Since $(g^*)^{1/p} \in L^1(G)$ and dominates $|F(x_0, x)|$ we can apply the dominated convergence theorem of Lebesgue to obtain the L^1-convergence of $F(x_0, \cdot)$ to the function f. From this, (3.4) and the fact that $d\mu(x) = f(x)\,dx$ follow immediately and Theorem 3.2 is established.

Many other results connected with H^p-space theory also admit similar extensions. For example, the theory involving the Lusin area function, as developed in chapter VII of Stein [8], can be carried out in this situation as well. In this connection see also the results of Fefferman and Stein [4], [7].

The Riesz transforms alluded to above are defined in Stein [9], where a proof is given of their boundedness as operators on $L^p(G)$, $1 < p < \infty$. These Riesz transforms add a novel feature to the harmonic analysis on semi-simple compact groups that does not appear in the commutative case. When $G = SU(2)$ there exist linear combinations of these transforms that are "shift" operators with respect to a "canonical" basis of $L^2(G)$ (see Coifman and Weiss [3]). The extension of the Riesz brothers' theorem obtained here implies a corresponding theorem for expansions of some Jacobi polynomials (which are connected with the elements of this basis). That this phenomenon is more general is evident from the fact that there exists a root-space decomposition of the associated Lie algebra.

References

[1] A. P. Calderón, *The behaviour of harmonic function at the boundary*, T. A. M. S. 68 (1950), pp. 47–54.

[2] R. R. Coifman and G. Weiss, *On subharmonicity inequalities involving solutions of generalized Cauchy–Riemann equations*, Studia Math. 36 (1970), p. 77–83.

[3] — *Analyse harmonique non-commutative sur certains espaces homogenes*, Lecture Notes in Mathematics, No. 242 Berlin 1971.

[4] C. Fefferman, *Characterization of bounded mean oscillation*, Bull. of the A. M. S. 77 (1971), pp. 587–588.

[5] G. Hochschild, *The structure of Lie Groups*, San Francisco 1965.

[6] T. Radó, *Subharmonic functions*, Chelsea, New York 1949.

[7] E. M. Stein, *L^p-boundedness of certain convolution operators*, Bull. Amer. Math. Soc. 77 (1971), pp. 404–405.

[8] — *Singular integrals and differentiability properties of functions*, New Jersey (1970).

[9] — *Topics in harmonic analysis*, Ann. of Math. Stud. 63 New Jersey 1970.

[10] — and G. Weiss, *On the theory of harmonic functions of several variables*, Acta Math. 103 (1960), pp. 26–62.

[11] — *Introduction to Fourier analysis on Euclidean spaces*, N. J. 1971.

[12] K.-O. Widman, *On the boundary behavior of solutions to a class of elliptic partial differential equations*, Ark. Math. 6. 26 (1966), pp. 485–533.

WASHINGTON UNIVERSITY, ST. LOUIS

Received September 2, 1971 (383)

STUDIA MATHEMATICA, T. XLIV. (1972)

On regular temperate distributions

by

Z. SZMYDT (Warszawa)

Abstract. There are given some conditions which imply that a locally Lebesgue integrable function u defines a temperate distribution (u, σ) by the relation $(u, \sigma) = \int_{\mathbf{R}^n} u(x)\sigma(x)\,dx$ with the integral converging absolutely for every function rapidly decaying at infinity. It is shown that the assertion included in [1] about the necessity of one of these conditions is not true.

1. Basic notations. The variable in the n-dimensional real Euclidean space $\mathbf{R}^n$ will be denoted by $x = (x_1, \ldots, x_n)$. By α we shall denote multi-indices, that is, n-tuples $(\alpha_1, \ldots, \alpha_n)$ of non-negative integers. We set $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ with $D_j = \dfrac{\partial}{\partial x_j}$. Similarly we write $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

A complex valued function φ defined in $\mathbf{R}^n$ is said to be a C^∞ function if it possesses continuous partial derivatives of all orders. By C_0^k we denote the set of all functions in C^∞ with compact support in $\mathbf{R}^n$.

By S or $S(\mathbf{R}^n)$ we denote the set of all functions $\sigma \in C^\infty$ such that

$$\sup_x |x^\beta D^\alpha \sigma(x)| < \infty \tag{1}$$

for all multi-indices α and β. The topology in S is defined by semi-norms in the left-hand side of (1).

A continuous linear functional (u, σ) on S is called a *temperate distribution*. The set of all temperate distributions is denoted by S'.

We denote by $L_1^{\mathrm{loc}}(\mathbf{R}^n)$ the space of locally Lebesgue integrable functions, i.e. Lebesgue integrable on any compact subset of $\mathbf{R}^n$. We identify every function $u \in L_1^{\mathrm{loc}}(\mathbf{R}^n)$ with the distribution u defined by:

$$(u, \varphi) = \int_{\mathbf{R}^n} u(x)\varphi(x)\,dx \quad \text{for} \quad \varphi \in C_0^\infty(\mathbf{R}^n). \tag{2}$$

A temperate distribution u is called *regular* if there exists a function $u \in L_1^{\mathrm{loc}}(\mathbf{R}^n)$ such that

$$(u, \sigma) = \int_{\mathbf{R}^n} u(x)\sigma(x)\,dx \quad \text{for} \quad \sigma \in S(\mathbf{R}^n) \tag{3}$$

and the integral

$$(4)\qquad \int_{\boldsymbol{R}^n} u(x)\sigma(x)\,dx$$

converges absolutely for every $\sigma \in S(\boldsymbol{R}^n)$.

2. Suppose that the locally integrable function $u(x)$ has a growth not higher than a power type growth at infinity, i.e. that the inequality

$$(5)\qquad |u(x)| \leqslant C(1+|x|)^k$$

is satisfied when $|x| > r$ for some $k \geqslant 0$ and $r > 0$.

If this condition is fulfilled the integral (4) converges absolutely for every $\sigma \in S$ and (3) defines a regular temperate distribution.

This statement can be found in the treatise [1], chapter II, 1.5. But the condition (5) is not necessary against the assertion in [1] and [2] (1) as it results from the following example:

EXAMPLE 1. Let Δ_m, $m = 1, 2, \ldots$ be a sequence of intervals in $\boldsymbol{R}^1$:

$$\Delta_m = (m - 2^{-(m+1)}\cdot e^{-m},\ m + 2^{-(m+1)}\cdot e^{-m}).$$

Let us define the function u on $\boldsymbol{R}^1$ as follows:

$$u(x) = \begin{cases} e^m & \text{if } x \in \Delta_m,\ m = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $u \in L_1^{\mathrm{loc}}(\boldsymbol{R}^1)$, and although condition (5) is not satisfied, the integral (4) converges absolutely for every $\sigma \in S(\boldsymbol{R}^1)$.

Note that in this example $u \in L_1(\boldsymbol{R}^1)$ what itself implies the regularity of the temperate distribution (3).

Some other sufficient conditions for the regularity of (3) will be given in the next section.

3. THEOREM 1. *If $u \in L_1^{\mathrm{loc}}(\boldsymbol{R}^n)$ and the function $u(x)/(1+x^2)^k$ belongs to $L_1(\boldsymbol{R}^n)$ for some $k \geqslant 0$, then the integral (4) converges absolutely for every $\sigma \in S(\boldsymbol{R}^n)$ and the functional u defined by (3) is a regular temperate distribution.*

Proof. Let us denote by v the function:

$$v(x) = u(x)/(1+x^2)^k \quad \text{for } x \in \boldsymbol{R}^n.$$

(1) Let us reproduce this statement from the translation [2] of [1]: For absolute convergence of the integral (4) in the space S for every fundamental function σ, it is sufficient (and necessary) that the locally integrable function u have a growth not higher than a power type at infinity.

Observe that $v \in L_1(\boldsymbol{R}^n)$ and that for an arbitrary function $\sigma \in S(\boldsymbol{R}^n)$ there is a constant $C < \infty$ such that $|(1+x^2)^k\sigma(x)| < C$ for $x \in \boldsymbol{R}^n$. So $u(x)\sigma(x) = v(x)(1+x^2)^k\sigma(x) \in L_1(\boldsymbol{R}^n)$, i.e. the integral (4) converges absolutely. Moreover, if $\lim\limits_{\nu\to\infty}\sigma_\nu(x) = 0$ in the space $S(\boldsymbol{R}^n)$, then

$$\lim_{\nu\to\infty}(1+x^2)^k\sigma_\nu(x) = 0 \quad \text{uniformly in } \boldsymbol{R}^n,$$

and consequently

$$\lim_{\nu\to\infty}(u, \sigma_\nu) = \lim_{\nu\to\infty}\int_{\boldsymbol{R}^n} v(x)(1+x^2)^k\sigma_\nu(x)\,dx = 0.$$

Remark 1. If u is a locally Lebesgue integrable function, condition (5) implies the assumptions of Theorem 1, but not conversely, as we have seen from the Example 1.

In the proof of the next theorem we shall use the following lemma.

LEMMA. *Let u be a locally integrable function, non-negative almost everywhere (a.e.) in $\boldsymbol{R}^n$ and such that the distribution u defined by (2) is temperate* (2).

Then the temperate distribution u is non-negative, i.e. $(u, \sigma) \geqslant 0$ for all non-negative functions $\sigma \in S(\boldsymbol{R}^n)$.

Proof. Let σ be a non-negative function belonging to $S(\boldsymbol{R}^n)$. Denote by ψ an arbitrary function satisfying the following conditions:

$$(6)\qquad \psi \in C_0^\infty(\boldsymbol{R}^n),\quad 0 \leqslant \psi(x) \leqslant 1,\quad \psi(x) = 1 \quad \text{for } |x| \leqslant 1.$$

It is easy to see that the functions $\varphi_\nu(x) = \sigma(x)\psi(x/\nu)$, $\nu = 1, 2, \ldots$, have the following properties:

$$(7)\qquad \begin{aligned} &0 \leqslant \varphi_\nu(x) \leqslant \sigma(x) \quad \text{for } x \in \boldsymbol{R}^n, \quad \varphi_\nu \in C_0^\infty(\boldsymbol{R}^n), \quad \nu = 1, 2, \ldots, \\ &\lim_{\nu\to\infty}\varphi_\nu = \sigma \quad \text{in } S(\boldsymbol{R}^n). \end{aligned}$$

Combining (7) with the assumptions of lemma we obtain:

$$(u, \sigma) = \lim_{\nu\to\infty}(u, \varphi_\nu) = \lim_{\nu\to\infty}\int_{\boldsymbol{R}^n} u(x)\varphi_\nu(x)\,dx \geqslant 0.$$

THEOREM 2. *Let u be a locally integrable function, non-negative a.e. in $\boldsymbol{R}^n$ and such that the distribution (2) is temperate.*

Then the relation (3) holds, and the integral (4) converges absolutely for every $\sigma \in S(\boldsymbol{R}^n)$.

(2) It means that there is a functional $\tilde{u} \in S'(\boldsymbol{R}^n)$ such that $(\tilde{u}, \varphi) = (u, \varphi)$ for every $\varphi \in C_0^\infty(\boldsymbol{R}^n)$. In the following we shall not apply the new symbol $\tilde{u}$ for the temperate distribution u given by (2).

Proof. Let $\sigma \in S(\boldsymbol{R}^n)$ be given arbitrarily. It can be proved that there exists a function $\tilde{\sigma} \in S(\boldsymbol{R}^n)$ such that

$$|\sigma(x)| \leqslant \tilde{\sigma}(x) \quad \text{for } x \in \boldsymbol{R}^n. \tag{8}$$

Denote by ψ a function satisfying conditions (6) and observe that the functions $\tilde{\varphi}_\nu(x) = \tilde{\sigma}(x)\psi(x/\nu)$, $\nu = 1, 2, \ldots$, belong to $C_0^\infty(\boldsymbol{R}^n)$ and have the following properties:

$$0 \leqslant \tilde{\varphi}_\nu(x) \leqslant \tilde{\sigma}(x) \quad \text{for } x \in \boldsymbol{R}^n, \quad \nu = 1, 2, \ldots, \tag{9}$$

$$\tilde{\varphi}_\nu(x) = \tilde{\sigma}(x) \quad \text{for } |x| \leqslant \nu. \tag{10}$$

In virtue of (9) the lemma stated before implies that $(u, \tilde{\sigma} - \tilde{\varphi}_\nu) \geqslant 0$; as $u \in L_1^{\text{loc}}(\boldsymbol{R}^n)$ we derive

$$\int_{\boldsymbol{R}^n} u(x)\tilde{\varphi}_\nu(x)\,dx = (u, \tilde{\varphi}_\nu) \leqslant (u, \tilde{\sigma}) < \infty, \quad \nu = 1, 2, \ldots$$

Since $u \geqslant 0$ a.e. by (10) and (9) we obtain:

$$\int_{|x|\leqslant\nu} u(x)\tilde{\sigma}(x)\,dx = \int_{|x|\leqslant\nu} u(x)\tilde{\varphi}_\nu(x)\,dx \leqslant \int_{\boldsymbol{R}^n} u(x)\tilde{\varphi}_\nu(x)\,dx, \quad \nu = 1, 2, \ldots,$$

so in virtue of (8) the integral (4) converges absolutely:

$$\int_{|x|\leqslant\nu} |u(x)\sigma(x)|\,dx \leqslant \int_{|x|\leqslant\nu} u(x)\tilde{\sigma}(x)\,dx \leqslant (u, \tilde{\sigma}) < \infty, \quad \nu = 1, 2, \ldots \tag{11}$$

We must now prove that the relation (3) holds. In order to do this observe that the functions $\varphi_\nu(x) = \sigma(x)\psi(x/\nu)$, $\nu = 1, 2, \ldots$, belong to $C_0^\infty(\boldsymbol{R}^n)$ and have the following properties:

$$|\varphi_\nu(x)| \leqslant |\sigma(x)|, \quad \nu = 1, 2, \ldots, \tag{12}$$

$$\lim_{\nu\to\infty} \varphi_\nu = \sigma \quad \text{in } S(\boldsymbol{R}^n). \tag{13}$$

In view of assumptions: $u \in S'(\boldsymbol{R}^n)$, $u \in L_1^{\text{loc}}(\boldsymbol{R}^n)$ we obtain from (13):

$$(u, \sigma) = \lim_{\nu\to\infty}(u, \varphi_\nu) = \lim_{\nu\to\infty} \int_{\boldsymbol{R}^n} u(x)\varphi_\nu(x)\,dx. \tag{14}$$

Taking into account (11)–(13) and Lebesgue's convergence theorem we obtain (3) from (14).

By the fundamental structure theorem for distributions belonging to $S'(\boldsymbol{R}^n)$[3] we can give to Theorem 2 the following form:

[3] Cf. [3], p. 239.

THEOREM 3. *Let u be a locally integrable function non-negative a.e. in $\boldsymbol{R}^n$ and such that*[4] $u = D^a h$ *for some multi-index a and some continuous function h having a growth not higher than a power type at infinity*[5].

Then u is a regular temperate distribution, i.e. for every function $\sigma \in S(\boldsymbol{R}^n)$ the integral (4) converges absolutely and the relation (3) defines a continuous linear functional on $S(\boldsymbol{R}^n)$.

Remark 2. Note that Theorem 3 (and also Theorem 2) would not be valid without the hypothesis on the sign of the function u.

EXAMPLE 2. Let u be the function defined by

$$u(x) = e^x \cos e^x = \frac{d}{dx}(\sin e^x) \quad \text{for } x \in \boldsymbol{R}^1.$$

It is easy to see that for every function $\sigma \in S(\boldsymbol{R}^1)$

$$\int_{-\infty}^{+\infty} e^x \cos e^x \sigma(x)\,dx = -\int_{-\infty}^{+\infty} \sin e^x \sigma'(x)\,dx. \tag{15}$$

If $\lim_{\nu\to\infty} \sigma_\nu = 0$ in $S(\boldsymbol{R}^1)$, then $(1+x^2)\sigma_\nu'(x)$ converges uniformly to zero and therefore by (15)

$$\lim_{\nu\to\infty} \int_{-\infty}^{+\infty} u(x)\sigma_\nu(x)\,dx = 0.$$

So we have shown that the functional u defined by (3) belongs to $S'(\boldsymbol{R}^1)$. But it is not a regular one. In fact if a non-negative function $\sigma \in S(\boldsymbol{R}^1)$ satisfies the conditions:

$$\sigma(x) = \begin{cases} e^{-x} & \text{for } x > \ln\dfrac{3\pi}{4} \\ 0 & \text{for } x < \ln\dfrac{\pi}{4}, \end{cases}$$

then

$$\int_{-\infty}^{+\infty} |e^x \cos e^x| \sigma(x)\,dx \geqslant \int_{\ln\frac{3\pi}{4}}^{\infty} |\cos e^x|\,dx$$

$$\geqslant \tfrac{1}{2}\sqrt{2} \sum_{k=1}^{\infty} \left(\ln\left(k\pi + \frac{\pi}{4}\right) - \ln\left(k\pi - \frac{\pi}{4}\right)\right) = \infty.$$

[4] The derivative D^a is obviously a derivative in the distribution sense.

[5] It means that $h(x) = (1+|x|)^k \tilde{h}(x)$ for some integer k and some continuous bounded function $\tilde{h}$.

References

[1] И. М. Гельфанд и Г. Е. Шилов, *Пространства основных и обобщенных функций*, Москва 1958.
[2] I. M. Gel'fand and G. E. Shilov, *Generalized functions*, vol. 2, *Spaces of fundamental and generalized functions*, 1968.
[3] L. Schwartz, *Théorie des distributions*, Paris 1966.

Received September 9, 1971 (384)

STUDIA MATHEMATICA T. XLIV. (1972)

Estimates for the spline orthonormal functions and for their derivatives

by

Z. CIESIELSKI (Sopot) and J. DOMSTA (Sopot)

Abstract. The aim of this note is to obtain local exponential estimates for the spline functions $f_j^{(m,k)}$ with $j \geqslant k-m$, $m \geqslant -1$ and $0 \leqslant k \leqslant m+1$ which were constructed in the previous paper [2] of the authors. These estimates are important in the investigations of order of approximation by partial sums of the expansions with respect to the set $(f_j^{(m,k)}, j \geqslant k-m)$, where $0 \leqslant k \leqslant m+1$.

1. Introduction. We are going to describe briefly the functions $f_j^{(m,k)}$ by means of the Haar functions. Let $I = \langle 0, 1\rangle$ and let $(t_n, n \geqslant 0)$ be the dyadic sequence in I associated to the Haar system, i.e. $t_0 = 0$, $t_1 = 1$ and for $n = 2^\mu + \nu$ with $\mu \geqslant 0, 1 \leqslant \nu \leqslant 2^\mu$

$$t_n = \frac{2\nu - 1}{2^{\mu+1}}. \tag{1}$$

For each $n > 1$ the subsequence $(t_i, i = 0, \ldots, n)$ can be ordered into an increasing sequence $0 = s_{n,0} < \ldots < s_{n,n} = 1$ and then it can be extended as follows

$$s_{n,i} = \begin{cases} \dfrac{i}{2^{\mu+1}} & \text{for } i = \ldots, 2\nu-1, 2\nu, \\[2mm] \dfrac{i-\nu}{2^\mu} & \text{for } i = 2\nu+1, 2\nu+2, \ldots, \end{cases} \tag{2}$$

where i is ranging over all the integers.

The n-th Haar function is defined as follows: $\chi_1 = 1$, and for $n = 2^\mu + \nu$ with $\mu \geqslant 0, 1 \leqslant \nu \leqslant 2^\mu$

$$\chi_n(t) = \begin{cases} 2^{\mu/2} & \text{for } t \in \langle s_{n,2\nu-2}, s_{n,2\nu-1}), \\ -2^{\mu/2} & \text{for } t \in \langle s_{n,2\nu-1}, s_{n,2\nu}), \\ 0 & \text{elsewhere in } \langle 0, 1), \end{cases}$$

and in addition $\chi_n(1) = \chi_n(1_-)$.