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Existence theorems in problems of optimal control

by

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Dedicated to Professor Antoni Zygmund

Abstract. The compactness in an appropriate topology of a set of trajectories of a control system is proved. The lower semi-continuity of certain functionals on certain sets of trajectories is also proved. Existence theorems for optimal controls in ordinary and relaxed problems are obtained as immediate consequences of these theorems. The conditions imposed on the control systems and the existence theorems obtained are essentially those of Cesari, but the proofs are different in many important aspects.

1. Introduction. In this paper we present a compactness and semi-continuity theorem, Theorem 1 below, for control systems governed by ordinary differential equations. From this theorem one can obtain general existence theorems for optimal control problems in Mayer and Lagrange form, as indicated in Section 6. Our existence theorems are not as general as those of Cesari [3] in that our output function θ, defined in Section 2, is real valued, while the analogous function in [3] can be vector valued. On the other hand, we require a version of Cesari's property (Q) that is weaker than that employed in [3]. Our theorems do cover the cases of interest in applications and we feel that the contribution of this paper lies in the simplification of the arguments in these cases.

Although our Theorem 1 can be viewed as a special case of the lower closure theorems of Cesari (See [4] for a full account), our theorem is general enough to yield the results we desire. Our methods, however, do yield lower closure very simply when the derivatives of the convergent sequence converge weakly in L_1. The various growth conditions used in variational and control problems guarantee that this is always true for minimizing sequences.

2. Notation and definitions. We shall use single letters to denote vectors, we shall use subscripts to distinguish vectors, and we shall use superscripts to denote components of vectors. The letter t will denote

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a real variable, which we call time, the letter x will denote a vector $x = (x^1, \ldots, x^n)$ in R^n, which we call the state variable, the letter y will denote a real number, which we call the output variable, and the letter w will denote a vector in R^m, which we call the control variable. By $x_1 \geqslant x_2$ we mean that every component of x_1 is greater than or equal to the corresponding component of x_2. The euclidean norm of a vector x will be denoted by $|x|$.

Let f be a real function $(t, x, w) \to f(t, x, w)$ defined on $R \times R^n \times R^m$ with range in R^1 and let g be a function $(t, x, w) \to g(t, x, w)$ defined on $R \times R^n \times R^m$ with range in R^n. Let $\mathscr{R}$ be a subset of (t, x)-space. Let Ω denote a mapping that assigns to each point (t, x) in $\mathscr{R}$ a subset $\Omega(t, x)$ of R^m. Let $\mathscr{B}$ be a set of points (t_1, x_1, t_2, x_2) in R^{2n+2} with $t_2 > t_1$.

An absolutely continuous function $\varphi = (\varphi^1, \ldots, \varphi^n)$ defined on an interval $[t_1, t_2]$ is said to be an *admissible trajectory* if there exists a measurable function $u = (u^1, \ldots, u^m)$ defined on the same interval such that the following hold:

$$
\begin{aligned}
&\text{(i)} && \big(t, \varphi(t)\big) \in \mathscr{R} \quad \text{for all } t \text{ in } [t_1, t_2],\\
&\text{(ii)} && \big(t_1, \varphi(t_1), t_2, \varphi(t_2)\big) \in \mathscr{B},\\
(2.1)\quad &\text{(iii)} && \varphi'(t) = g\big(t, \varphi(t), u(t)\big) \text{ a.e. in } [t_1, t_2],\\
&\text{(iv)} && u(t) \in \Omega\big(t, \varphi(t)\big) \text{ a.e. in } [t_1, t_2],\\
&\text{(v)} && f\big(t, \varphi(t), u(t)\big) \text{ is in } L_1[t_1, t_2].
\end{aligned}
$$

The function u is said to be an *admissible control* and the pair (φ, u) is said to be an *admissible pair*. By virtue of (2.1)-(v) we can define an absolutely–continuous *output-function* θ corresponding to each admissible pair (φ, u) as follows:

$$
(2.2) \qquad \theta(t) = \int_{t_1}^{t} f\big(s, \varphi(s), u(s)\big)\, ds \qquad t_1 \leqslant t \leqslant t_2.
$$

Output functions and output variables arise in applications and in the reduction of control problems in Lagrange formulation to control problems in Mayer formulation. We define the number $\theta(t_2)$ to be the *output of the system* or simply the *output*.

3. Assumptions. In this section we list two sets of assumptions for our theorems. One of the statements in A requires the introduction of a function Q^+ that assigns to each point (t, x) in $\mathscr{R}$ a subset $Q^+(t, x)$ of R^{n+1} as follows:

$$
Q^+(t, x) = \{(\eta, \xi)\colon \eta \geqslant f(t, x, w),\ \xi = g(t, x, w),\ w \in \Omega(t, x)\},
$$

where η is a scalar and ξ is an n-vector.

Assumption A. (1) $\mathscr{R}$ is a compact subset of $R \times R^n$. (2) The set $\mathscr{B}$ is compact. (3) For each (t, x) in $\mathscr{R}$ the set $\Omega(t, x)$ is closed. (4) The set $\mathscr{D} = \{(t, x, w)\colon (t, x) \in \mathscr{R},\ w \in \Omega(t, x)\}$ is closed. (5) For each (t, x) in $\mathscr{R}$ the set $Q^+(t, x)$ is closed and convex. (6) For all (t, x, w) in $\mathscr{D}$, $f(t, x, w) \geqslant 0$.

From (2.1)-(i) and Assumption A(1) it follows that the graphs of all trajectories lie in a compact subset $\mathscr{R}$ of $R \times R^n$. Thus, all of the intervals $[t_1, t_2]$ corresponding to admissible pairs (φ, u) are contained in a fixed compact interval I. It further follows that all points $\big(t_1, \varphi(t_1), t_2, \varphi(t_2)\big)$ corresponding to admissible trajectories will lie in a compact subset of R^{2n+2}. Hence the assumption that $\mathscr{B}$ is compact rather than closed entails no loss of generality.

Assumption A(4) is equivalent to the assumption that Ω is upper semi-continuous in the sense of Kuratowski ([8], pp. 32–34) on $\mathscr{R}$. Assumption A(5) was introduced by Cesari in [2].

The second set of assumptions involves a slight generalization of Cesari's property (Q).

Let $\delta > 0$, let $(t_0, x_0) \in \mathscr{R}$, and let $N_x(t_0, x_0, \delta)$ denote the set of points (t_0, x) in $\mathscr{R}$ such that $|x - x_0| < \delta$. Let $Q^+\big(N_x(t_0, x_0, \delta)\big) = \bigcup \{Q^+(t_0, x)\colon (t_0, x) \in N_x(t_0, x_0, \delta)\}$. For a set A let cl co A denote the closure of the convex hull of A. The mapping Q^+ is said to *satisfy property* (Q*) *at* (t_0, x_0) if

$$
Q^+(t_0, x_0) = \bigcap_{\delta > 0} \operatorname{cl}\ \operatorname{co} Q^+\big(N_x(t_0, x_0, \delta)\big).
$$

It is readily verified that if property (Q) holds at a point then so does property (Q*). An extension of property (Q) similar to ours was also used by Olech [12].

Assumption B. (1) The function f is lower semicontinuous and g is continuous. (2) The mapping Q^+ satisfies property (Q*) at each point in $\mathscr{R}$.

4. A compactness and semi-continuity theorem. Let $\mathscr{Z}$ denote the set of all continuous functions z defined on subintervals $[t_1, t_2]$ of I. We extend the definition of z to a function $\tilde{z}$ defined on all of I by setting $\tilde{z}(t) = z(t_1)$ if $a \leqslant t \leqslant t_1$ and $\tilde{z}(t) = z(t_2)$ if $t_2 \leqslant t \leqslant b$. If $t \in [t_1, t_2]$, then $\tilde{z}(t) = z(t)$. For z in $\mathscr{Z}$ defined on $[t_1, t_2]$ and $z_0 \in \mathscr{Z}$ defined on $[t_{01}, t_{02}]$, we define

$$
\varrho(z, z_0) = |t_{01} - t_1| + |t_{02} - t_2| + \max |\tilde{z}_0(t) - \tilde{z}(t)|,
$$

where the max is taken over all t in I. It is easily checked that ϱ is a metric and that $\mathscr{Z}$ is complete under this metric. A sequence of functions $\{z_k\}$ defined on intervals $[t_{1k}, t_{2k}]$ converges to a function z defined on an interval $[t_1, t_2]$ if and only if $t_{ik} \to t_i$, $i = 1, 2$ and $\tilde{z}_k \to \tilde{z}$ uniformly on I.

Let $\mathscr{A}$ denote the class of all admissible pairs and let $\mathscr{A}_T$ denote the class of all admissible trajectories. Clearly $\mathscr{A}_T \subset \mathscr{X}$. Note that since we do not assume that the differential equation $x' = g(t, x, u(t))$ has a unique solution for a given set of initial conditions, different admissible pairs can have the same admissible control. The set $\mathscr{A}_T$ is a metric space in the ϱ-metric.

THEOREM 1. *Let Assumptions* A *and* B *hold. Let $\mathscr{A}_0$ be a set of admissible pairs (φ, u) such that the trajectories φ are equi-absolutely continuous and such that the outputs $\theta(t_2)$ are uniformly bounded. Then there exists a sequence (φ_k, u_k) in $\mathscr{A}_0$, a real number γ and an admissible pair (φ, u) in $\mathscr{A}$ with the following properties*: (i) *$\varphi_k \to \varphi$ in the ϱ metric.* (ii)

$$\gamma = \lim_{k\to\infty} \theta_k(t_{2k}) \geqslant \theta(t_2), \tag{4.1}$$

where θ_k is the output function associated with (θ_k, u_k) and θ is the output function associated with (φ, u).

The conclusion of the theorem says that the set $\mathscr{A}_{0T}$ of trajectories corresponding to admissible pairs in $\mathscr{A}_0$ is conditionally compact in $\mathscr{A}_T$ and that the output $\theta(t_2)$ is lower semicontinuous on $\mathscr{A}_0$.

Cesari [3], [5], gives very general conditions that guarantee equi-absolute continuity of the components of φ. In our notation the most important of these is the following.

LEMMA 1. *Let there exist a non-negative lower semicontinuous function H^i: $(t, x, w) \to H^i(t, x, w)$ and a constant A^i such that for each admissible pair (φ, u) in $\mathscr{A}_0$, $H^i(t, \varphi(t), u(t))$ is integrable on $[t_1, t_2]$ and*

$$\int_{t_1}^{t_2} H^i(t, \varphi(t), u(t))\,dt \leqslant A^i, \tag{4.2}$$

where $[t_1, t_2]$ is the interval of definition of (φ, u). For each $\varepsilon > 0$ let there exist a function M^i_ε defined and integrable on I such that

$$|\varphi^{i'}(t)| = |g^i(t, \varphi(t), u(t))| \leqslant M^i_\varepsilon(t) + \varepsilon H^i(t, \varphi(t), u(t)). \tag{4.3}$$

Then the functions φ^i are equi-absolutely continuous.

Another condition that ensures equi-absolute continuity of the integrals of $\varphi^{i'}$, and hence of φ^i, is that of de la Valée–Poussin (See [11] Theorem 7, p. 159). In particular this condition implies that if there is a $p_i > 1$ and a constant $C_i > 0$ such that for all (φ, u) in $\mathscr{A}_0$,

$$\int_{t_1}^{t_2} |\varphi^{i'}(t)|^{p_i}\,dt \leqslant C^i \tag{4.4}$$

then the functions φ^i are equi-absolutely continuous. It can also be shown directly that (4.4) implies (4.2) and (4.3) with $H^i = |g^i|^{p_i}$.

Theorem 1 remains true if we weaken the hypotheses somewhat. First, we can replace A(6) by the requirement that f be bounded below. Second, we can replace B(1) by the following condition.

ASSUMPTION C. For each t in I, the function $F = (f, g)$ is a continuous function of (x, w) on R^{n+1} and for each (x, w) in R^{n+1} the function F is measurable with respect to t in I.

We can weaken B(2) by requiring the property (Q^*) to hold at every point of $\mathscr{R}$ with the possible exception of a set of points whose t-coordinate lies in a set of measure zero in I.

At the appropriate places in the proof of Theorem 1 we shall indicate the modifications that must be made to accomodate the weakened hypotheses.

5. Proof of Theorem 1. The proof will exploit the weak convergence in L_1 of a sequence of derivatives φ_k' and Mazur's theorem which states that a strongly closed convex set in a Banach space is weakly closed. In the proof we shall select subsequences of various sequences. Unless stated otherwise, we shall relabel the subsequence with the labeling of the original sequence. We break the proof up into several steps.

Step 1. There is a sequence $\{(\varphi_k, u_k)\}$ of elements in $\mathscr{A}_0$, a real γ, points t_1 and t_2 in I with $t_2 > t_1$, and points x_{01} and x_{02} in R^n such that for $i = 1, 2$, $(t_i, x_{0i}) \in \mathscr{R}$, $(t_1, x_{01}, t_2, x_{02}) \in \mathscr{B}$, and

$$t_{ik} \to t_i, \qquad \varphi_k(t_{ik}) \to x_{0i}, \qquad \theta_k(t_{2k}) \to \gamma. \tag{5.1}$$

Since the set of outputs $\theta(t_2)$ corresponding to pairs (φ, u) in $\mathscr{A}$ is bounded it follows that there is a real number γ and a sequence $\{(\varphi_k, u_k)\}$ of admissible pairs in $\mathscr{A}_0$ such that $\theta_k(t_{2k}) \to \gamma$, where θ_k is the output corresponding to (φ_k, u_k). Since $\mathscr{B}$ is compact there is a subsequence of this sequence such that $(t_{1k}, \varphi_k(t_{1k}), t_{2k}, \varphi_k(t_{2k}))$ converges to a point $(t_1, x_{01}, t_2, x_{02})$ in $\mathscr{B}$. Hence $t_2 > t_1$. Since $(t_{ik}, \varphi_k(t_{ik})) \in \mathscr{R}$, $i = 1, 2$, it follows that $(t_i, x_{0i}) \in \mathscr{R}$.

Step 2. There exists an absolutely continuous function φ defined on $[t_1, t_2]$ and a subsequence $\{\varphi_k\}$ such that $\varphi_k \to \varphi$ in the ϱ metric and for the extended functions, $\tilde{\varphi}_k' \to \tilde{\varphi}'$ weakly in $L_1[I]$. Moreover, φ satisfies (2.1)-(i) and (2.1)-(ii).

Since the graphs of all trajectories in $\mathscr{A}_0$ lie in the compact set $\mathscr{R}$, the functions φ_k are uniformly bounded and so are their extensions $\tilde{\varphi}_k$. Since the functions φ_k are equi-absolutely continuous, the same is true of their extensions $\tilde{\varphi}_k$. It therefore follows from Ascoli's theorem that there exists a subsequence $\{\tilde{\varphi}_k\}$ and a function $\tilde{\varphi}$ defined on I such that

$\tilde{\varphi}_k$ converges to $\tilde{\varphi}$ uniformly on I. Moreover, the function $\tilde{\varphi}$ is absolutely continuous, so that $\tilde{\varphi}'$ exists and is in L_1. Thus,

$$(5.2)\qquad \tilde{\varphi}(t) = \tilde{\varphi}(a) + \int_a^t \tilde{\varphi}'(s)\,ds \qquad a \leqslant t \leqslant b.$$

The function $\tilde{\varphi}$ is readily seen to be the extension of the function φ defined by

$$\varphi(t) = x_{01} + \int_{t_1}^t \tilde{\varphi}'(s)\,ds \qquad t_1 \leqslant t \leqslant t_2.$$

Thus $\tilde{\varphi}(a) = x_{01}$. Since $t_{ik} \to t_i$, $i = 1, 2$, we have shown that $\varphi_k \to \varphi$ in the ϱ metric.

From (5.2) with $\tilde{\varphi}(a)$ replaced by x_{01}, from (5.1), the relation,

$$\tilde{\varphi}_k(t) = \varphi_k(t_{1k}) + \int_a^t \tilde{\varphi}_k'(s)\,ds \qquad a \leqslant t \leqslant b,$$

and the convergence of $\tilde{\varphi}_k$ to φ it follows that for all t in $[a, b]$

$$\int_a^t \tilde{\varphi}_k'(s)\,ds \to \int_a^t \tilde{\varphi}'(s)\,ds.$$

Since the functions $\tilde{\varphi}_k$ are equi-absolutely continuous, their derivatives $\tilde{\varphi}_k'$ have equi-absolutely continuous integrals. Hence (See Banach [1], page 136) $\tilde{\varphi}_k' \to \tilde{\varphi}'$ weakly in $L_1[I]$.

Since φ is the uniform limit of functions for which (2.1)-(i) and (ii) hold, these conditions hold for φ.

Step 3. There exists a function λ that is integrable on $[t_1, t_2]$ such that $(\lambda(t), \varphi'(t)) \in Q^+(t, \varphi(t))$ for a.e. t in $[t_1, t_2]$ and such that

$$(5.3)\qquad \int_{t_1}^{t_2} \lambda(s)\,ds \leqslant \gamma.$$

Since $\tilde{\varphi}_k' \to \tilde{\varphi}'$ weakly in L_1 we obtain the following statement from a corollary to Mazur's theorem ([6], Corollary, Theorem 2.9.3, p. 36). For each integer j there exists an integer n_j, a set of integers $i = 1, \ldots, k$ where $k = k(j)$ depends on j, and a set of numbers $a_{1j}, \ldots, a_{kj}$ satisfying

$$(5.4)\qquad a_{ij} \geqslant 0, \quad i = 1, \ldots, k, \quad \sum_{i=1}^k a_{ij} = 1,$$

such that $n_{j+1} > n_j + k(j)$ and

$$(5.5)\qquad \int_a^b \Big| \tilde{\varphi}' - \sum_{i=1}^k a_{ij} \tilde{\varphi}_{n_j+i}' \Big|\, dt < 1/j.$$

Let $\psi_j = \sum_{i=1}^k a_{ij} \tilde{\varphi}_{n_j+i}'$. If for $t \notin [t_{1q}, t_{2q}]$ we define $g(t, \varphi_q(t), u_q(t)) = 0$, (recall u_q is only defined on $[t_{1q}, t_{2q}]$), we write ψ_j as

$$(5.6)\qquad \psi_j(t) = \sum_{i=1}^k a_{ij} g(t, \varphi_{n_j+i}(t), u_{n_j+i}(t)).$$

In terms of ψ_j, (5.5) says that $\psi_j \to \tilde{\varphi}'$ in $L_1[I]$. Hence there is a subsequence $\{\psi_j\}$ such that

$$(5.7)\qquad \psi_j(t) \to \tilde{\varphi}'(t) \quad \text{a.e. in } I.$$

We suppose that (5.6) is now this subsequence. Corresponding to the sequence (5.6) we define a sequence $\{\lambda_j\}$ as follows:

$$(5.8)\qquad \lambda_j(t) = \sum_{i=1}^k a_{ij} f(t, \varphi_{n_j+i}(t), u_{n_j+i}(t)),$$

where if $t \notin [t_{1q}, t_{2q}]$, $f(t, \varphi_q(t), u_q(t)) = 0$ and where for each j the numbers a_{ij}, the indices $n_j + i$ and the functions φ_{n_j+i} and u_{n_j+i} are as in (5.6). Note that if $t \notin [t_1, t_2]$, there exists a j_0 such that if $j > j_0$ then $\psi_j(t) = 0$, and $\lambda_j(t) = 0$.

Define

$$(5.9)\qquad \lambda(t) = \liminf \lambda_j(t).$$

Since $f \geqslant 0$, it follows that $\lambda \geqslant 0$. Moreover if $t \notin [t_1, t_2]$, then $\lambda(t) = 0$. From this and from Fatou's theorem we get, upon setting $f_q = f(t, \varphi_q(t), u_q(t))$,

$$(5.10)\qquad \int_{t_1}^{t_2} \lambda\,dt = \int_a^b \lambda\,dt \leqslant \liminf \sum_{i=1}^k a_{ij} \int_a^b f_{n_j+i}\,dt$$
$$= \liminf \sum_{i=1}^k a_{ij} \int_{t_{1,n_j+i}}^{t_{2,n_j+i}} f_{n_j+i}\,dt = \liminf \sum_{i=1}^k a_{ij}\, \theta_{n_j+i}(t_{2,n_j+i}).$$

The inequality (5.3) now follows from (5.1) and (5.4). Since $\lambda \geqslant 0$ it follows that λ is in $L_1[I]$ and is finite a.e. on $[t_1, t_2]$. We note that the arguments just made could also be carried out if we merely assumed f to be bounded from below.

We now show that $(\lambda(t), \varphi'(t)) \in Q^+(t, \varphi(t))$ a.e. Let T_1 denote the set of points in $[t_1, t_2]$ at which $\lambda(t)$ is finite and $\psi_j(t) \to \varphi'(t)$. If we weaken B(2) then we also assume that property (Q*) holds for all t in T_1. For each k define a set E_k as follows: $E_k = \{t\colon t \in [t_{1k}, t_{2k}],\ u_k(t) \notin \Omega(t, \varphi_k(t))\}$. Then by (2.1)-(iv), meas $E_k = 0$. Let E denote the union of the sets E_k and let T_2 denote the set of points in $[t_1, t_2]$ that do not belong to E. Let $T' = T_1 \cap T_2$. Clearly, meas $T' = t_2 - t_1$.

Let t_0 be a fixed element in T', $t_0 \neq t_i$, $i = 1, 2$. There exists a subsequence $\{\lambda_j(t_0)\}$, which depends on t_0, such that $\lambda_j(t_0) \to \lambda(t_0)$. For the corresponding subsequence $\{\psi_j(t_0)\}$ we have, by (5.7), that $\psi_j(t_0) \to \varphi'(t_0)$. Since t_0 is interior (t_1, t_2), and $t_{ik} \to t_i$, $i = 1, 2$, it follows that there exists a j_0 such that if $j > j_0$, then $t_0 \in (t_{1,n_j+i}, t_{2,n_j+i})$. For each $\delta > 0$, there exists an integer k_0, depending on δ such that if $k > k_0$, $|\varphi_k(t_0) - \varphi(t_0)| < \delta$. Hence, for $k > k_0$, $(t_0, \varphi_k(t_0)) \in N_x(t_0, \varphi(t_0), \delta)$. Therefore for j sufficiently large

$$\Big(f\big(t_0, \varphi_{n_j+i}(t_0), u_{n_j+i}(t_0)\big)\Big), g\big(t_0, \varphi_{n_j+i}(t_0), u_{n_j+i}(t_0)\big) \in Q^+\Big(N_x\big(t_0, \varphi(t_0), \delta\big)\Big).$$

Therefore, by (5.6), (5.8), and (5.4),

$$\big(\lambda_j(t_0), \psi_j(t_0)\big) \in \operatorname{co} Q^+\Big(N_x\big(t_0, \varphi(t_0), \delta\big)\Big).$$

Since $\lambda_j(t_0) \to \lambda(t_0)$ and $\psi_j(t_0) \to \varphi'(t_0)$, we have that

$$\big(\lambda(t_0), \varphi'(t_0)\big) \in \operatorname{cl}\operatorname{co} Q^+\Big(N_x\big(t_0, \varphi(t_0), \delta\big)\Big).$$

Since δ is arbitrary, $\big(\lambda(t_0), \varphi'(t_0)\big)$ is in $\operatorname{cl}\operatorname{co} Q^+\big(N_x(t_0), \varphi(t_0), \delta\big)$ for every $\delta > 0$ and hence in the intersection of these sets. Therefore, by Assumption B(2), we get that $\big(\lambda(t_0), \varphi'(t_0)\big) \in Q^+\big(t_0, \varphi(t_0)\big)$. Since t_0 was an arbitrary point in T' different from t_1 or t_2, we get the desired result.

Step 4. There exists a measurable function u defined on $[t_1, t_2]$ such that for almost all t: (i) $\varphi'(t) = g\big(t, \varphi(t), u(t)\big)$; (ii) $u(t) \in \Omega\big(t, \varphi(t)\big)$; (iii) $\lambda(t) \geqslant f\big(t, \varphi(t), u(t)\big)$.

The existence of a function v satisfying the conclusion of Step 4 is a restatement of $\big(\lambda(t), \varphi'(t)\big) \in Q^+\big(t, \varphi(t)\big)$. We show that there exists a measurable function u with this property. Let $T = \{t: \big(\lambda(t), \varphi'(t)\big) \in Q^+\big(t, \varphi(t)\big)\}$, let $Z = R^1 \times R^n \times R^n \times R^1$ and let $D = \{(t, x, w, \eta): (t, x, w) \in \mathscr{D}, \eta \geqslant f(t, x, w)\}$. The functions φ' and λ are measurable. Clearly, T is measurable and Z is Hausdorff. Since by Assumption A(4) $\mathscr{D}$ is closed and by Assumption B(1) f is lower semicontinuous, the set D is closed and hence can be written as the union of a countable number of compact sets. Let $\Gamma: t \to \big(t, \varphi(t), \varphi'(t), \lambda(t)\big)$; thus Γ is a measurable map from T to Z. Let $G: (t, x, w, \eta) \to \big(t, x, g(t, x, w), \eta\big)$, Then G is a continuous map from D to Z and $\Gamma(T) \subset G(D)$. Thus, the hypotheses of the McShane–Warfield extension of Filippov's Lemma [9] are satisfied. Hence there exists a measurable mapping $\mu: T \to D$, say $\mu: t \to \big(t(t), x(t), u(t), \eta(t)\big)$ such that

$$G\big(\mu(t)\big) = \Big(t(t), x(t), g\big(t(t), x(t), u(t)\big), \eta(t)\Big) = \Gamma(t) = \big(t, \varphi(t), \varphi'(t), \lambda(t)\big).$$

From this Step 4 follows.

If we replace the continuity assumptions on F in B by Assumption C then by a well known theorem ([11], Thm. 18.2, p. 142 or [13]) there exists for each $\varepsilon > 0$ an open set $E \subset I$ such that meas. $(E) < \varepsilon$ and such that F is continuous on $(I - E) \times R^n \times R^m$. We proceed as above to obtain a measurable u on $T - (E \cap T)$, and since ε is arbitrary we obtain the desired result. For details see [7] or [12].

Step 5. Completion of proof. Statements (i) and (ii) of Step 4 assert that (φ, u) satisfies (2.1)-(iii) and (iv). Since φ and u are measurable and f is either lower semicontinuous or satisfies Assumption C, it follows that $f\big(t, \varphi(t), u(t)\big)$ is measurable. Since λ is integrable and f is bounded from below, it follows from (iii) of Step 4 that $f\big(t, \varphi(t), u(t)\big)$ is integrable. Hence (φ, u) satisfies (2.1)-(v). From this and from Step 2 it follows that (φ, u) is admissible and that $\varphi_k \to \varphi$ in the ϱ metric. It therefore only remains to prove (4.1).

The equality in (4.1) was established in Step 1. Let θ be the output corresponding to (φ, u). Then from (iii) of Step 4 and (5.3) it follows that $\gamma \geqslant \theta(t_2)$ and (4.1) is thereby established.

We now show that if we require the components of admissible pairs to satisfy (4.2) and if we take $f = H^i$, then the pair (φ, u) of Theorem 1 is admissible. From (5.10) we get

$$\int_{t_1}^{t_2} \lambda(t)\,dt \leqslant A,$$

and the assertion now follows from (iii) of Step 4.

If (4.4) is required to hold for all admissible pairs then it holds for (φ, u) by virtue of the fact that (4.4) implies that for a suitable subsequence $\varphi_k' \to \varphi$ weakly in L_p as well as in L_1.

6. Existence theorems. We illustrate how the preceding results are applied to obtain existence theorems by considering the following control problem. Let the class $\mathscr{A}$ of admissible pairs be non-empty. If (φ, u) is in $\mathscr{A}$ and θ is the corresponding output let $\eta(\varphi, u) = \big(t_1, \varphi(t_1), t_2, \varphi(t_2), \theta(t_2)\big)$. Let e be a real valued function on $\mathscr{B} \times R^1$; $e: (t_1, x_1, t_2, x_2, y_2) \to e(t_1, x_1, t_2, x_2, y_2)$. Define a functional J on $\mathscr{A}$ as follows: $J(\varphi, u) = e\big(\eta(\varphi, u)\big)$. Let $\mu = \inf\{J(\varphi, u): (\varphi, u) \in \mathscr{A}\}$. The problem is to find an element (φ^*, u^*) in $\mathscr{A}$ such that $J(\varphi^*, u^*) = \mu$. Such a pair is called an *optimal pair.*

ASSUMPTION D. The function e is lower semicontinuous on $\mathscr{B} \times R^1$ and is nondecreasing in y_2.

THEOREM 2. *Let Assumptions* A, B *and* D *hold and let the outputs* $\theta(t_2)$ *lie in a bounded set. Let* $\mathscr{A}$ *be non empty and let there exist a minimizing sequence such that the functions* φ_k *are equi-absolutely continuous. Then* μ *is finite and there exists an optimal pair* (φ^*, u^*).

Since for admissible (φ, u), $f\big(t, \varphi(t), u(t)\big)$ is in $L_1[t_1, t_2]$, it follows that $\mu < +\infty$. Let (φ_k, u_k) be a minimizing sequence with equi-absolutely continuous functions φ_k. Then by the definition of a minimizing sequence, $e(\eta(\varphi_k, u_k)) \to \mu$. On the other hand, by Theorem 1 with $\mathscr{A}_0 = \{(\varphi_k, u_k)\}$, there exists a (φ^*, u^*) in $\mathscr{A}$ defined on an interval $[t_1^*, t_2^*]$ and a subsequence $\{(\varphi_k, u_k)\}$ such that $\eta(\varphi_k, u_k) \to (t_1^*, \varphi^*(t_1^*), t_2^*, \varphi^*(t_2^*), \gamma) \equiv P^*$. By the lower semi-continuity of e, $\liminf e(\eta(\varphi_k, u_k)) \geqslant e(P^*)$. Hence $\mu \geqslant e(P^*) > -\infty$. But since $\gamma \geqslant \theta^*(t_2)$, it follows from the monotonicity of e that $\mu \geqslant e(P^*) \geqslant e(\eta(\varphi^*, u^*)) \geqslant \mu$. Hence $\mu = e(\eta(\varphi^*, u^*))$.

If the sets $Q^+(t, x)$ are not convex one can replace the original problem with a "relaxed problem" in which the derivatives of the trajectories lie in $\mathrm{co}Q^+(t, x)$. This can be done in several ways ([3], [10], [15], [16]). If it is done as in [3] the relaxed problem is cast as a new control problem in which the set that plays the role of $Q^+(t, x)$ is convex. An existence theorem for the relaxed problem is then obtained as a straightforward application of Theorem 1 to the new control problem. The statement of the theorem and the details should be clear to the reader who has consulted [3].

Existence theorems for Lagrange problems are obtained by transforming them into Mayer problems in the standard way. See e.g. [3].

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