References

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Estimates for the spline orthonormal functions and for their derivatives

by

Z. CIESIELSKI (Sopot) and J. DOMSTA (Sopot)

Abstract. The aim of this note is to obtain local exponential estimates for the spline functions $f_j^{(m,k)}$ with $j \geqslant k-m$, $m \geqslant -1$ and $0 \leqslant k \leqslant m+1$ which were constructed in the previous paper [2] of the authors. These estimates are important in the investigations of order of approximation by partial sums of the expansions with respect to the set $(f_j^{(m,k)}, j \geqslant k-m)$, where $0 \leqslant k \leqslant m+1$.

1. Introduction. We are going to describe briefly the functions $f_j^{(m,k)}$ by means of the Haar functions. Let $I = \langle 0, 1\rangle$ and let $(t_n, n \geqslant 0)$ be the dyadic sequence in I associated to the Haar system, i.e. $t_0 = 0$, $t_1 = 1$ and for $n = 2^\mu + \nu$ with $\mu \geqslant 0, 1 \leqslant \nu \leqslant 2^\mu$

$$t_n = \frac{2\nu-1}{2^{\mu+1}}. \tag{1}$$

For each $n > 1$ the subsequence $(t_i, i = 0, \ldots, n)$ can be ordered into an increasing sequence $0 = s_{n,0} < \ldots < s_{n,n} = 1$ and then it can be extended as follows

$$s_{n,i} = \begin{cases} \dfrac{i}{2^{\mu+1}} & \text{for } i = \ldots, 2\nu-1, 2\nu, \\ \dfrac{i-\nu}{2^\mu} & \text{for } i = 2\nu+1, 2\nu+2, \ldots, \end{cases} \tag{2}$$

where i is ranging over all the integers.

The n-th Haar function is defined as follows: $\chi_1 = 1$, and for $n = 2^\mu + \nu$ with $\mu \geqslant 0, 1 \leqslant \nu \leqslant 2^\mu$

$$\chi_n(t) = \begin{cases} 2^{\mu/2} & \text{for } t \in \langle s_{n,2\nu-2}, s_{n,2\nu-1}), \\ -2^{\mu/2} & \text{for } t \in \langle s_{n,2\nu-1}, s_{n,2\nu}), \\ 0 & \text{elsewhere in } \langle 0, 1), \end{cases}$$

and in addition $\chi_n(1) = \chi_n(1_-)$.

Now, for given $m \geqslant -1$ let us take the sequence of linearly independent functions $(1, t, \ldots, t^{m+1}, G^{m+1}\chi_n, n = 2, 3, \ldots)$, where

$$(Gf)(t) = \int_0^t f(s)\,ds \quad \text{for } t \in I, \tag{3}$$

and $G^0 f = f$. Applying to this sequence the Schmidt orthonormalization procedure we obtain the orthonormal set $(f_j^{(m)}, j = -m, -m+1, \ldots)$ of splines of order m (of degree $m+1$). Differentiating k times each function $f_j^{(m)}$ we obtain a new sequence of linearly independent spline functions of order $m-k$ and we use for them the notation

$$f_j^{(m,k)} = D^k f_j^{(m)}, \quad j \geqslant k-m,\ 0 \leqslant k \leqslant m+1.$$

Notice, for $m = -1$ the $f_j^{(m)}, j > 0$, are the Haar functions i.e. $f_j^{(m)} = \chi_j$ and for $m = 0$ the $f_j^{(m)}, j \geqslant 0$, are the Franklin functions i.e. $f_j^{(m)} = f_j$ (cf. [1]).

Our main task is to establish the following

THEOREM. *To each $m, m \geqslant -1$, there exist constants $C_m > 0$ and q_m, $0 < q_m < 1$, depending on m only such that the inequality*

$$|f^{(m,k)}(t)| < C_m \cdot n^{k+1/2} \cdot q_m^{n|t-t_n|} \tag{4}$$

holds for all $t \in I$, $n > 1$ and k, $0 \leqslant k \leqslant m+1$.

This Theorem in the particular case of $m = 0$ and $k = 0$ wase stablished earlier in [1] and for $m = -1$ it is trivial.

We refer to the paper [2] for the terminology and notation used in this note.

This seems to be the most convenient place to mention that the definition of splines can be extended in a nutural way to cover the step functions i.e. the splines of order -1. This makes possible to extend the results of [2] to $0 \leqslant k \leqslant m+1$ and $m \geqslant -1$ instead of $0 \leqslant k \leqslant m$ and $m \geqslant 0$.

2. The case of $k = 0$. To prove our Theorem for $k = 0$ it is sufficient to combine inequality (21) of [2] with the following

LEMMA 1. *Let $n > 1$ and $m \geqslant -1$. Then*

$$f_n^{(m)}(t) = \sum_{i=2\nu-2-m}^{2\nu} \underline{a}_{n,i}^{(m)} \underline{N}_{n,i}^{(m)}(t), \quad t \in I, \tag{5}$$

where the numbers $\underline{a}_{n,i}^{(m)}$ are uniquely determined and with some constant $A_m > 0$

$$|\underline{a}_{n,i}^{(m)}| < A_m \cdot n^{-1/2}, \quad 2\nu-2-m \leqslant i \leqslant 2\nu. \tag{6}$$

Proof. Let us start with the Dirichlet kernel of the orthonormal system $(f_j^{(m)}, j = -m, -m+1, \ldots)$

$$K_n^{(m)}(t,s) = \sum_{i=-m}^{n} f_j^{(m)}(t) f_j^{(m)}(s) = \sum_{i,j=-m}^{n} A_{n;i,j}^{(m)} N_{n,j}^{(m)}(t) N_{n,j}^{(m)}(s) \tag{7}$$

with $A_n^{(m)}$ being the inverse to the Gram matrix $G_n^{(m)}$ with the entries

$$G_{n;i,j}^{(m)} = (N_{n,j}^{(m)}, N_{n,i}^{(m)}), \quad (i,j) \in J_n^m \times J_n^m,$$

where $J_n^m = \{-m, -m+1, \ldots, n\}$. The functions $N_{n,i}^{(m)}, i \in J_n^m$, form a partition of unity over I corresponding to the partition $(s_{n,i}, i = 0, \pm 1, \ldots)$ of the real line.

These functions are the B-splines of order m (of degree $m+1$) and they form a basis in the space $C_n^m(I)$ of all splines of order m corresponding to the partition $(s_{n,i}, i = 0, \ldots, n)$.

Let us denote by D^T the transpose to the matrix D and let $N_n^{(m)}(t) = (N_{n,i}^{(m)}, i \in J_n^m)$ be the one column matrix.

In the matrix notation formula (7) becomes

$$K_n^{(m)}(t,s) = (N_n^{(m)}(t))^T \circ A_n^{(m)} \circ N_n^{(m)}(s), \tag{8}$$

where $\circ$ stands for the usual matrix multiplication.

Now, according to (8), we have for $n > -m$

$$\begin{aligned} f_n^{(m)}(t) \cdot f_n^{(m)}(s) &= K_n^{(m)}(t,s) - K_{n-1}^{(m)}(t,s) \\ &= (N_n^{(m)}(t))^T \circ A_n^{(m)} \circ N_n^{(m)}(s) - (N_{n-1}^{(m)}(t))^T \circ A_{n-1}^{(m)} \circ N_{n-1}^{(m)}(s). \end{aligned} \tag{9}$$

However, $N_{n-1;i}^{(m)} \in C_n^m(I)$ for $i \in J_{n-1}^m$ and therefore there exist numbers $B_{n;i,j}^{(m)}$ such that

$$N_{n-1,i}^{(m)}(t) = \sum_{j=-m}^{n} B_{n;i,j}^{(m)} N_{n,j}^{(m)}(t) \tag{10}$$

or in the matrix notation

$$N_{n-1}^{(m)} = B_n^{(m)} \circ N_n^{(m)}, \tag{11}$$

where the indices (i,j) of the entries of $B_n^{(m)}$ run over the set $J_{n-1}^m \times J_n^m$. Substituting (11) into (9) we obtain

$$\begin{aligned} f_n^{(m)}(t) f_n^{(m)}(s) = (N_n^{(m)}(t))^T \circ A_n^{(m)} \circ N_n^{(m)}(s) - \\ - (N_n(t))^T \circ (B_n^{(m)})^T \circ A_{n-1}^{(m)} \circ B_n^{(m)} \circ N_n^{(m)}(s). \end{aligned} \tag{12}$$

Since $A_n^{(m)}$ is inverse to $G_n^{(m)}$ it follows from (12) that

$$f_n^{(m)}(t)f_n^{(m)}(s) = \big(A_n^{(m)}\circ N_n^{(m)}(t)\big)^T\circ X_n^{(m)}\circ N_n^{(m)}(s), \tag{13}$$

where

$$X_n^{(m)} = E - G_n^{(m)}\circ (B_n^{(m)})^T\circ A_{n-1}^{(m)}\circ B_n^{(m)} \tag{14}$$

with E being the identity matrix.

There is a simple relation between $X_n^{(m)}$ and $B_n^{(m)}$. First notice that from (10) follows

$$G_{n-1}^{(m)} = B_n^{(m)}\circ G_n^{(m)}\circ (B_n^{(m)})^T,$$

which combined with (14) gives

$$B_n^{(m)}\circ X_n^{(m)} = 0. \tag{15}$$

It follows also from (10) that

$$B_{n;i,j}^{(m)} = \begin{cases} \delta_{i,j} & \text{for } (i,j)\in (J_{n-1}^m\times J_n^m)\setminus(\{2\nu-2-m,\dots,n-1\}\times \\ & \qquad\times\{2\nu-2-m,\dots,n\}), \\ \delta_{i+1,j} & \text{for } (i,j)\in (J_{n-1}^m\times J_n^m)\setminus(\{-m,\dots,2\nu-1\}\times \\ & \qquad\times\{-m,\dots,2\nu\}). \end{cases} \tag{16}$$

The combination of (15) and (16) gives

$$X_{n;i,j}^{(m)} = 0 \quad \text{for } (i,j)\in (J_n^m\setminus\{2\nu-2-m,\dots,2\nu\})\times J_n^m. \tag{17}$$

Introducing the dual basis

$$\underline{N}_{n,i}^{(m)}(t) = \sum_{j=-m}^{n} A_{n;i,j}^{(m)} N_{n,j}^{(m)}(t), \qquad i\in J_n^m,$$

which in the matrix notation is represented as one column matrix $\underline{N}_n^{(m)}(t)$ i.e.

$$\underline{N}_n^{(m)}(t) = A_n^{(m)}\circ N_n^{(m)}(t).$$

Applying this to (13) we obtain

$$f_n^{(m)}(t)\cdot f_n^{(m)}(s) = \big(\underline{N}_n^{(m)}(t)\big)^T\circ X_n^{(m)}\circ N_n^{(m)}(s). \tag{18}$$

Since $f_n^{(m)}\in C_n^{(m)}(I)$ we have

$$\begin{aligned} f_n^{(m)}(s) &= \sum_{j=-m}^{n} a_{n,j}^{(m)} N_{n,j}^{(m)}(s), \\ f_n^{(m)}(t) &= \sum_{i=-m}^{n} \underline{a}_{n,i}^{(m)} \underline{N}_{n,i}^{(m)}(t), \end{aligned} \tag{19}$$

with the constants $a_{n,j}^{(m)}$ and $\underline{a}_{n,i}^{(m)}$ uniquely determined.

Substitution of (19) into (18) leads to the equalities

$$X_{n;i,j}^{(m)} = \underline{a}_{n,i}^{(m)}\cdot a_{n,j}^{(m)}, \qquad (i,j)\in J_n^m\times J_n^m,$$

which together with (17) imply

$$\underline{a}_{n,i}^{(m)} = 0 \quad \text{for } i\in J_n^m\setminus\{2\nu-2-m,\dots,2\nu\},$$

and this completes the proof of (5).

To establish (6) we need the formula $a_{n,i}^{(m)} = (f_n^{(m)}, N_{n,i}^{(m)})$. Applying to its right-hand side the Schwartz inequality we obtain

$$|\underline{a}_{n,i}^{(m)}| \leqslant \|N_{n,i}^{(m)}\|_2 \leqslant |\operatorname{supp} N_{n,i}^{(m)}|^{1/2} \leqslant A_m\cdot n^{-1/2} \quad \text{for } i\in J_n^m$$

with some constant $A_m > 0$.

3. The estimates for arbitrary k, $0\leqslant k\leqslant m+1$. The proof of our Theorem in this case is based on

LEMMA 2. *Let $n = 2^\mu+\nu$ with $\mu\geqslant 0$ and $1\leqslant\nu\leqslant 2^\mu$, and let $m\geqslant -1$ be given. Moreover, let*

$$f_m^{(m,k)}(t) = \sum_{j=k-m}^{n} a_{n,j}^{(m,k)} N_{n,j}^{(m-k)}(t).$$

Then there exists constants $C_m^{(k)} > 0$ and r_m, $0 < r_m < 1$, such that

$$|a_{n,j}^{(m,k)}| < C_m^{(k)}\cdot n^{k+1/2}\cdot r_m^{|j-2\nu|} \tag{20}$$

holds for all $j\in J_n^{m-k}$ and k, $0\leqslant k\leqslant m+1$.

The proof goes by induction with respect to k. For $k=0$, $a_{n,j}^{(m,0)} = a_{n,j}^{(m)}$, hence by Lemma 1

$$\begin{aligned} a_{n,j}^{(m)} &= (f_n^{(m)}, \underline{N}_{n,j}^{(m)}) \\ &= \sum_{i=-m}^{n} A_{n;j,i}^{(m)}(f_n^{(m)}, N_{n,i}^{(m)}) \\ &= \sum_{i=-m}^{n} A_{n;j,i}^{(m)}\underline{a}_{n,i}^{(m)} \\ &= \sum_{i=2\nu-2-m}^{2\nu} A_{n;j,i}^{(m)}\underline{a}_{n,i}^{(m)}, \end{aligned}$$

and by the main result of [3]

$$|a_{n,j}^{(m)}| \leqslant A_m \cdot n^{-1/2} \cdot \sum_{i=2\nu-2-m}^{2\nu} |A_{n;i,j}^{(m)}| \leqslant B_m \cdot n^{1/2} \cdot \sum_{i=2\nu-2-m}^{2\nu} r_m^{|i-j|} \leqslant C_m^{(0)} \cdot n^{1/2} \cdot r^{|j-2\nu|},$$

with some constants $B_m > 0$ and $C_m^{(0)} > 0$.

Now, suppose that (20) holds for given k, $0 \leqslant k \leqslant m$. Then according to the differentiation formula (P.4) of [2] we obtain

$$a_{n,j}^{(m,k+1)} = \frac{a_{n,j}^{(m,k)} - a_{n,j-1}^{(m,k)}}{s_{n,j+m-k} - s_{n,j-1}} (m-k+1)$$

and therefore

$$|a_{n,j}^{(m,k+1)}| < (m-k+1) \cdot 2n \cdot C_n^{(k)} \cdot n^{k+1/2} \cdot (r_m^{|j-2\nu|} + r_m^{|j-1-2\nu|}), < C_m^{(k+1)} \cdot n^{(k+1)+1/2} \cdot r_m^{|j-2\nu|}.$$

with some constant $C^{(k+1)}$ and this proves (20) for all k, $0 \leqslant k \leqslant m+1$.

We are ready now to prove the Theorem in its complete generality.

Let us denote by $\langle t \rangle$ the integer determined by the inequalities $s_{n,\langle t\rangle -1} \leqslant t < s_{n,\langle t \rangle}$.

Then since supp $N_{n,j}^{(m-k)} = \langle s_{n,j-1}, s_{n,j+m-k+1} \rangle$, we have

$$f_n^{(m,k)}(t) = \sum_{j=\langle t\rangle-(m-k)-1}^{\langle t \rangle} a_{n,j}^{(m,k)} N_{n,j}^{(m-k)}(t)$$

and therefore

$$|f_n^{(m,k)}(t)| < C_m \cdot n^{k+1/2} \sum_{j=\langle t\rangle-(m-k)-1}^{\langle t \rangle} r_m^{|j-2\nu|} \leqslant D_m n^{k+1/2} r_m^{|\langle t\rangle - 2\nu|}$$

with some constants $C_m > 0$ and $D_m > 0$ whence (4) follows with $q_m = r_m^{1/2}$.

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICS INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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Series trigonometriques speciales et corps quadratiques

par

Yves MEYER (Orsay, France)

En hommage au Professeur Zygmund

Sommaire. Soit E l'espace de toutes les fonctions presque-périodiques dont les fréquences sont des nombres de Pisot irrationnels d'un corps quadratique réel. Deux remarquables propriétés fonctionnelles de E sont étudiées.

Introduction. On construit un espace vectoriel E de fonctions continues et bornées $f: \boldsymbol{R} \to \boldsymbol{C}$(1), invariant par translation, et possédant les deux propriétés suivantes:

(a) toute fonction continue $g: \boldsymbol{R} \to \boldsymbol{C}$ coïncide sur tout compact S de mesure de Lebesgue strictement inférieure à 1 avec une certaine fonction f de E

(b) pour tout compact S intégrable au sens de Riemann et dont la mesure de Lebesgue est strictement supérieure à 1, les normes $\sup_S |f|$ et $\sup_{\boldsymbol{R}} |f|$ sont équivalentes sur E.

En utilisant, au lieu de fréquences entières, l'ensemble Λ des nombres de Pisot d'un corps quadratique réel et en appelant E l'espace de Banach de toutes les fonctions presque périodiques dont les fréquences appartiennent à Λ, on obtient cette généralisation surprenante de l'espace des fonctions périodiques de période donnée.

Les § 1, 2 et 3 sont consacrés aux énoncés des définitions et des théorèmes. Les démonstrations occupent les § 4, 5 et 6. Enfin le § 7 contient des indications pour diverses généralisations à des espaces de fonctions à valeurs complexes définies sur $\boldsymbol{R}^n$ ou sur $\boldsymbol{Q}_p$.

1. Définitions et notations. Une somme trigonométrique est une fonction continue $f: \boldsymbol{R} \to \boldsymbol{C}$ qui s'écrit $f(t) = \sum_{\lambda \in F} a_\lambda \exp 2\pi i \lambda t$. Les λ tels que $a_\lambda \neq 0$ s'appellent les *fréquences de* f. L'ensemble F des fréquences de f est le spectre de f.

(1) $\boldsymbol{R}$ désigne le corps des nombres réels et $\boldsymbol{C}$ celui des nombres complexes.