From (3.30), we have that (3.33) will follow once we show both

(3.34) $$\limsup_{t\to 0}\int_t^\infty |A_n^\nu(r/t)-1|r^{-1}dr < \infty$$

and

(3.35) $$\limsup_{t\to 0} t^{-1}\int_t^\infty |A_n^{\nu'}(r/t)|\,dr < \infty.$$

From (3.31), we have that

$$\int_t^\infty |A_n^\nu(r/t)-1|r^{-1}dr \leqslant b_{n,\nu}\int_1^\infty s^{-3/2}ds,$$

and (3.34) is established.

We next establish (3.35). From (3.23) and (3.26), we see that for $n\geqslant 2$, $A_n^{\nu'}(r)$ is a constant multiple of $A_{n-1}^{\nu+1}(r)/r^2$. But then from (3.24), we have

(3.36) $$|A_n^{\nu'}(r)| \leqslant constant/r^2 \quad \text{for } n\geqslant 2.$$

On the other hand, for $n=1$, we see from (3.23) that $A_1^{\nu'}(r)$ is a constant multiple of $\int_0^\infty e^{-s/r}J_{\nu+1}(s)s^{\nu+2}ds/r^2$. We conclude from ([7], p. 386) that

(3.37) $$|A_1^{\nu'}(r)| \leqslant \text{constant}/r^3.$$

(3.35) follows immediately from (3.36) and (3.37), and the proof of Lemma 3 is complete.

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The range of a random function defined in the unit disk

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Abstract. This is a continuation of the investigation begun in *'The distributiou of the values of a random function in the unit disk'*, Studia Mathematica 41 (1972). A family of domains is defined such that all members of the family are congruent and have the following properties. Each point $e^{i\theta}$ on the unit circumference is the apex of a member $\mathscr{D}(\theta)$ of the family and the closure of $\mathscr{D}(\theta)$ less its apex lies entirely within the unit disk. It is then shown that almost all functions of the family considered have the property that in every $\mathscr{D}$ their range at the apex of $\mathscr{D}$ is the complex plane.

§ 1. Introduction. This paper like an earlier one is concerned with the behaviour of a power series whose coefficients are random variables. As in the previous paper [3] we shall for the most part restrict ourselves to the Steinhaus family

(1.1) $$f(z,\omega) = \sum_0^\infty e^{2\pi i\vartheta_n(\omega)}a_n z^n$$

where the $\vartheta_n(\omega)$ are independent random variables uniformly distributed on the unit interval. We suppose that

(1.2) $$\limsup_{n\to\infty}(|a_n|)^{1/n} = 1$$

and

(1.3) $$\sum_0^\infty |a_n|^2 = \infty.$$

In the last paragraph we shall discuss various extensions of our results to other probability distributions.

It was shown in [3] that almost all functions of (1.1) take every value infinitely often in every sector of the unit circle. This result can conveniently be expressed in terms of the notions of cluster set and range (cf. [1] pp. 1 and 7). The cluster set of a function f at a point z_0 is defined as the set of values ζ such that to each ζ there exists a sequence $\{z_n\}$ such that $f(z_n)$ tends to ζ as z_n tends to z_0. The range of f at z_0 is defined as the set of values ζ such that to each ζ there exists a sequence $\{z_n\}$ such that z_n tends to z_0 and $f(z_n)=\zeta$. It follows from the above result that almost all functions of (1.1) have the complex plane for their range at all points z_0 on the unit circumference. A priori it follows that the cluster set at all points will be the complex sphere.

However something more is true. We write

$$\mathfrak{M}(r) = \Big(\sum_{0}^{\infty} |a_n|^2 r^{2n}\Big)^{1/2}, \tag{1.4}$$

then in view of (1.2) and (1.3) this function is an increasing function of r which tends to infinity as r tends to unity. We show that in terms of this function we can define a family of domains $\mathscr{D}$. Each of these domains has a single point z_0, its apex, on the unit circumference is symmetric about the radius vector Oz_0 and its closure less the point z_0 lies entirely within the unit disk.

We can now speak of the range $R(f, \mathscr{D}, z_0)$ of f at z_0 relative to $\mathscr{D}$. It is the set of points ζ on the w-sphere such that there exists a sequence $\{z_n\} \subset \mathscr{D}(z_0) - z_0$ for which $z_n \to z_0$ and $f(z_n) = \zeta$ for all n. We show that denoting the family (1.1) by $\mathscr{F}$ there exists a set $\mathscr{E}$ of measure zero such that for $\omega \in \mathscr{F} \setminus \mathscr{E}$ and all z_0 on the unit circumference the range $R(f, \mathscr{D}, z_0)$ is the complex plane. The domain $\mathscr{D}$ is determined solely by (1.4) and the more rapidly $\mathfrak{M}(r)$ tends to infinity the smaller we can take $\mathscr{D}$.

To define $\mathscr{D}$ we first choose $\psi_1 = \psi_1(r)$ such that

$$\mathfrak{M}(r\cos\psi_1) = \big(\mathfrak{M}(r)\big)^{1/2}$$

and then write

$$\psi = 3\max\big(\psi_1, (\log \mathfrak{M}(r)^{-1}\big) \tag{1.5}$$

so that $\psi = \psi(r)$ tends to zero as r tends to unity. If $r_0 \leqslant r < 1$ and r_0 is chosen appropriately the curve $C(z_0)$ given by the polar coordinates $(r, \arg z_0 \pm \psi)$ will lie in the unit disk. The domain $\mathscr{D}(z_0)$ is that bounded by $C(z_0)$ and $|z| = r_0$, and $\mathscr{D}$ is the family of all $\mathscr{D}(z_0)$. If inparticular $\log\mathfrak{M}(r) = \exp\big(1/(1-r)\big)$ then for $\mathscr{D}(z_0)$ we can take a sector vertex z_0 defined by

$$\arg z_0 - \pi/4 \leqslant \arg(z_0 - z) \leqslant \arg z_0 + \pi/4, \qquad |z_0 - z| < \delta.$$

We state our theorem as follows

THEOREM 1. *There is a set $\mathscr{E}$ of measure zero such that if $\mathscr{F}$ denotes the family* (1.1) *subject to* (1.2) *and* (1.3) *and if $\omega \in \mathscr{F} \setminus \mathscr{E}$ then $f(z, \omega)$ takes every complex value infinitely often in every $\mathscr{D}(z_0)$.*

The proof of this theorem uses the methods of [3] extensively and wherever the argument is similar to that used in [3] we refer the reader to that paper.

§ 2. Preliminary lemmas. We define D_α as the domain common to $|z| < r$ and $|z - r\sec\alpha| < r\tan\alpha$. The Green's function of this domain with respect to the point $r\cos\alpha$ is

$$G(z;\ r\cos\alpha) = -\log\left|\frac{r(z - r\cos\alpha)}{z(z\cos\alpha - r)}\right|.$$

If $g(z)$ has no zeros in D_α then $\log|g(z)|$ is harmonic in D_α and by Green's theorem

$$\log|g(r\cos\alpha)| = \frac{1}{2\pi}\int_C \log|g(z)|\frac{\partial G}{\partial n}\,ds$$

where C is the frontier of D_α. Putting in the value of $\frac{\partial G}{\partial n}$ we find

$$\log|g(r\cos\alpha)| = \int_{-\alpha}^{\alpha} \log|g(re^{i\theta})|K(\theta, \alpha)\,d\theta + \int_{-\pi/2+\alpha}^{\pi/2-\alpha} \log|g(z)|K(\theta, \pi/2-\alpha)\,d\theta \tag{2.1}$$

where

$$K(\theta, \alpha) = \frac{\cos\alpha}{\pi}\,\frac{\cos\theta - \cos\alpha}{1 - 2\cos\theta\cos\alpha + \cos^2\alpha} \tag{2.2}$$

and in the second integral $z = r\sec\alpha - e^{i\theta} r\tan\alpha$. We shall show that if E is an ω-set such that to each $\omega \in E$ $f(z, \omega)$ omits a value $b(\omega)$ then the measure of E cannot exceed $(\log\mathfrak{M}(r))^{-2}$. In order to show this we make use of the equality (2.1). First of all we have

LEMMA 1. *If E is any ω-set and $b(\omega)$ any measurable function of ω, satisfying $|b(\omega)| \leqslant \log\mathfrak{M}(r\cos\alpha)$, then*

$$\int_E \log|f(r\cos\alpha, \omega) - b(\omega)|\,d\mu \leqslant \big(1-\eta(r)\big)\mathfrak{M}(r\cos\alpha)\mu(E) - C\mu(E)\log\mu(E)$$

where $\eta(r)$ tends to zero as r tends to unity.

Proof. We have

$$\log|f(r\cos\alpha, \omega) - b(\omega)|\,d\mu \leqslant \log^+|f(r\cos\alpha, \omega)| + \log\log\mathfrak{M}(r\cos\alpha).$$

The desired conclusion now follows from Lemma 2.3 of [3].

LEMMA 2. *Let $\{z_j\}$, $j = 1, 2, \ldots, N$ be a set of complex numbers; $C_j^{(1)}$ the disk $|z - z_j| < \delta$ and $C_j^{(2)}$ the disk $|z - z_j| < 4\delta$. Denote by D_1 the domain $\bigcup_1^N C_j^{(1)}$ and by D_2 the domain $\bigcup_1^N C_j^{(2)}$. Suppose that, in D_2, $g(z)$ is* (i) *regular* (ii) *nowhere zero* (iii) *such that $|g(z)| \leqslant M$ and suppose further that $|g'(z_j)| \geqslant A$ for all z_j. Then we have in D_1*

$$\log|g(z)| \geqslant -4\log M + 5\log(\delta A).$$

Proof. We have

$$g'(z_j) = \frac{1}{2\pi i}\int_{|z-z_j|=\delta}\frac{g(z)}{(z-z_j)^2}\,dz$$

and so

$$\sup_{|z-z_j|=\delta}|g(z)| \geqslant \delta A.$$

Let ζ_j be a point on $|z-z_j| = \delta$ where $|g(z)|$ attains its maximum. We apply Lemma 5.3 of [3] to the disks $C_j^{(1)}$ and $C_j^{(2)}$ to obtain

$$\log|g(z)| \geqslant -4\log M + 5\log|g(\zeta_j)| \geqslant -4\log M + 5\log\delta A$$

within $|z-\zeta_j| \leqslant 2\delta$ and so within $|z-z_j| \leqslant \delta$ and therefore within D_1.

In the next Lemma we shall use D_a^+ for a domain obtained by expanding D_a by an amount 4δ all round where δ will be $(\mathfrak{M}(r))^{-1/2}$ or $(1-r)^2$ as the case may be. We use $\eta(r)$ for a quantity which tends to zero as r tends to unity and write

$$I_1(\omega) = \int_{-a}^{a}\log|f(re^{i\theta})-b(\omega)|K(\theta, a)\,d\theta \tag{2.3}$$

where $a = \pi/k$ and k is an integer given by $\pi/(k+1) < \frac{1}{3}\psi \leqslant \pi/k$ where ψ is defined by (1.5). In consequence

$$a \geqslant \left(\log\mathfrak{M}(r)\right)^{-1}. \tag{2.4}$$

We have to distinguish two cases. If $\mathfrak{M}(r)$ grows fast enough so that the hypothesis of Lemma 5.1 of [3] are satisfied we can choose r so that simultaneously

$$\left(\mathfrak{M}(r)\right)^{-1/2} \leqslant \tfrac{1}{8}(1-r), \qquad \mathfrak{M}\left(r+4\left(\mathfrak{M}(r)\right)^{-1/2}\right) \leqslant 4\mathfrak{M}(r). \tag{2.5}$$

Otherwise the hypotheses of Lemma 6.1 are satisfied and for appropriate r

$$\mathfrak{M}\left(\frac{1+r}{2}\right) \leqslant 8\mathfrak{M}(r). \tag{2.6}$$

In case (2.5) we take $\delta = \left(\mathfrak{M}(r)\right)^{-1/2}$ and in case (2.6) $\delta = (1-r)^2$. We choose points $\{z_j\}$ on the frontier of D_a equally spaced and so that

$$\delta \leqslant |z_j - z_{j-1}| < 2\delta$$

and write θ_j for $\arg z_j$ if $|z_j| = r$ and $\theta_j = \arg(r\sec a - z)$ on the remaining arc. We denote by E_2 the ω-set for which

$$\sup_j|f'(z_j, \omega)| \leqslant \left(\mathfrak{M}(r)\right)^{-1} \tag{2.7}$$

in case (2.5) and

$$\sup_j|f'(z_j, \omega)| \leqslant (1-r)^7 \tag{2.8}$$

in case (2.6). We have

LEMMA 3. *If the ω-set E is such that to each $\omega\in E$ there exists $b(\omega)$ satisfying $|b(\omega)| \leqslant \log\mathfrak{M}(r)$ and if $|f(z, \omega)-b(\omega)|$ has no zeros in D_a^+ and if $E_1 = E\setminus E_2$ then*

$$\int_{E_1} I(\omega)\,d\mu \geqslant (1-\eta)\log\mathfrak{M}(r)\,\mu(E) + C\mu(E)\log\mu(E) - \left(\mathfrak{M}(r)\right)^{-1/8}.$$

Proof. We write

$$E_{1j} = \{\omega \mid \omega\in E_1 \mid\ |f(z_j, \omega)| \geqslant \left(\mathfrak{M}(r)\right)^{1/2}\}$$

$$E_{2j} = E_1 \setminus E_{1j}$$

then, for $|z-z_j| \leqslant \delta$,

$$|f(z, \omega)-b(\omega)| \geqslant \tfrac{1}{2}|f(z_j, \omega)|$$

as in Lemmas 5.4, 6.3 and 7.3 of [3]. Whence

$$\int_{E_1} I(\omega)\,d\mu \geqslant \sum_{j=1}^{N}\int_{\theta_j-\delta}^{\theta_j+\delta}K(\theta, a)\,d\theta\int_{E_{1j}}\log\left(\tfrac{1}{2}|f(z_j, \omega)|\right)d\mu +$$
$$+\sum_{j=1}^{N}\int_{E_{2j}}d\mu\int_{\theta_j-\delta}^{\theta_j+\delta}\log|f(z, \omega)-b(\omega)|K(\theta, a)\,d\theta = \Sigma_1+\Sigma_2.$$

But by Lemma 2.1 of [3]

$$\int_{E_{1j}}\log|f(z_j, \omega)|\,d\mu \geqslant \log\mathfrak{M}(r)\,\mu(E_{1j}) + C\mu(E_{1j})\log\mu(E_{1j}).$$

And by Lemma 3.3 of [3]

$$\mu(E_{2_j}) \leqslant \left(\mathfrak{M}(r)\right)^{-1/6} \tag{2.9}$$

and so the second member is at least

$$\log\mathfrak{M}(r)\,\mu(E_1) + C\mu(E_1)\log\mu(E_1) - \left(\mathfrak{M}(r)\right)^{-1/8}.$$

Now

$$\int_{-a}^{a}K(\theta, a)\,d\theta = \left(1-\frac{2a}{\pi}\right)$$

and so

$$\sum\nolimits_1 \geqslant \left(1-\frac{2a}{\pi}\right)\log\mathfrak{M}(r)\,\mu(E_1) + C\mu(E_1)\log\mu(E_1) - \left(\mathfrak{M}(r)\right)^{-1/8}.$$

The treatment of Σ_2 differs in the two cases. In case (2.5) we use (2.9) and we apply Lemma 1 with $\delta = \left(\mathfrak{M}(r)\right)^{-1/2}$ to obtain

$$\log|f(z, \omega)-b(\omega)| \geqslant -C\log\mathfrak{M}(r) \qquad \text{for } |z-z_j| \leqslant \delta,$$

whence

$$\sum_2 \geqslant -C\left(\mathfrak{M}(r)\right)^{-1/6}\log\mathfrak{M}(r)\int_{-a}^{a}K(\theta, a)\,d\theta \geqslant -\left(\mathfrak{M}(r)\right)^{-1/8}.$$

In case (2.6) we use the argument of Lemma 7.3 of [3]. The only change is that the sum

$$\frac{1}{N}\sum_{j=1}^{N}\mu(E_{pj})$$

is increased by a factor α^{-2}, but in view of (2.4) this makes no difference to the final result.

We write

$$I_2(\omega) = \int_{-\pi/2+\alpha}^{\pi/2-\alpha} \log|f(z,\omega)-b(\omega)|K\left(\theta,\frac{\pi}{2}-\alpha\right)d\theta$$

and we have

LEMMA 4. *Under the hypotheses of Lemma 3*

$$\int_{E_1} I_2(\omega)\,d\mu \geqslant -C\alpha\log\mathfrak{M}(r)\mu(E_1) - C\alpha\big(\mathfrak{M}(r)\big)^{-1/8}.$$

Proof. In case (2.5) this follows from Lemma 2 on taking $\delta = \big(\mathfrak{M}(r)\big)^{-1/2}$ and using the inequality $K\left(\theta,\frac{\pi}{2}-\alpha\right)\leqslant\alpha$.

Case (2.6) requires the argument of Lemma 7.3 of [3]. We write

$$E_{1j} = \{\omega|\ \omega\in E_1\ \sup_{|z-z_j|\leqslant 2\delta}|f(z,\omega)-b(\omega)|\geqslant\big(\mathfrak{M}(r)\big)^{-1}\}$$

then with $\delta = (1-r)^2$ we have by Lemma 2

$$\sum_j\int_{E_{1j}}d\mu\int_{\theta_j-\delta}^{\theta_j+\delta}\log|f(z,\omega)-b(\omega)|K\left(\theta,\frac{\pi}{2}-\alpha\right)d\theta\geqslant -C\alpha\log\mathfrak{M}(r).$$

Next as in Lemma 7.3 we write

$$E_1\setminus E_{1j} = \bigcup_{p=2}^{P}E_{pj}$$

where the sets E_{pj} are disjoint and in E_{pj}

$$\Lambda_p^{-1}\leqslant\sup_{|z-z_j|\leqslant\delta}|f(z,\omega)-b(\omega)|<\Lambda_{p-1}^{-1}.$$

Also $\Lambda_1 = \big(\mathfrak{M}(r)\big)^{-1}$ and

$$\Lambda_p = 2^{p-1}\Lambda_1,\quad \Lambda_p^{-1}\leqslant(1-r)^7<\Lambda_{p-1}^{-1}.$$

Using Lemma 2 we have as in Lemma 7.3 of [3]

$$(2.10)\quad \sum_p = \sum_{j=1}^{N}\int_{E_{pj}}d\mu\int_{\theta_j-\delta}^{\theta_j+\delta}\log|f(z,\omega)-b(\omega)|K\left(\theta,\frac{\pi}{2}-\alpha\right)d\theta$$

$$\geqslant -4\pi\alpha\log\Lambda_p\frac{1}{N}\sum_{j=1}^{N}\mu(E_{pj}).$$

Write

$$z_j = r\sec\alpha - e^{i\theta_j}r\tan\alpha$$

so that for all j

$$r_j = |z_j|\geqslant r(\sec\alpha-\tan\alpha).$$

Then as in Lemma 7.3 of [3]

$$(2.11)\quad \sum_{pj}\mu(E_{pj})\leqslant\Big(\sum_{j,l}\sum\mu(E_{pj}\cap E_{pl})\Big)^{1/2}$$

and

$$\mu(E_{pj}\cap E_{pl})\leqslant C\Lambda_{p-1}^{-1/3}\left(\sum_1^\infty|a_n|^2r_j^nr_l^n\sin^2\left(\frac{\theta_j-\theta_l}{2}\right)\right)^{-1/6}.$$

If the sum in the second number is not less than Λ_{p-1}^{-1}, then

$$(2.12)\quad \mu(E_{pj}\cap E_{pl})\leqslant C\Lambda_{p-1}^{-1/6}$$

and the proof proceeds as before.

If not let a_k be the first non-vanishing term in the sequence $a_1, a_2, \ldots$ Then

$$\sum_{n=1}^{\infty}|a_n|^2r_j^nr_l^n\sin^2n\left(\frac{\theta_j-\theta_l}{2}\right)\geqslant|a_k|^2r^{2k}(\sec\alpha-\tan\alpha)^{2k}\sin^2k\left(\frac{\theta_j-\theta_l}{2}\right)$$

so that

$$\left|\sin k\left(\frac{\theta_j-\theta_l}{2}\right)\right|\leqslant\big(\Lambda_{p-1}^{1/2}|a_n|r^k(\sec\alpha-\tan\alpha)^k\big)^{-1}.$$

Since k is fixed independent of r and since α tends to zero as r tends to unity we may suppose

$$\sec\alpha-\tan\alpha>1-k^{-1}$$

and then the number of terms which satisfy the above inequality is at most

$$CN^2\Lambda_{p-1}^{-1}|a_k|^{-1}\leqslant KN^2\Lambda_{p-1}^{-1/2}$$

where K depends on $|a_k|$ but not on either r or p. Hence from (2.11) and (2.12) we have

$$\frac{1}{N}\sum_j\mu(E_{pj})\leqslant C\Lambda_{p-1}^{-1/12}$$

and on inserting this in (2.10)

$$\sum_p\Big(\sum_j\Big)\geqslant -C\alpha\sum_{p=2}^{P}\Lambda_{p-1}^{-1/12}\log\Lambda_p\geqslant -C\alpha\big(\mathfrak{M}(r)\big)^{-1/8}.$$

Whence

$$\int_E I_2(\omega)\,d\mu \geqslant -Ca\log\mathfrak{M}(r)\mu(E) - C\big(\mathfrak{M}(r)\big)^{-1/8}$$

as desired.

§ 3. Proof of Theorem 1. By Lemmas 5.1, 6.1 and 7.1 of [3] we can find a sequence $\{r_\nu\}$ such that $r_\nu \to 1$ and either

$$\big(\mathfrak{M}(r_\nu)\big)^{-1} < \left(\frac{1-r_\nu}{4}\right)^2 \quad \text{and} \quad \mathfrak{M}\Big(r_\nu + 4\big(\mathfrak{M}(r_\nu)\big)^{-1/2}\Big) \leqslant 4\mathfrak{M}(r_\nu)$$

or

$$\mathfrak{M}\left(\frac{1+r_\nu}{2}\right) < 8\mathfrak{M}(r_\nu)$$

that is so that one of the conditions (2.5) and (2.6) is satisfied for an infinity of r. With one of these values for r we divide the circumference $|z| = r$ into π/a equal arcs. If the mid points of these arcs are $\beta_1, \beta_2, \ldots,$ where $\beta_1 = 0$ then D_k is the domain bounded by the arcs

$$|z| = r_\nu, \qquad \beta_k - a \leqslant \arg z \leqslant \beta_k + a,$$

$$z = e^{i\beta_k} r(\sec a - e^{i\theta}\tan a) \qquad -\frac{\pi}{2} + a \leqslant \theta \leqslant \frac{\pi}{2} - a.$$

We show that if there is an ω-set E such that to each $\omega \in E$ $f(z, \omega)$ omits a value $b(\omega)$ in the domain D_k^+ then

$$\mu(E) \leqslant C\big(\log\mathfrak{M}(r)\big)^{-2}.$$

It will clearly be sufficient to consider one domain only namely the domain D_1 for which $\beta = 0$. With the notation of Lemmas 3 and 4 we have

$$E \subset E_1 \cup E_2.$$

By Lemma 3.3 of [3] the measure of the ω-set for which $|f'(z_k, \omega)|$ satisfies (2.7) or (2.8) is at most $\big(\mathfrak{M}(r)\big)^{-2/3}$ or $(1-r)^{7/3}\big(\mathfrak{M}(r)\big)^{-1/3}$ according to whether (2.5) or (2.6) hold. In case (2.5) the number of points z_j is of the order of $\big(\mathfrak{M}(r)\big)^{1/2}$ and so the measure of E_2 is at most of the order $\big(\mathfrak{M}(r)\big)^{-1/6}$. In case (2.6) the number of points z_j is of the order of $(1-r)^{-2}$ and so the measure of E_2 is at most of order $(1-r)^{1/3}\big(\mathfrak{M}(r)\big)^{-1/3}$.

We now have to find an upper bound for the measure of E_1. Under the hypothesis that for $\omega \in E_1$ $f(z, \omega) - b(\omega)$ does not vanish in D_a^+ for some $b(\omega)$ satisfying $|b(\omega)| \leqslant \log\mathfrak{M}(r\cos a)$ it follows from Lemmas 1, 3 and 4 that

$$(1-\eta)\mu(E_1)\log\mathfrak{M}(r\cos a) - C\mu(E_1)\log\mu(E_1) \geqslant (1-\eta)\log\mathfrak{M}(r)\mu(E_1) - \big(\mathfrak{M}(r)\big)^{-1/8}$$

where η is a quantity which tends to zero with $1-r$. Now by (1.5) $a = a(r)$ was chosen so that

$$\mathfrak{M}(r\cos a) \leqslant \big(\mathfrak{M}(r)\big)^{1/2}$$

so that we have

$$C\mu(E_1)\log\big(1/\mu(E_1)\big) \geqslant (\tfrac{1}{2}-\eta)\mu(E_1)\log\mathfrak{M}(r) - \big(\mathfrak{M}(r)\big)^{-1/8}.$$

If $\mu(E_1) \leqslant \big(\mathfrak{M}(r)\big)^{-1/8}$ there is nothing to prove. If otherwise then

$$C\log\big(1/\mu(E_1)\big) \geqslant (\tfrac{1}{2}-\eta)\log\mathfrak{M}(r)$$

or

$$\mu(E_1) \leqslant \big(\mathfrak{M}(r)\big)^{-c}$$

for some positive number c. Whence for r near enough to unity

$$\mu(E_1) \leqslant \tfrac{1}{2}\big(\log\mathfrak{M}(r)\big)^{-2}$$

and so for the domain D_1

$$\mu(E) \leqslant \big(\log\mathfrak{M}(r)\big)^{-2}.$$

If now we denote by $\mathscr{E}_\nu$ the union of these sets E for all the domains D_k corresponding to $|z| = r_\nu$ then we have in view of (2.4).

$$\mu(\mathscr{E}_\nu) \leqslant \big(\log\mathfrak{M}(r_\nu)\big)^{-1}.$$

By hypothesis $\mathfrak{M}(r)$ tends to infinity as r tends to unity and so by choosing a sub-sequence of the $\{r_\nu\}$ we can arrange that

$$\sum\big(\log\mathfrak{M}(r_\nu)\big)^{-1}$$

is convergent and so

$$\mu\Big(\bigcup_{\nu\geqslant n}\mathscr{E}_\nu\Big)$$

tends to zero. The set $\mathscr{E} = \bigcap_n\big(\bigcup_{\nu\geqslant n}\mathscr{E}_\nu\big)$ is the desired exceptional set. If $\omega \in \mathscr{F}\setminus\mathscr{E}$ then we can find r_ν such that what ever b $f(z, \omega)$ will take this value b in every domain $D_k(r_\nu)$. But whatever the value of z_0 the domain $\mathscr{D}(z_0)$ must contain at least one domain $D_k(r_\nu)$ for each ν and so whatever the value of z_0 $f(z, \omega)$ must take the value b in $\mathscr{D}(z_0)$ infinitely often. This completes the proof of Theorem 1.

§ 4. Further remarks. The results of this paper are not restricted to the family (1.1). In [2] we considered the family

$$f(z, \omega) = \sum_0^\infty a_n(\omega)z^n \tag{4.1}$$

where the $a_n(\omega)$ are independent random variables in one or two dimensions. We denote the characteristic function of a_n by $e^{i\beta_n\xi + i\gamma_n\eta}\varphi_n(\xi, \eta)$ where β_n

and γ_n are respectively the expectations of the real and imaginary parts of a_n. We write $a_n = \beta_n + i\gamma_n$ and for the variances and co-variance

$$\sigma_{n,1}^2 = V(\operatorname{Re} a_n), \quad \sigma_{n,2}^2 = V(\operatorname{Im} a_n), \quad \sigma_n^2 = \sigma_{n,1}^2 + \sigma_{n,2}^2,$$
$$\varkappa_n = \operatorname{Cov}(\operatorname{Re} a_n, \operatorname{Im} a_n).$$

We write $S_n^2(\vartheta)$ for the positive definite quadratic form

$$\tfrac{1}{2}(\sigma_{n,1}^2 \cos^2\vartheta + \sigma_{n,2}^2 \sin^2\vartheta + 2\varkappa_n \cos\vartheta \sin\vartheta).$$

We assume that the characteristic functions $\varphi_n(\xi, \eta)$ are such that for $\xi = \varrho\cos\vartheta$, $\eta = \varrho\sin\vartheta$

(i) $|\varphi_n(\xi, \eta)| \leqslant 1 - \tfrac{1}{2}s_n^2\varrho^2$ for $s_n\varrho \leqslant \delta_1$,

(ii) $|\varphi_n(\xi, \eta) \leqslant k < 1$ for $s_n\varrho \geqslant \delta_1 > 0$,

(iii) $|\varphi_n(\xi, \eta)| \leqslant M(s_n\varrho)^{-\delta}$ for all ξ, η and some $\delta > 0$.

The following theorems may be proved by the methods of [2] and [3] and the present paper.

THEOREM 2. *If $\sum_0^\infty \sigma_n^2 r^{2n}$ converges for $r < 1$ and diverges for $r \geqslant 1$, if the radius of convergence of $\sum_0^\infty a_n z^n$ is not less than 1, and if the characteristic functions $\varphi_n(\xi, \eta)$ of the independent random variables $\{a_n - a_n\}$ satisfy conditions* (i), (ii) *and* (iii) *then almost all functions of the family* (4.1) *are such that their range $R(f, z_0)$ at all points z_0 of the unit circumference is the complex plane.*

For the analogue of Theorem 1 of the present paper we replace the function $\mathfrak{M}(r)$ of (1.4) by $(\sum_0^\infty \sigma_n^2 r^{2n})^{1/2}$ and define the domains $\mathscr{D}(z_0)$ just as in §1 but in terms of this new function. We have

THEOREM 3. *If the conditions of Theorem 2 are satisfied then almost all functions* (4.1) *take every value infinitely often in every $\mathscr{D}(z_0)$.*

We conclude with one more remark. We have assumed that the coefficients of the power series are independent. This in certain cases can be replaced by the hypothesis that their differences are independent. The condition needed is that

$$\sum_1^\infty V(a_n - a_{n-1})$$

shall be divergent. Indeed

$$g(z) = (1-z)f(z) = a_0 + \sum_1^\infty (a_n - a_{n-1})z^n$$

and it can be proved just as in Theorem 1 that $g(z) - b(1-z)$ has for all b an infinity of zeros in every $\mathscr{D}(z_0)$.

References

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[2] A. C. Offord, *The distribution of zeros of power series whose coefficients are independent random variables*, Indian J. Math., 9 (1967), pp. 175–196.

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