

[9] — *On the existence and uniqueness of solutions of an ordinary differential equation in the case of Banach space*, ibidem, pp. 667–673.

[10] S. Saks, *Theory of the integral*, New York 1937.

[11] J. Szarski, *Differential inequalities*, Warsaw 1966.

[12] T. Ważewski, *Sur l'existence et l'unicité des intégrales des equations différentielles ordinaires au cas de l'espace de Banach*, Bull. Acad. Polon. Sci. Série des Sci. Math. Astr. Phys. 8 (1960), pp. 301–305.

Received September 20, 1971 (391)

STUDIA MATHEMATICA T. XLIV. (1972)

Spherical convergence and integrability of multiple trigonometric series on hypersurfaces

by

M. J. KOHN (Minneapolis, Minn.)

Abstract. Let T be a trigonometric series in k variables and let Γ be a subset of E^k of $k-1$ dimensional character. We investigate conditions on the coefficients of T and on the structure of Γ so that T converges spherically a.e. on Γ and can be formally termwise integrated over Γ. We apply our results to improve a k-dimensional version of Riemann's Theorem for formally integrated series.

1. Introduction. Let

$$(1.1)\qquad \sum c_n e^{in\cdot x}$$

be a trigonometric series in k variables, and let Γ be a subset of E^k of Hausdorff dimension $k-1$ (e.g., a hypersurface). We investigate conditions on Γ and on the coefficients c_n of (1.1) so that the series converges spherically almost everywhere on Γ and can be integrated formally over Γ. We apply our results to improve a theorem of Shapiro, [5], on Riemann summability of multiple trigonometric series.

The results of this paper form part of the author's Doctoral dissertation. The author wishes to record his debt to Professor A. Zygmund, under whose direction the dissertation was written.

2. Notation. We denote points of Euclidean space E^k, $k \geqslant 2$, by $x = (x_1, \ldots, x_k)$ and integral lattice points by $n = (n_1, \ldots, n_k)$. We write $n\cdot x = n_1x_1 + \ldots + n_kx_k$ and $|x| = (x_1^2 + \ldots + x_k^2)^{\frac{1}{2}}$. We put $T^k = \{x \in E^k: |x_i| \leqslant \pi, i = 1, \ldots, k\}$.

For the series (1.1) we write

$$S_R(x) = \sum_{|n|<R} c_n e^{in\cdot x}$$

and we say the series converges spherically at x to sum s if

$$\lim_{R\to\infty} S_R(x) = s.$$

We denote by $H_\beta(\Gamma)$ the Hausdorff measure of order β of a set Γ. We are concerned with sets of Hausdorff dimension $k-1$, that is, sets Γ