STUDIA MATHEMATICA, T. XLIV (1972)

Oscillatory integrals and a multiplier problem for the disc *

by

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Dedicated to A. Zygmund

Abstract. Let $0 < a < \frac{1}{2}$ and let $m(x)$ be defined for $x \in \boldsymbol{R}^2$ by $m(x) = (1-|x|^2)^a$, $|x| \leqslant 1$, and $m(x) = 0$, $|x| > 1$. The problem of determining all values of p for which m is a multiplier for $L^p(\boldsymbol{R}^2)$ is treated by means of a study of certain general oscillatory integrals.

Introduction. Let a be a positive number and let $m(x)$ be defined for $x \in \boldsymbol{R}^2$ by $m(x) = (1-|x|^2)^a$, $|x| \leqslant 1$, and $m(x) = 0$, $|x| > 1$. Define the operator T on $L^2(\boldsymbol{R}^2)$ by setting $(Tf)\hat{}\,(x) = m(x)\hat{f}(x)$, $x \in \boldsymbol{R}^2$, where $\hat{f}$ denotes the Fourier transform of f. We say that m is a multiplier for $L^p(\boldsymbol{R}^2)$ if $\|Tf\|_p \leqslant C\|f\|_p$, $f \in L^p \cap L^2(\boldsymbol{R}^2)$. We shall study the problem of determining all values of p for which T is a bounded operator on $L^p(\boldsymbol{R}^2)$. T can clearly be written as a convolution operator, $Tf = \hat{m} * f$, and it is wellknown that $\hat{m}(x) = CJ_{1+a}(|x|)|x|^{-1-a}$, where C is a constant and J_{1+a} denotes the usual Bessel function of order $1+a$. From the asymptotic expansion for Bessel functions it follows that

$$\hat{m}(x) = C\cos(|x|+a)|x|^{-3/2-a} + O\{|x|^{-5/2-a}\}, \qquad |x| \to \infty,$$

for some constants C and a. If $a > 1/2$ then $\hat{m} \in L^1(\boldsymbol{R}^2)$ and hence T is a bounded operator on $L^p(\boldsymbol{R}^2)$ for $1 \leqslant p \leqslant \infty$. For $a = 1/2$ E.M. Stein [3] has proved that T is bounded on $L^p(\boldsymbol{R}^2)$ if $1 < p < \infty$. The corresponding result in odd dimensions was obtained by Calderón and Zygmund [1]. To study the case $0 < a < 1/2$ we introduce the kernels K_λ and operators T_λ defined by $K_\lambda(x) = e^{i|x|}|x|^{-\lambda}$, $x \in \boldsymbol{R}^2$, and $T_\lambda f = K_\lambda * f$, $3/2 < \lambda < 2$. These operators have been studied by C. Fefferman [2]. Fefferman observed that if χ is the characteristic function of the set $\{x \in \boldsymbol{R}^2;\ |x| \leqslant 1/10\}$, then $T_\lambda \chi \in L^p(\boldsymbol{R}^2)$ only if $p > 2/\lambda$. Hence a necessary condition for T_λ to be bounded on $L^p(\boldsymbol{R}^2)$ is $2/\lambda < p < 2/(2-\lambda)$. Fefferman proved that if

* This work has been supported by the Swedish Natural Science Research Council.

$\lambda > 5/3$, then this condition is also sufficient. We shall here prove that the above condition is sufficient for all $\lambda > 3/2$. The same condition with λ replaced by $3/2+\alpha$ will then be necessary and sufficient for m to be a multiplier for $L^p(\mathbf{R}^2)$ in the case $0<\alpha<1/2$. During the preparation of this paper we have learnt that Fefferman has also obtained this result.

The estimates for the operators T_λ will be a consequence of our study of oscillatory integrals of the form $S_N^\varphi f(x,y) = N^{1/2}\int_0^1 e^{iN\varphi(t,x,y)}f(t)\,dt$, $(x,y)\in D$, $f\in L^1(I)$, $N\geqslant 1$, where I is the interval $[0,1]$ and D is a square in $\mathbf{R}^2$. For example let $D = I\times[2,3]$. Assuming that φ is real-valued and belongs to $C^\infty(\Omega)$, where $\Omega\subset\mathbf{R}^3$ is an open set containing $I\times D$, and that the determinant

$$J = \begin{vmatrix} \dfrac{\partial^2\varphi}{\partial t\,\partial x} & \dfrac{\partial^2\varphi}{\partial t\,\partial y} \\ \dfrac{\partial^3\varphi}{\partial t^2\,\partial x} & \dfrac{\partial^3\varphi}{\partial t^2\,\partial y}\end{vmatrix}$$

is different from zero on $I\times D$, we shall prove that

$$\|S_N^\varphi f\|_{L^2(D)}\leqslant C\|f\|_{L^2(I)}$$

and

(1) $$\|S_N^\varphi f\|_{L^p(D)}\leqslant C_\varepsilon N^\varepsilon\|f\|_{L^p(I)},\quad 2<p\leqslant 4,\quad \varepsilon>0,$$

where C depends only on φ and C_ε on φ and ε. The proof of (1) in the case $p=4$ is the main difficulty in our study of the operators S_N^φ and we shall now describe how this estimate can be obtained. Since J is continuous there exists a square D', which has the same center as D and is larger than D, such that J does not vanish on $I\times D'$. Let $w\in C^\infty(\mathbf{R}^2)$ be non-negative and have its support in D' and assume that $w(x,y)=1$ for $(x,y)\in D$. We have

$$\|S_N^\varphi f\|_{L^4(D)}^4 \leqslant \iint_{R^2} |S_N^\varphi f(x,y)|^4 w(x,y)\,dx\,dy$$
$$= N^2\iiiint_{I^4} f(\alpha)f(\beta)\overline{f(\alpha_1)}\,\overline{f(\beta_1)}\,\times$$
$$\times\Big\{\iint_{R^2} e^{iN(\varphi(\alpha,x,y)+\varphi(\beta,x,y)-\varphi(\alpha_1,x,y)-\varphi(\beta_1,x,y))}w(x,y)\,dx\,dy\Big\}\,d\alpha\,d\beta\,d\alpha_1\,d\beta_1.$$

We set $F(\alpha,\beta,\alpha_1,\beta_1;x,y) = \varphi(\alpha,x,y)+\varphi(\beta,x,y)-\varphi(\alpha_1,x,y)-\varphi(\beta_1,x,y)$ and let grad F denote the gradient of F with respect to (x,y). We shall prove that for most choices of $(\alpha,\beta,\alpha_1,\beta_1)$ we have $N^{-1+\varepsilon}\leqslant C|\mathrm{grad}\,F|$ on D' and that the higher derivatives of F can be majorized by a constant multiple of $|\mathrm{grad}\,F|$. It follows that we can get good estimates of the inner integral above by performing repeated partial integrations with respect to one of the variables x and y. These estimates can be used to prove (1) for $p=4$. The proof is contained in Section 1.

The results for the operators T_λ follow from the estimates of the oscillatory integrals in the following way. The inequality (1) with $\varphi(t,x,y) = \{(x-t)^2+y^2\}^{1/2}$ and $p=4$ will be used to estimate the operators S_N^λ defined by

$$S_N^\lambda f(x) = \int_{I^2} N^2K_\lambda(N(x-\xi))f(\xi)\,d\xi,\quad x\in D,\quad f\in L^1(I^2).$$

Using interpolation we shall prove that if $4\leqslant p<2/(2-\lambda)$ then there exists a number $\delta>0$ such that $\|S_N^\lambda f\|_{L^p(D)}\leqslant CN^{-\delta}\|f\|_{L^p(I^2)}$. From this it will follow that convolution with the dilatation $N^2K_\lambda(Nx)$ of $K_\lambda(x)$ is a bounded operator on $L^p(I^2)$, $2/\lambda<p<2/(2-\lambda)$, with a norm which is uniformly bounded in N. By a change of scale it then follows that T_λ is a bounded operator on $L^p(\mathbf{R}^2)$ if $2/\lambda<p<2/(2-\lambda)$. The details are carried out in Section 2. We stress that our method is limited to the case of two dimensions.

In the case $\varphi(t,x,y) = \{(x-t)^2+y^2\}^{1/2}$ the method for estimating S_N^φ which we described above actually gives a result which is stronger than (1), namely $\|S_N^\varphi f\|_{L^4(D)}\leqslant C(\log N)^A\|f\|_{L^4(I)}$, $N\geqslant 2$, for some constant A. This estimate can be used to prove that if m is a function on $\mathbf{R}^2$ which is infinitely differentiable for $|x|<3/4$, vanishes for $|x|\geqslant 1$ and satisfies $m(x) = \Big\{\log\dfrac{1}{1-|x|}\Big\}^{-k}$ for $1/2\leqslant|x|<1$, then m is a multiplier for $L^p(\mathbf{R}^2)$, $4/3\leqslant p\leqslant 4$, provided k is large enough.

A particularly simple case of integrals of the form $N^{1/2}\int_0^1 e^{iN\varphi(t,x,y)}f(t)\,dt$ arises when $\varphi(t,x,y) = xt+y\psi(t)$ for some function ψ on I. For $N=1$ this integral can then be regarded as a two-dimensional Fourier transform of a measure on the curve $\{(t,u)\in\mathbf{R}^2;\ t\in I,\ u=\psi(t)\}$. For instance let $\psi(t)=t^2$ and set

$$P_Nf(x,y) = N^{1/2}\int_0^1 e^{iN(xt+yt^2)}f(t)\,dt,\quad f\in L^1(I),\ N\geqslant 1.$$

Then the inequality

(2) $$\|P_Nf\|_{L^4(I^2)}\leqslant CN^\varepsilon\|f\|_{L^4(I)},\quad \varepsilon>0,$$

corresponding to the estimate (1) for $p=4$, can easily be obtained in the following way. Performing a change of variable $u=t+s$, $v=t^2+s^2$ we get

$$\{P_1f(x,y)\}^2 = \int_0^1\int_0^1 e^{i\{x(t+s)+y(t^2+s^2)\}}f(t)f(s)\,dt\,ds =$$
$$= \iint_U e^{i(xu+yv)}f(t)f(s)\,|t-s|^{-1}\,du\,dv,$$

where in the last integral t and s are functions of u and v and U is the region $\{(u,v)\in \boldsymbol{R}^2;\ 0<u<2,\ u^2/2<v<u^2\}$. The Hausdorff–Young theorem and another change of variable yield

$$\Big\{\iint_{\boldsymbol{R}^2} |P_1 f(x,y)|^{4+2\varepsilon}\,dx\,dy\Big\}^{1/(2+\varepsilon)}$$
$$\leqslant C\Big\{\iint_U |f(t)|^{2-\delta}|f(s)|^{2-\delta}|t-s|^{-2+\delta}\,du\,dv\Big\}^{1/(2-\delta)}$$
$$= C\Big\{\int_0^1\int_0^1 |f(t)|^{2-\delta}|f(s)|^{2-\delta}|t-s|^{-1+\delta}\,dt\,ds\Big\}^{1/(2-\delta)} \quad \text{where } 0<\delta<1.$$

Defining H by $H(t)=\int_0^1 |f(s)|^{2-\varrho}|t-s|^{-1+\delta}\,ds$, $t\in I$, we know from the theory of fractional integration (see [4], ch. XII) that $\|H\|_{L^2(I)}\leqslant C\,\||f|^{2-\delta}\|_{L^2(I)}$. Schwarz's inequality combined with this estimate yields

$$\|P_1 f\|^2_{L^{4+2\varepsilon}(\boldsymbol{R}^2)} \leqslant C\{\||f|^{2-\delta}\|_{L^2(I)}\|H\|_{L^2(I)}\}^{1/(2-\delta)} < C\|f\|^2_{L^{4-2\delta}(I)}$$

and hence

$$\|P_1 f\|_{L^{4+\varepsilon}(\boldsymbol{R}^2)} < C\|f\|_{L^4(I)},\qquad \varepsilon>0. \tag{3}$$

(2) now follows from this fact and the relation $P_N f(x,y)=N^{1/2}P_1 f(Nx,Ny)$ by a change of scale.

Inequalities of the type (3) were used by Fefferman in [2] to estimate the operators T_λ and their analogues in higher dimensions. He then considered the operator R defined by $Rf=(f\,d\theta)\hat{}$, where $f\in L^1(S^{n-1})$, $n\geqslant 2$, and θ denotes the surface measure on S^{n-1}. Among other things Fefferman proved, using a method similar to the one which gave (3), that $\|Rf\|_{L^{4+\varepsilon}(\boldsymbol{R}^2)}\leqslant C\|f\|_{L^4(S^1)}$, $\varepsilon>0$. We remark that the technique which gave (3) can also be used to study the operator R and that for $n\geqslant 3$ it yields the result $\|Rf\|_{L^4(\boldsymbol{R}^n)}\leqslant C\|f\|_{L^2(S^{n-1})}$, which is best possible for $n=3$ in the sense that $L^4(\boldsymbol{R}^3)$ cannot be replaced by $L^p(\boldsymbol{R}^3)$ for any $p<4$.

1. A theorem on oscillatory integrals. In this section we shall study the operators S_N^φ defined by

$$S_N^\varphi f(x,y)=N^{1/2}\int_0^1 e^{iN\varphi(t,x,y)}f(t)\,dt,\qquad (x,y)\in D,\ f\in L^1(I),\ N\geqslant 1.$$

The following theorem will be proved.

THEOREM I. *Assume that $\varphi\in C^\infty(\Omega)$, where $\Omega\subset\boldsymbol{R}^3$ is an open set containing $I\times D$, and that the determinant*

$$J=\begin{vmatrix}\dfrac{\partial^2\varphi}{\partial t\,\partial x} & \dfrac{\partial^2\varphi}{\partial t\,\partial y}\\[2mm] \dfrac{\partial^3\varphi}{\partial t^2\,\partial x} & \dfrac{\partial^3\varphi}{\partial t^2\,\partial y}\end{vmatrix}$$

does not vanish on $I\times D$. Then

$$\|S_N^\varphi f\|_{L^2(D)}\leqslant C\|f\|_{L^2(I)}$$

and

$$\|S_N^\varphi f\|_{L^p(D)}\leqslant CN^\varepsilon\|f\|_{L^p(I)},\qquad 2<p\leqslant 4,\ \varepsilon>0,$$

where C depends only on φ and ε.

We shall first prove a lemma which will be needed in the proof of Theorem I. Choose D' and w as in the introduction and set $\xi=(\alpha,\beta,\alpha_1,\beta_1)$ and

$$F(x,y)=F(\xi;x,y)=\varphi(\alpha,x,y)+\varphi(\beta,x,y)-\varphi(\alpha_1,x,y)-\varphi(\beta_1,x,y),$$
$$\xi\in I^4,\qquad (x,y)\in D'.$$

Also let $I_N(\xi)=\iint_{\boldsymbol{R}^2} e^{iNF(\xi;x,y)}w(x,y)\,dx\,dy$, $\xi\in I^4$, $N\geqslant 1$. Assuming that φ satisfies the same conditions as in Theorem I, we shall prove the following lemma.

LEMMA 1. *Suppose $\varepsilon>0$. There exists a number $d>0$ such that if $E\subset I$ is a measurable set and $\operatorname{diam} E<d$ then*

$$\int_{E^4}|I_N(\xi)|\,d\xi\leqslant CN^{-2+\varepsilon}mE,$$

where C depends only on φ, ε and w.

Proof. Let l be a large positive number. We shall prove that outside a set of small measure $I_N(\xi)$ is majorized by CN^{-l}. Let $A_1=\{\xi\in I^4;\ |\alpha_1-\beta_1|\leqslant|\alpha-\beta|$ and $\max(|\alpha_1-\alpha|,\ |\beta_1-\beta|)\leqslant\max(|\alpha_1-\beta|,\ |\beta_1-\alpha|)\}$. Because of the symmetric way in which α, β, α_1 and β_1 enter in F we have

$$\int_{E^4}|I_N(\xi)|\,d\xi\leqslant C\int_{E^4\cap A_1}|I_N(\xi)|\,d\xi.$$

Now set $m=\max(|\alpha_1-\alpha|,\ |\beta_1-\beta|)$, let M be a large number and define A_2 as $\{\xi\in I^4,\ m\leqslant M|\alpha-\beta|\}$. We also set $E_1=E^4\cap A_1\cap A_2$ and $E_2=(E^4\cap A_1)\setminus E_1$. We shall first estimate the integral of $|I_N(\xi)|$ over E_1.

From Fubini's theorem it follows that

$$\int_{E_1}|I_N(\xi)|\,d\xi\leqslant\iint_{E^2}g(\alpha,\beta)\,d\alpha\,d\beta,$$

where

$$g(\alpha,\beta)=\iint_{A(\alpha,\beta)}|I_N(\xi)|\,d\alpha_1\,d\beta_1,$$

$$A=A_1\cap A_2\quad\text{and}\quad A(\alpha,\beta)=\{(\alpha_1,\beta_1);\ \xi\in A\}.$$

We claim that for $(\alpha,\beta)\in E^2$

(1.1) $$g(\alpha,\beta)\leqslant CN^{-2+2\varepsilon}|\alpha-\beta|^{-1},\qquad |\alpha-\beta|>N^{-1/2+\varepsilon}$$

and

(1.2) $$g(\alpha,\beta)\leqslant CN^{-1+\varepsilon}|\alpha-\beta|+CN^{-l},\qquad |\alpha-\beta|\leqslant N^{-1/2+\varepsilon}.$$

Set

$$P(\alpha,\beta)=\{(\alpha_1,\beta_1)\in A(\alpha,\beta);\ m<N^{-1+\varepsilon}|\alpha-\beta|^{-1},\ |\alpha+\beta-\alpha_1-\beta_1|<N^{-1+\varepsilon}\}.$$

It is easy to see that the measure of $P(\alpha,\beta)$ satisfies the same estimates as $g(\alpha,\beta)$ in (1.1) and (1.2). Hence (1.1) and (1.2) will be proved if we can show that

(1.3) $$|I_N(\xi)|\leqslant CN^{-l}\ \text{ if }\ (\alpha_1,\beta_1)\in A(\alpha,\beta)\diagdown P(\alpha,\beta),\ (\alpha,\beta)\in E^2.$$

To prove (1.3) we study the function F in the formula for $I_N(\xi)$. We set $\varrho=(\alpha+\beta)/2$ and expand $\varphi(\alpha,x,y)$ in a Taylor series $\varphi(\alpha,x,y)=\varphi(\varrho,x,y)+\dfrac{\partial\varphi}{\partial t}(\varrho,x,y)(\alpha-\varrho)+\dfrac{1}{2}\dfrac{\partial^2\varphi}{\partial t^2}(\varrho,x,y)(\alpha-\varrho)^2+B(\alpha,x,y)$, where $B(\alpha,x,y)=\dfrac{1}{2}\int_\varrho^\alpha(\alpha-s)^2\dfrac{\partial^3\varphi}{\partial t^3}(s,x,y)\,ds$ and $\dfrac{\partial}{\partial t}$ denotes differentiation with respect to the first variable. Using the same expansion for the other terms in F and setting $\alpha'=\alpha-\varrho$, $\beta'=\beta-\varrho$, $\alpha_1'=\alpha_1-\varrho$ and $\beta_1'=\beta_1-\varrho$, we obtain $F(x,y)=H(x,y)+R(x,y)$, where

$$H(x,y)=\frac{\partial\varphi}{\partial t}(\varrho,x,y)(\alpha'+\beta'-\alpha_1'-\beta_1')+\frac{1}{2}\frac{\partial^2\varphi}{\partial t^2}(\varrho,x,y)(\alpha'^2+\beta'^2-\alpha_1'^2-\beta_1'^2)$$

and

$$R(x,y)=B(\alpha,x,y)+B(\beta,x,y)-B(\alpha_1,x,y)-B(\beta_1,x,y).$$

We set $\delta_1=\alpha'+\beta'-\alpha_1'-\beta_1'$, $\delta_2=\alpha'^2+\beta'^2-\alpha_1'^2-\beta_1'^2$ and $\delta=(\delta_1,\delta_2)$. Using the fact that α_1' and β_1' are solutions of the equation

$$\alpha_1'^2-(\alpha'+\beta')\alpha_1'+\alpha'\beta'+O(|\delta|)=0$$

we can easily prove that

(1.4) $$m|\alpha-\beta|\leqslant C|\delta|.$$

It follows that

(1.5) $$\left|\frac{\partial^{i+j}F}{\partial x^i\partial y^j}\right|\leqslant C|\delta|,\qquad 2\leqslant i+j\leqslant l/\varepsilon+2,$$

and $|\operatorname{grad}R|\leqslant Cm|\alpha-\beta|^2$ on the square D'. Since

$$\frac{\partial H}{\partial x}(x,y)=\frac{\partial^2\varphi}{\partial t\partial x}(\varrho,x,y)\delta_1+\frac{1}{2}\frac{\partial^3\varphi}{\partial t^2\partial x}(\varrho,x,y)\delta_2$$

and

$$\frac{\partial H}{\partial y}(x,y)=\frac{\partial^2\varphi}{\partial t\partial y}(\varrho,x,y)\delta_1+\frac{1}{2}\frac{\partial^3\varphi}{\partial t^2\partial y}(\varrho,x,y)\delta_2$$

and since the absolute value of the determinant

$$J=\begin{vmatrix}\dfrac{\partial^2\varphi}{\partial t\partial x} & \dfrac{\partial^2\varphi}{\partial t\partial y}\\[2ex] \dfrac{\partial^3\varphi}{\partial t^2\partial x} & \dfrac{\partial^3\varphi}{\partial t^2\partial y}\end{vmatrix}$$

is bounded below, we obtain $|\delta|\leqslant C|\operatorname{grad}H|$ and using (1.4) $m|\alpha-\beta|\leqslant C|\operatorname{grad}H|$.

From the estimates of $\operatorname{grad}R$ and $\operatorname{grad}H$ it follows that if $\operatorname{diam}E$ and hence $|\alpha-\beta|$ is small enough, then

(1.6) $$|\delta|\leqslant C|\operatorname{grad}F|$$

on D'. From (1.5) and (1.6) we conclude that there exists a positive number γ, depending only on φ, such that if ω is a square contained in D' with side length less than γ, then either $|\delta|\leqslant C\left|\dfrac{\partial F}{\partial x}\right|$ on ω or $|\delta|\leqslant C\left|\dfrac{\partial F}{\partial y}\right|$ on ω. We now choose φ_j, $j=1,2,\dots,K$, such that $\varphi_i\in C^\infty(\boldsymbol{R}^2)$, $\sum_1^K\varphi_j=1$ on D' and the support of each φ is contained in a square with side length less than γ. We have

$$I_N(\xi)=\sum_1^K\iint e^{iNF(x,y)}w(x,y)\varphi_j(x,y)\,dx\,dy$$

and to estimate the jth term in this sum we may assume that $|\delta|\leqslant C\left|\dfrac{\partial F}{\partial x}\right|$ on the support of $w\varphi_j$. Performing k partial integrations with respect to x we obtain

$$\iint e^{iNF}w\varphi_j\,dx\,dy=\iint e^{iNF}\left\{iN\frac{\partial F}{\partial x}\right\}^{-k}g_k\,dx\,dy,$$

where g_k is a linear combination of functions of the form

$$\frac{\partial^s(w\varphi_j)}{\partial x^s}\prod_{p=1}^r\frac{\partial^{i_p}F}{\partial x^{i_p}}\left\{\frac{\partial F}{\partial x}\right\}^{-r},\qquad 2\leqslant i_p\leqslant k+1,\ 0\leqslant s,r\leqslant k.$$

Choosing $k=[l/\varepsilon]+1$ and using (1.5) we obtain $|I_N(\xi)|\leqslant C(N|\delta|)^{-k}$. From (1.4) and the fact that $(\alpha_1,\beta_1)\in A(\alpha,\beta)\diagdown P(\alpha,\beta)$ it follows that $N^{-1+\varepsilon}\leqslant C|\delta|$ and hence $|I_N(\xi)|\leqslant C(N^\varepsilon)^{-k}\leqslant CN^{-l}$. This proves (1.3) and the proof of (1.1) and (1.2) is complete. We have

$$\int_{E_1} |I_N(\xi)|\,d\xi \leqslant \int_E \Big\{ \int_E g(\alpha,\beta)\,d\alpha \Big\} d\beta$$

and from (1.1) and (1.2) it follows that the inner integral is less than

$$CN^{-1+\varepsilon} \int_0^{N^{-1/2+\varepsilon}} \gamma\,d\gamma + CN^{-2+2\varepsilon} \int_{N^{-1/2+\varepsilon}}^{1} \gamma^{-1}\,d\gamma + CN^{-l},$$

which can be majorized by $CN^{-2+3\varepsilon}$. Hence $\int_{E_1} |I_N(\xi)|\,d\xi \leqslant CN^{-2+3\varepsilon} mE$.

It remains to estimate $\int_{E_2} |I_N(\xi)|\,d\xi$. Let $\xi \in E_2$. We may assume without loss of generality that $m = |\alpha_1 - \alpha|$. We set

$F(x, y) = K(x, y) + L(x, y)$, where $K(x, y) = 2\{\varphi(\alpha, x, y) - \varphi(\alpha_1, x, y)\}$

and

$$L(x, y) = \varphi(\beta, x, y) - \varphi(\alpha, x, y) - \varphi(\beta_1, x, y) + \varphi(\alpha_1, x, y).$$

It is easy to see that $\left| \dfrac{\partial^{i+j} F}{\partial x^i \partial y^j} \right| \leqslant Cm$, $2 \leqslant i+j \leqslant l/\varepsilon + 2$, and $|\operatorname{grad} L| \leqslant C|\alpha - \beta|$ on the square D'.

We have

$$\frac{\partial K}{\partial x}(x, y) = 2\left\{ \frac{\partial \varphi}{\partial x}(\alpha, x, y) - \frac{\partial \varphi}{\partial x}(\alpha_1, x, y) \right\} = 2\frac{\partial^2 \varphi}{\partial t\,\partial x}(\alpha_1, x, y)(\alpha - \alpha_1) + O(m^2)$$

and

$$\frac{\partial K}{\partial y}(x, y) = 2\left\{ \frac{\partial \varphi}{\partial y}(\alpha, x, y) - \frac{\partial \varphi}{\partial y}(\alpha_1, x, y) \right\} = 2\frac{\partial^2 \varphi}{\partial t\,\partial y}(\alpha_1, x, y)(\alpha - \alpha_1) + O(m^2)$$

and from the fact that the determinant J does not vanish it follows that $\max\left(\left| \dfrac{\partial^2 \varphi}{\partial t\,\partial x} \right|, \left| \dfrac{\partial^2 \varphi}{\partial t\,\partial y} \right| \right)$ is bounded below and hence $m \leqslant C|\operatorname{grad} K|$ on D'.

Since $\xi \in E_2$ implies $m > M|\alpha - \beta|$ we conclude that $m \leqslant C|\operatorname{grad} F|$ on D'. Set $E_3 = \{\xi \in E_2;\, m \leqslant N^{-1+\varepsilon}\}$. Then the measure of E_3 is less than $N^{-3+3\varepsilon} mE$ and for $\xi \in E_2 \setminus E_3$ we can use repeated partial integrations as above to prove that $|I_N(\xi)| \leqslant CN^{-l}$. Hence $\int_{E_2} |I_N(\xi)|\,d\xi \leqslant CN^{-3+3\varepsilon} mE$. This completes the proof of Lemma 1.

We shall now prove Theorem I.

Proof of Theorem I. We first prove the L^2 estimate. We have

$$\|S_N^\varphi f\|_{L^2(D)}^2 \leqslant \iint_{\mathbf{R}^2} |S_N^\varphi f(x, y)|^2 w(x, y)\,dx\,dy = N \iint_{E^2} f(\alpha)\overline{f(\beta)} \Big\{ \iint_{\mathbf{R}^2} e^{iNG(\alpha,\beta;x,y)} w(x, y)\,dx\,dy \Big\}\,d\alpha\,d\beta,$$

where $G(\alpha, \beta; x, y) = G(x, y) = \varphi(\alpha, x, y) - \varphi(\beta, x, y)$ and $E = \{x \in I;\, f(x) \neq 0\}$. Without loss of generality we may assume that $\operatorname{diam} E$ is small. It then follows as in the last part of the proof of Lemma 1 that $|\alpha - \beta| \leqslant C|\operatorname{grad} G|$ if $(\alpha, \beta) \in E^2$. Setting

$$I_N(\alpha, \beta) = \iint_{\mathbf{R}^2} e^{iNG(\alpha,\beta;x,y)} w(x, y)\,dx\,dy$$

and integrating by parts twice in this integral we obtain

$$|I_N(\alpha, \beta)| \leqslant CN^{-2}|\alpha - \beta|^{-2}.$$

We have

$$\|S_N^\varphi f\|_{L^2(D)}^2 \leqslant \int_0^1 |f(\alpha)| \Big\{ N \int_0^1 |f(\beta)|\,|I_N(\alpha, \beta)|\,d\beta \Big\}\,d\alpha$$

and using the above estimate of $I_N(\alpha, \beta)$ we can easily prove that the expression in brackets is less than $Cf^*(\alpha)$, where f^* is the Hardy–Littlewood maximal function of f. Hence

$$\|S_N^\varphi f\|_{L^2(D)}^2 \leqslant C\|f\|_{L^2(I)} \|f^*\|_{L^2(I)} \leqslant C\|f\|_{L^2(I)}^2$$

and the L^2 inequality is proved. We shall now use Lemma 1 to prove the L^4 estimate. We first prove that if $f \in L^\infty(I)$ and $E = \{x \in I;\, f(x) \neq 0\}$ then

(1.7) $$\|S_N^\varphi f\|_{L^4(D)} \leqslant CN^\varepsilon \|f\|_{L^\infty(I)} (mE)^{1/4}.$$

Without loss of generality we may assume that $\operatorname{diam} E < d$, where d is the same as in Lemma 1. Using Lemma 1 we obtain

$$\|S_N^\varphi f\|_{L^4(D)}^4 \leqslant \iint_{\mathbf{R}^2} |S_N^\varphi f(x, y)|^4 w(x, y)\,dx\,dy$$

$$= N^2 \iint_{\mathbf{R}^2} \Big\{ \iiiint_{I^4} e^{iNF(\alpha,\beta,\alpha_1,\beta_1;x,y)} f(\alpha) f(\beta) \overline{f(\alpha_1)}\,\overline{f(\beta_1)}\,d\alpha\,d\beta\,d\alpha_1\,d\beta_1 \Big\} \times w(x, y)\,dx\,dy$$

$$= N^2 \iiiint_{E^4} f(\alpha) f(\beta) \overline{f(\alpha_1)}\,\overline{f(\beta_1)} \times \Big\{ \iint_{\mathbf{R}^2} e^{iNF(\alpha,\beta,\alpha_1,\beta_1;x,y)} w(x, y)\,dx\,dy \Big\}\,d\alpha\,d\beta\,d\alpha_1\,d\beta_1$$

$$\leqslant N^2 \|f\|_{L^\infty(I)}^4 \int_{E^4} |I_N(\xi)|\,d\xi \leqslant CN^\varepsilon \|f\|_{L^\infty(I)}^4 mE,$$

and (1.7) follows from this estimate.

Let f be non-negative and $\|f\|_{L^4(I)} = 1$. We set

$$F_1 = \{x \in I;\ f(x) \leqslant N^{-1}\},$$
$$F_2 = \{x \in I;\ N^{-1} < f(x) < N\},$$
$$F_3 = \{x \in I;\ f(x) \geqslant N\}$$

and let χ_i denote the characteristic function of F_i, $i = 1, 2, 3$. Setting $f_i = \chi_i f$, $i = 1, 2, 3$, we then have $f = f_1 + f_2 + f_3$.

Obviously $\|S_N^\varphi f_1\|_{L^4(D)} \leqslant 1$ and we can easily obtain the same estimate for $S_N^\varphi f_3$ if we use the fact that $mF_3 \leqslant N^{-4}$. We have $f_2 = \sum_n g_n$, where $g_n(x) = f_2(x)$ if $x \in E_n = \{x \in I;\ 2^{n-1} < f_2(x) \leqslant 2^n\}$ and $g_n(x) = 0$ otherwise. Applying (1.7) and Hölder's inequality we obtain

$$\|S_N^\varphi f_2\|_{L^4(D)} \leqslant \sum_n \|S_N^\varphi g_n\|_{L^4(D)} \leqslant CN^\varepsilon \sum_n 2^n (mE_n)^{1/4}$$
$$\leqslant CN^\varepsilon (\log N)^{3/4} \Big(\sum_n 2^{4n} mE_n\Big)^{1/4} \leqslant CN^{2\varepsilon} \|f_2\|_{L^4(I)} \leqslant CN^{2\varepsilon}.$$

This completes the proof of the L^4 estimate in Theorem I and the L^p estimate for $2 < p < 4$ follows from interpolation between this result and the L^2 inequality.

2. The operators T_λ. We shall prove the following result for the operators T_λ defined in the introduction.

THEOREM II. *If $2/\lambda < p < 2/(2-\lambda)$ then T_λ is a bounded operator on $L^p(\mathbf{R}^2)$.*

Before proving Theorem II we shall prove two lemmas. First define the operator S_N^λ by

$$S_N^\lambda f(x) = N^{2-\lambda} \int_{I^2} \frac{e^{iN|x-\xi|}}{|x-\xi|^\lambda} f(\xi)\, d\xi, \quad x \in D,\ f \in L^1(I^2),\ N > 0.$$

LEMMA 2. *If $4 \leqslant p < 2/(2-\lambda)$ then there exists a number $\delta > 0$ such that $\|S_N^\lambda f\|_{L^p(D)} \leqslant C_p N^{-\delta} \|f\|_{L^p(I^2)}$, $N > 0$.*

Proof. We first observe that Theorem I holds if we define φ by

$$\varphi(t, x, y) = \{(x-t)^2 + y^2\}^{1/2}, \quad t \in I,\ (x, y) \in D.$$

This follows from the fact that in this case the determinant J equals $y^3\{(x-t)^2+y^2\}^{-3}$ and does not vanish on $I \times D$. It is easy to see that Theorem I holds also if we replace S_N^φ by the operator S_N defined by

$$S_N f(x, y) = N^{1/2} \int_0^1 \frac{e^{iN\{(x-t)^2+y^2\}^{1/2}}}{\{(x-t)^2+y^2\}^{\lambda/2}} f(t)\, dt, \quad (x, y) \in D,\ f \in L^1(I).$$

This can be proved in the same way as Theorem I. Hence the above S_N is a bounded operator from $L^4(I)$ to $L^4(D)$ with norm not greater than CN^ε and it is obviously a bounded operator from $L^\infty(I)$ to $L^\infty(D)$ with norm less than $N^{1/2}$. Interpolation yields

(2.1) $$\|S_N f\|_{L^p(D)} \leqslant CN^{1/2+4(\varepsilon-1/2)/p} \|f\|_{L^p(I)}, \quad 4 \leqslant p \leqslant \infty.$$

We have

$$S_N^\lambda f(x, y) = N^{3/2-\lambda} \int_I \Big\{ N^{1/2} \int_I \frac{e^{iN\{(x-t)^2+(y-u)^2\}^{1/2}}}{\{(x-t)^2+(y-u)^2\}^{\lambda/2}} f(t, u)\, dt \Big\}\, du$$

and using Minkowski's inequality, (2.1) and the Hölder inequality we obtain

$$\|S_N^\lambda f\|_{L^p(D)} \leqslant N^{3/2-\lambda} CN^{1/2+4(\varepsilon-1/2)/p} \int_I \Big\{ \int_I |f(t, u)|^p\, dt \Big\}^{1/p} du$$
$$\leqslant CN^{2-\lambda+4(\varepsilon-1/2)/p} \|f\|_{L^p(I^2)}, \quad 4 \leqslant p \leqslant \infty.$$

If $p < 2/(2-\lambda)$ we can choose ε so small that $2-\lambda+4(\varepsilon-1/2)/p < 0$ and Lemma 2 follows.

We shall now apply Lemma 2 to the study of the operators T_N^λ defined by

$$T_N^\lambda f(x) = N^{2-\lambda} \int_{I^2} \frac{e^{iN|x-\xi|}}{|x-\xi|^\lambda} f(\xi)\, d\xi, \quad x \in I^2,\ f \in L^1(I^2),\ N > 0.$$

LEMMA 3. *If $2/\lambda < p < 2/(2-\lambda)$ then $\|T_N^\lambda f\|_{L^p(I)^2} \leqslant C_p \|f\|_{L^p(I^2)}$, $N > 0$.*

Proof. First assume $4 \leqslant p < 2/(2-\lambda)$. If ω is a square in $\mathbf{R}^2$ let $\frac{1}{2}\omega$ denote the square with the same center as ω and a side length which equals half the side length of ω. Let Ω_μ, $\mu = 0, 1, 2, \ldots$, denote the set of all dyadic squares in $(-2, 2) \times (-2, 2)$ with side length $2^{-\mu}$, and let Ω_μ^* denote the set of all squares which are the union of four squares in Ω_μ. Let $f \in L^p(I^2)$ and set f equal to zero outside I^2. If $x \in I^2$ and x does not belong to the boundary of any dyadic square let $\omega_\mu^*(x)$ be the unique element of Ω_μ^* which satisfies $x \in \frac{1}{2}\omega_\mu^*(x)$, $\mu \geqslant 0$, and set $\omega_{-1}^*(x) = (-2, 2) \times \times (-2, 2)$.

For measurable sets S we define $A(x, S)$ by

$$A(x, S) = N^{2-\lambda} \int_S \frac{e^{iN|x-\xi|}}{|x-\xi|^\lambda} f(\xi)\, d\xi, \quad x \in I^2,$$

and we also set $A_\mu(x) = A[x, \omega_{\mu-1}^*(x) \setminus \omega_\mu^*(x)]$, $\mu \geqslant 0$. Defining μ_N by $2^{-\mu_N - 1} < N^{-1} \leqslant 2^{-\mu_N}$ we have

(2.2) $$T_N^\lambda f(x) = \sum_{\mu=0}^{\mu_N} A_\mu(x) + A[x, \omega_{\mu_N}^*(x)].$$

From the construction of $\omega_\mu^*(x)$ it follows that $A_\mu(x) = \sum_{\omega\in\Omega_\mu} A(x,\omega)\chi_{F_\omega}(x)$, where F_ω is the union of squares in $\Omega_{\mu+1}$, with the property that the distance from each square to ω is approximately $2^{-\mu}$, and χ_{F_ω} is the characteristic function of F_ω. Since $\sum_{\omega\in\Omega_\mu}\chi_{F_\omega}(x) = 12$ Hölder's inequality yields $|A_\mu(x)|^p \leqslant C\sum_{\omega\in\Omega_\mu}|A(x,\omega)|^p\chi_{F_\omega}(x)$ and hence

$$\int_{I^2}|A_\mu(x)|^p dx \leqslant C\sum_{\omega\in\Omega_\mu}\int_{F_\omega}|A(x,\omega)|^p dx. \tag{2.3}$$

Performing a change of scale we can use Lemma 2 to estimate the last integral. We obtain

$$\int_{F_\omega}|A(x,\omega)|^p dx \leqslant C_p(N2^{-\mu})^{-\delta p}\int_\omega |f(x)|^p dx.$$

A combination of this inequality and (2.3) yields

$$\|A_\mu\|_{L^p(I^2)} \leqslant C_p N^{-\delta}2^{\delta\mu}\|f\|_{L^p(I^2)}.$$

The last term in (2.2) can be majorized by the Hardy–Littlewood maximal function of f and we get

$$\|T_N^\lambda f\|_{L^p(I^2)} \leqslant C_p N^{-\delta}\sum_{\mu=0}^{\mu_N} 2^{\delta\mu}\|f\|_{L^p(I^2)} + C_p\|f\|_{L^p(I^2)} \leqslant C_p\|f\|_{L^p(I^2)}.$$

This completes the proof of Lemma 3 in the case $4 \leqslant p < 2/(2-\lambda)$, and the general case follows from this by interpolation and a standard duality argument.

Theorem II now follows from Lemma 3 by a change of scale. This completes the proof of Theorem II.

ADDENDUM

Using ideas from this paper the second author has proved that the above multiplier result holds also if the unit disc is replaced by a general compact set C in the plane with the property that its boundary ∂C is a simple closed C^∞ curve which has a tangent with finite order of contact at each point. In this case the function $(1-|x|^2)^a$ is replaced by a function which equals $[\mathrm{dist}(x,\partial C)]^a$ when $x\in C$ and is close to ∂C.

The following result on restrictions of Fourier transforms can be obtained by a modification of the argument at the end of the introduction. Let Γ be a C^{n+1} curve in the plane, for some integer $n \geqslant 3$, which has positive curvature except at finitely many points. Assume that the highest order of contact of the tangent at these points is $n-1$. Then, if $1\leqslant p$, $q\leqslant\infty$ and $1/(n+1)p+1/q > 1$, the Fourier transform of a function in $L^q(\boldsymbol{R}^2)$ restricts to a function in $L^p(\Gamma; ds)$, where s denotes the arc length. If $1/(n+1)p+1/q<1$ this does not hold.

The proofs will appear elsewhere.

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Received August 20, 1971 (378)