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Singular integrals and spherical convergence

by

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Dedicated to Professor Antoni Zygmund on the occasion of his 50th year of mathematical publication

Abstract. With $K(x)$ designating a spherical harmonic kernel of Calderón–Zygmund type and letting $f(x)$ be in L^1 on the N-torus, this paper studies the connection between the convergence of the singular integral $\int f(x-y)K(y)\,dy$ and the spherical convergence of the multiple trigonometric series $\sum f^{\wedge}(m)K^{\wedge}(m)e^{i(m,x)}$.

1. Introduction. Let $f(x)$ be a real-valued 2π-periodic function in $L^1[-\pi,\pi)$, and for m an integer, set $f^{\wedge}(m) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)e^{-imx}dx$. Also, let $K(x) = x^{-1}$, and let $K^{\wedge}(x)$ designate its principal-valued Fourier transform. In particular, $K^{\wedge}(0) = 0$, and $K^{\wedge}(m) = -i(\operatorname{sgn} m)/2$. Suppose that at a fixed point x^0, there exists a positive constant A such that for $m \neq 0$

$$f^{\wedge}(m)K^{\wedge}(m)e^{imx^0}+f^{\wedge}(-m)K^{\wedge}(-m)e^{imx^0} \geqslant -A/|m|. \tag{1.1}$$

Then Hardy and Littlewood showed in [3] that a necessary and sufficient condition that

$$\lim_{R\to\infty}\sum_{|m|\leqslant R} f^{\wedge}(m)K^{\wedge}(m)e^{imx^0} = \alpha, \tag{1.2}$$

where α is finite-valued, is that

$$\Big[\lim_{R\to\infty}(2\pi)^{-1}\int_{\varepsilon\leqslant|x|\leqslant R} f(x^0-x)K(x)\,dx\Big] \to \alpha \quad \text{as } \varepsilon\to 0. \tag{1.3}$$

Motivated by our recent paper [4], we intend to show here that a similar situation prevails in Euclidean N-space, E_N, $N\geqslant 2$, when $K(x)$ is a spherical harmonic kernel of the Calderón–Zygmund type.

From now on. $x = (x_1,\ldots,x_N)$, $(x,y) = x_1y_1+\ldots+x_Ny_N$, $T_N = \{x\colon -\pi\leqslant x_j<\pi,\ j = 1,\ldots,N\}$ and

$$f^{\wedge}(m) = (2\pi)^{-N}\int_{T_N} f(x)e^{-i(m,x)}dx \tag{1.4}$$

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where m is an integral lattice point and f is a real-valued periodic function of period 2π in each variable which is in $L^1(T_N)$.

$P_n(x)$ is called a *spherical harmonic of order* n, $n \geqslant 1$, if $P_n(x)$ is an homogeneous polynomial of degree n which is harmonic in E_N. If $K(x) = P_n(x)/|x|^{N+n}$, $K(x)$ is called a *spherical harmonic kernel of Calderón–Zygmund type*. Throughout this paper, $K(x)$ will designate a kernel of this type. $K\hat{}\,(y)$ will designate the principal-valued Fourier transform of K, i.e., with $B(y, R) = \{x\colon |x-y| < R\}$,

$$K\hat{}\,(y) = \lim_{\varepsilon\to 0}\lim_{R\to\infty}(2\pi)^{-N}\int_{B(0,R)-B(0,\varepsilon)} K(x)e^{-i(x,y)}\,dx.$$

In particular, we have that

$$K\hat{}\,(0) = 0 \tag{1.5}$$

and from ([5], p. 69) that

$$K\hat{}\,(m) = \gamma(n, N)P_n(m/|m|) \quad \text{for } m \neq 0 \tag{1.6}$$

where $\gamma(n, N) = (-i)^n\Gamma(n/2)/2^N\Gamma[(n+N)/2]\pi^{N/2}$.

We note also that f real-valued implies that $[f\hat{}\,(m)K\hat{}\,(m)e^{i(m,x)} + f\hat{}\,(-m)K\hat{}\,(-m)e^{-i(m,x)}]$ is real valued. Consequently, we see that an N-dimensional version of (1.1) is the following:

There exists a positive constant A such that for $m \neq 0$

$$f\hat{}\,(m)K\hat{}\,(m)e^{i(m,x^0)} + f\hat{}\,(-m)K\hat{}\,(-m)e^{-i(m,x^0)} \geqslant -A/|m|^N. \tag{1.7}$$

Using (1.7), we intend to establish the following N-dimensional generalization of the Hardy–Littlewood result:

THEOREM 1. *Let $f(x)$ be a real-valued function which is of period 2π in each variable and which is in $L^1(T_N)$. Let $K(x)$ be a spherical harmonic kernel of Calderón–Zygmund type. Suppose that the condition* (1.7) *holds at x^0. Then a necessary and sufficient condition that*

$$\lim_{R\to\infty}\sum_{|m|\leqslant R} f\hat{}\,(m)K\hat{}\,(m)e^{i(m,x^0)} = a$$

where a is finite-valued is that

$$\Big[\lim_{R\to\infty}(2\pi)^{-N}\int_{B(0,R)-B(0,\varepsilon)} f(x^0-x)K(x)\,dx\Big] \to a \quad \textit{as } \varepsilon \to 0.$$

Comparing (1.2) and (1.3) with the corresponding statements in the above theorem, we see that this theorem is indeed an N-dimensional generalization of the Hardy–Littlewood result.

It turns out that condition (1.7) actually is not needed to establish the necessary condition of the above theorem. In particular, we shall establish the following result:

THEOREM 2. *Let $f(x)$ and $K(x)$ be as in the hypothesis of Theorem* 1 *(with condition* (1.7) *not assumed. Suppose that*

$$\lim_{R\to\infty}\sum_{|m|\leqslant R} f\hat{}\,(m)K\hat{}\,(m)e^{i(m,x^0)} = a$$

where a is finite-valued. Then

$$\Big[\lim_{R\to\infty}(2\pi)^{-N}\int_{B(0,R)-B(0,\varepsilon)} f(x^0-x)K(x)\,dx\Big] \to a \quad \textit{as } \varepsilon \to 0.$$

2. Proof of Theorem 2. With no loss in generality, we can assume $x^0 = 0$ and $f\hat{}\,(0) = 0$. Also, since Theorem 2 is true in the special case when f is a trigonometric polynomial (see [5], pp. 44–45) we can assume with no loss in generality that $a = 0$. Consequently with $K(x) = P_n(x)/|x|^{n+N}$ for $x \neq 0$, it follows from (1.6) that

$$\lim_{R\to\infty}\sum_{1\leqslant|m|\leqslant R} f\hat{}\,(m)P_n(m/|m|) = 0. \tag{2.1}$$

Next, we set for $t > 0$, n a positive integer, and $\nu = (N-2)/2$,

$$\chi_{n,\nu}(t) = \int_t^\infty J_{\nu+n}(r)/r^{\nu+1}\,dr \tag{2.2}$$

where $J_{\nu+n}(r)$ is the familiar Bessel function of the first kind of order $\nu+n$.

For $R > 0$, we set

$$S(R) = \sum_{0<|m|\leqslant R} f\hat{}\,(m)P_n(m/|m|) \tag{2.3}$$

and observe that for $R > 1$

$$\sum_{1\leqslant|m|\leqslant R} f\hat{}\,(m)P_n(m/|m|)\chi_{n,\nu}(|m|t) = \int_0^R \chi_{n,\nu}(rt)\,dS(r)$$
$$= S(R)\chi_{n,\nu}(Rt) + t^{-\nu}\int_1^R S(r)J_{\nu+n}(rt)r^{-(\nu+1)}\,dr.$$

Since $\chi_{n,\nu}(t)$ is bounded on the interval $(0, \infty)$, it follows from this last computation and from (2.1), (2.3), and well-known properties of Bessel functions that for $t > 0$

$$\lim_{R\to\infty}\sum_{1\leqslant|m|\leqslant R} f\hat{}\,(m)P_n(m/|m|)\chi_{n,\nu}(|m|t) = t^{-\nu}\int_1^\infty S(r)J_{\nu+n}(rt)r^{-(\nu+1)}\,dr. \tag{2.4}$$

Next we set for $s>0$,

$$f(x,s)=\sum_m f^\wedge(m)e^{i(m,x)}e^{-|m|s}. \tag{2.5}$$

It follows from well-known facts (see [6], p. 56) that

$$\lim_{s\to 0}\int_{B(0,R)-B(0,\varepsilon)}|f(x,s)-f(x)|\,dx=0 \quad \text{for } 0<\varepsilon<R<\infty. \tag{2.6}$$

Observing that the series in (2.5) is absolutely convergent, we have using ([7], p. 368) and ([5], p. 5) that

$$(2\pi)^{-N/2}\int_{B(0,R)-B(0,\varepsilon)} f(-x,s)K(x)\,dx = (-i)^n\sum_{m\neq 0} f^\wedge(m)P_n(m/|m|)e^{-|m|s}[\chi_{n,\nu}(|m|\varepsilon)-\chi_{n,\nu}(|m|R)]. \tag{2.7}$$

Next, we set

$$g(t)=\lim_{R\to\infty}(-i)^n\sum_{1\leqslant|m|\leqslant R} f^\wedge(m)P_n(m/|m|)\chi_{n,\nu}(|m|t) \tag{2.8}$$

and observe from (2.4) that $g(t)$ is well-defined and finite for $t>0$. Passing to the limit as $s\to 0$ in (2.7), we obtain from (2.6) and (2.8) that

$$(2\pi)^{-N/2}\int_{B(0,R)-B(0,\varepsilon)} f(-x)K(x)\,dx = g(\varepsilon)-g(R). \tag{2.9}$$

Now from (2.4) and (2.8) and the fact that $S(R)\to 0$ as $R\to\infty$, we have that

$$|g(R)|\leqslant cR^{-(\nu+1/2)} \tag{2.10}$$

where c is a constant.

We conclude from (2.9) and (2.10)

$$\lim_{R\to\infty}(2\pi)^{-N/2}\int_{B(0,R)-B(0,\varepsilon)} f(-x)K(x)\,dx = g(\varepsilon). \tag{2.11}$$

It follows from (2.4), (2.8), and (2.11) that the proof of the theorem will be complete if we show

$$\lim_{t\to 0} t^{-\nu}\int_1^\infty S(r)J_{\nu+n}(rt)r^{-(\nu+1)}\,dr=0. \tag{2.12}$$

To establish (2.12), we recall that $S(r)=o(1)$ as $r\to\infty$. Next, we note since $n\geqslant 1$, that $|J_{\nu+n}(s)|\leqslant cs^{\nu+1}$ for $0<s\leqslant 1$ where c is a constant. Consequently,

$$\left|t^{-\nu}\int_1^{1/t} S(r)J_{\nu+n}(rt)r^{-(\nu+1)}\,dr\right|\leqslant t\int_1^{1/t} o(1)\,dr=o(1) \quad \text{as } t\to 0. \tag{2.13}$$

Also,

$$t^{-\nu}\left|\int_{1/t}^\infty S(r)J_{\nu+n}(rt)r^{-(\nu+1)}\,dr\right| \leqslant t^{-(\nu+1/2)}\int_{1/t}^\infty o(1)r^{-(\nu+3/2)}\,dr=o(1) \quad \text{as } t\to 0. \tag{2.14}$$

(2.12) follows immediately from (2.13) and (2.14), and the proof to the theorem is complete.

Before passing on to the next section, we would like to point out that a close look at the proof just given will show that in dimension $N\geqslant 4$, spherical convergence can be replaced by higher orders of Bochner–Riesz summability, (see [6], p. 49). For example in dimension 4, Bochner–Riesz summability of order 1 will work. We shall deal with this and related matter in a subsequent paper.

3. Proof of the sufficiency condition of Theorem 1. We first need the following Tauberian lemma:

LEMMA 1. *Suppose that for $m\neq 0$, a_m is real-valued and that $a_m=0(m^j)$ as $|m|\to\infty$ for some non-negative integer j. Suppose furthermore that there exists a positive constant A such that $a_m\geqslant -A/|m|^N$ for $m\neq 0$. For $t>0$, set $I(t)=\sum_{m\neq 0}a_me^{-|m|t}$ and suppose also that $\lim_{t\to 0}I(t)=a$ where a is finite-valued. Then $\lim_{R\to\infty}\sum_{1\leqslant|m|\leqslant R}a_m=a$.*

To establish the above lemma, with no loss in generality we can assume $A=1$. Consequently, we have

$$I(t) \text{ is in class } C^\infty \quad \text{for } t>0; \tag{3.1}$$

$$\lim_{t\to 0}I(t)=a; \tag{3.2}$$

$$d^2I(t)/dt^2\geqslant -\sum_{m\neq 0}e^{-|m|t}/|m|^{N-2} \quad \text{for } t>0. \tag{3.3}$$

Next, we observe from the familiar Poisson summation formula ([1], p. 32), that for $t>0$

$$\sum_m e^{-|m|t}=B_N\sum_m t[t^2+|2\pi m|^2]^{-(N+1)/2} \tag{3.4}$$

where B_N is a constant. It follows from (3.4) and L'Hospital's rule that

$$\lim_{t\to 0}t^2\sum_{m\neq 0}e^{-|m|t}/|m|^{N-2} \quad \text{exists and is finite.} \tag{3.5}$$

We consequently have from (3.3) and (3.5) that there is a positive constant B_N' such that

$$d^2I(t)/dt^2\geqslant -B_N't^{-2} \quad 0<t<1. \tag{3.6}$$

But then it follows from (3.1), (3.2), (3.6) and ([2], p. 158) that

(3.7) $$\lim_{t\to 0} t\,dI(t)/dt = 0.$$

Since $A = 1$, we have by hypothesis

(3.8) $$|m|\,a_m + |m|^{-(N-1)} \geqslant 0 \quad \text{for } m \neq 0.$$

Also, we have from the (3.5) and L'Hospital's rule that there is a constant β_N such that

(3.9) $$\lim_{t\to 0} t \sum_{m\neq 0} e^{-|m|t}/|m|^{N-1} = \beta_N.$$

For $R > 0$, set $S_1(R) = \sum_{0<|m|\leqslant R} |m|\,a_m + |m|^{-(N-1)}$ and $S_2(R) = \sum_{0<|m|\leqslant R} |m|^{-(N-1)}$. Then we observe from (3.9) that

(3.10) $$\lim_{t\to 0} t \int_0^\infty e^{-rt}\,dS_2(r) = \beta_N$$

and from (3.7) and (3.9) that

(3.11) $$\lim_{t\to 0} t \int_0^\infty e^{-rt}\,dS_1(r) = \beta_N.$$

Consequently, it follows from (3.10) and ([2], p. 156 that

(3.12) $$R^{-1} \sum_{0<|m|\leqslant R} |m|^{-N-1} \to \beta_N \quad \text{as } R \to \infty$$

and from (3.8), (3.11), and ([2], p. 156) that

(3.13) $$R^{-1} \sum_{0<|m|\leqslant R} |m|\,a_m + |m|^{-(N-1)} \to \beta_N \quad \text{as } R \to \infty.$$

We conclude from (3.12) and (3.13) that

(3.14) $$\lim_{R\to\infty} R^{-1} \sum_{1\leqslant|m|\leqslant R} |m|\,a_m = 0.$$

Next, we set $S_3(R) = \sum_{0<|m|\leqslant R} |m|\,a_m$, for $R > 0$ and observe that

$$\sum_{0<|m|\leqslant R} a_m[1-e^{-|m|t}]$$
$$= R^{-1}S_3(R)[1-e^{-Rt}] - t\int_1^R S_3(r)e^{-rt}r^{-1}dr + \int_1^R S_3(r)[1-e^{-rt}]r^{-2}dr.$$

We conclude from this computation and (3.14) that

(3.15) $$\lim_{t\to 0} \sum_{0<|m|\leqslant t^{-1}} a_m[1-e^{-|m|t}] = 0.$$

Next, we observe that

$$\sum_{R<|m|} a_m e^{-|m|t} = -S_3(R)R^{-1}e^{-Rt} + t\int_R^\infty S_3(r)e^{-rt}r^{-1}dr + \int_R^\infty S_3(r)e^{-rt}r^{-2}dr.$$

We conclude from this computation and (3.14) that

(3.16) $$\lim_{t\to 0} \sum_{t^{-1}<|m|} a_m e^{-|m|t} = 0.$$

Since by hypothesis, $\lim_{t\to 0} \sum_{m\neq 0} a_m e^{-|m|t} = a$, we obtain from (3.15) and (3.16) that

$$\lim_{t\to 0} \sum_{0<|m|\leqslant t^{-1}} a_m = a,$$

and the proof to the lemma is complete.

We now proceed with the proof of the sufficiency condition of Theorem 1. If we can show

(3.17) $$\lim_{t\to 0} \sum_{m\neq 0} f^\wedge(m)K^\wedge(m)e^{i(m,x^0)}e^{-|m|t} = a$$

we are done. For

$$\sum_{m\neq 0} f^\wedge(m)K^\wedge(m)e^{i(m,x^0)}e^{-|m|t}$$
$$= 2^{-1}\sum_{m\neq 0}[f^\wedge(m)K^\wedge(m)e^{i(m,x^0)} + f^\wedge(m)K^\wedge(-m)e^{-i(m,x^0)}]e^{-|m|t}.$$

Consequently it would follow from (1.7), (3.17), and Lemma 1 that

$$\lim_{R\to\infty} \sum_{1\leqslant|m|\leqslant R} f^\wedge(m)K^\wedge(m)e^{i(m,x^0)}$$
$$= \lim_{R\to\infty} 2^{-1}\sum_{1\leqslant|m|\leqslant R}[f^\wedge(m)K^\wedge(m)e^{i(m,x^0)} + f\ \ (m)K^\wedge(-m)e^{-i(m,x^0)}] = a$$

and the sufficiency condition of Theorem 2 would be established.

(3.17) however follows immediately once the following two lemmas are established.

LEMMA 2. *Let $f(x)$ and $K(x)$ be as in the hypothesis of Theorem 1. Suppose*

(3.18) $$\Big[\lim_{R\to\infty} (2\pi)^{-N} \int_{B(0,R)-B(0,\varepsilon)} f(x^0-x)K(x)\,dx\Big] \to a \quad \text{as } \varepsilon\to 0$$

where a is finite-valued. Then

(3.19) $$\lim_{r\to 0} r^{-N} \int_{B(0,r)} f(x^0-x)|x|^N K(x)\,dx = 0.$$

LEMMA 3. *Let $f(x)$ and $K(x)$ be as in Theorem 1. Suppose (3.19) holds. Then*

$$\lim_{t\to 0}\Big\{\sum_{m\neq 0} f^\wedge(m)K^\wedge(m)e^{i(m,x^0)}e^{-|m|t} - \lim_{R\to\infty}(2\pi)^{-N}\int_{B(0,R)-B(0,t)} f(x^0-x)K(x)\,dx\Big\} = 0.$$

(Note that in the above lemmas we are not assuming that condition (1.7) holds at x^0.)

To prove Lemma 2, we let $S(0,r)$ represent the $(N-1)$-sphere with center 0 and radius r and we let $dS(x)$ represent its natural $(N-1)$-dimensional volume element. Then we define almost everywhere for $r>0$

$$h(r) = \int_{S(0,r)} f(x^0-x)K(x)\,dS(x).$$

Then h meets the conditions in the hypothesis of ([8], Lemma 7.23, p. 104). Consequently, $\int_0^r h(t)t^N dt = o(r^N)$ as $r\to 0$; (3.19) is established, and the proof to Lemma 2 is complete.

To establish Lemma 3, with no loss in generality we can assume $x^0=0$. Next for $r>0$, we set

$$g(r) = (2\pi)^{-N}\int_{B(0,r)} f(-x)|x|^N K(x)\,dx \tag{3.20}$$

and observe by assumption that

$$g(r) = o(r^N) \quad \text{as } r\to 0. \tag{3.21}$$

Now from ([5], p. 64 and p. 67), we have that

$$\sum_m f^\wedge(m)K^\wedge(m)e^{-|m|t} = \lim_{R\to\infty}\int_0^R A_n^\nu(r/t)r^{-N}dg(r) \tag{3.22}$$

where $\nu=(N-2)/2$ and

$$A_n^\nu(r) = \Gamma(n/2)\int_0^\infty e^{-s/r}J_{\nu+n}(s)s^{\nu+1}ds/2^{N/2}\Gamma[(n+N)/2]. \tag{3.23}$$

From ([5], p. 64), we also have that

$$0\leqslant A_n^\nu(r)\leqslant 1 \quad \text{for } r>0 \tag{3.24}$$

and

$$|A_n^\nu(r)|\leqslant \Gamma(N)r^N \quad \text{for } r>0. \tag{3.25}$$

Next, we set

$$A_n^{\nu\prime}(r) = dA_n^\nu(r)/dr \tag{3.26}$$

and observe from the fact that $|J_{\nu+n}(s)|\leqslant s^\nu$ that $|A_n^{\nu\prime}(r)|$ is majorized by $r^{-2}\Gamma(n/2)\int_0^\infty e^{-s/r}s^N ds/\Gamma[(n+N)/2]$. Consequently,

$$|A_n^{\nu\prime}(r)|\leqslant r^{N-1}\Gamma(n/2)\Gamma[N+1]/\Gamma[(n+N)/2]. \tag{3.27}$$

Integrating by parts and using (3.26) we next obtain the following formula:

$$\int_{r_1}^{r_2} A_n^\nu(r/t)r^{-N}dg(r) = g(r)A_n^\nu(r/t)r^{-N}\Big|_{r_1}^{r_2} + \int_{r_1}^{r_2} g(r)r^{-N}[NA_n^\nu(r/t)r^{-1} - t^{-1}A_n^{\nu\prime}(r/t)]\,dr. \tag{3.28}$$

We conclude from (3.28), (3.21), (3.24), (3.25), and (3.27) that $\int_0^t A_n^\nu(r/t)r^{-N}dg(r) = o(1)+o(1)\int_0^t r^{N-1}t^{-N}dr = o(1)$ as $t\to 0$. Consequently we have from (3.20) and (3.22) that the proof to the lemma will be complete once we show

$$\lim_{t\to 0}\Big\{\lim_{R\to\infty}\int_t^R [A_n^\nu(r/t)-1]r^{-N}dg(r)\Big\} = 0. \tag{3.29}$$

To establish (3.29), let $\eta>0$ be given. Using (3.21), choose $\delta>0$ such that

$$|g(r)|<\eta r^N \quad \text{for } 0<r\leqslant\delta. \tag{3.30}$$

Next from ([5], p. 55), we see that there is a constant $b_{n,\nu}$ (depending on n and ν) such that

$$|A_n^\nu(r)-1|\leqslant b_{n,\nu}r^{-1/2} \quad \text{for } r\geqslant 1. \tag{3.31}$$

Consequently, we have from (3.20) and (3.31) that

$$\Big|\lim_{R\to\infty}\int_\delta^R [A_n^\nu(r/t)-1]r^{-N}dg(r)\Big| \leqslant (2\pi)^{-N}b_{n,\nu}t^{1/2}\int_{E_N-B(0,\delta)} |f(-x)||K(x)||x|^{-1/2}dx. \tag{3.32}$$

But the integral on the right hand side of the inequality in (3.32) is finite. Therefore the expression on the left hand side of the inequality in (3.32) is $o(1)$ as $t\to 0$. We conclude from this fact, (3.24), (3.28), and (3.30) that (3.29) will be established once we show

$$\limsup_{t\to 0}\Big|\int_t^\delta g(r)r^{-N}\{N[A_n^\nu(r/t)-1]r^{-1} - t^{-1}A_n^{\nu\prime}(r/t)\}\,dr\Big| \leqslant c\eta \tag{3.33}$$

where c is a constant independent of δ.

From (3.30), we have that (3.33) will follow once we show both

(3.34) $$\limsup_{t\to 0}\int_t^\infty |A_n^\nu(r/t)-1|r^{-1}dr < \infty$$

and

(3.35) $$\limsup_{t\to 0} t^{-1}\int_t^\infty |A_n^{\nu\prime}(r/t)|dr < \infty.$$

From (3.31), we have that

$$\int_t^\infty |A_n^\nu(r/t)-1|r^{-1}dr \leqslant b_{n,\nu}\int_1^\infty s^{-3/2}ds,$$

and (3.34) is established.

We next establish (3.35). From (3.23) and (3.26), we see that for $n\geqslant 2$, $A_n^{\nu\prime}(r)$ is a constant multiple of $A_{n-1}^{\nu+1}(r)/r^2$. But then from (3.24), we have

(3.36) $$|A_n^{\nu\prime}(r)| \leqslant constant/r^2 \quad \text{for } n\geqslant 2.$$

On the other hand, for $n=1$, we see from (3.23) that $A_1^{\nu\prime}(r)$ is a constant multiple of $\int_0^\infty e^{-s/r}J_{\nu+1}(s)s^{\nu+2}ds/r^2$. We conclude from ([7], p. 386) that

(3.37) $$|A_1^{\nu\prime}(r)| \leqslant \text{constant}/r^3.$$

(3.35) follows immediately from (3.36) and (3.37), and the proof of Lemma 3 is complete.

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The range of a random function defined in the unit disk

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Abstract. This is a continuation of the investigation begun in *'The distributiou of the values of a random function in the unit disk'*, Studia Mathematica 41 (1972). A family of domains is defined such that all members of the family are congruent and have the following properties. Each point $e^{i\theta}$ on the unit circumference is the apex of a member $\mathscr{D}(\theta)$ of the family and the closure of $\mathscr{D}(\theta)$ less its apex lies entirely within the unit disk. It is then shown that almost all functions of the family considered have the property that in every $\mathscr{D}$ their range at the apex of $\mathscr{D}$ is the complex plane.

§ 1. Introduction. This paper like an earlier one is concerned with the behaviour of a power series whose coefficients are random variables. As in the previous paper [3] we shall for the most part restrict ourselves to the Steinhaus family

(1.1) $$f(z,\omega) = \sum_0^\infty e^{2\pi i\vartheta_n(\omega)}a_n z^n$$

where the $\vartheta_n(\omega)$ are independent random variables uniformly distributed on the unit interval. We suppose that

(1.2) $$\limsup_{n\to\infty}(|a_n|)^{1/n} = 1$$

and

(1.3) $$\sum_0^\infty |a_n|^2 = \infty.$$

In the last paragraph we shall discuss various extensions of our results to other probability distributions.

It was shown in [3] that almost all functions of (1.1) take every value infinitely often in every sector of the unit circle. This result can conveniently be expressed in terms of the notions of cluster set and range (cf. [1] pp. 1 and 7). The cluster set of a function f at a point z_0 is defined as the set of values ζ such that to each ζ there exists a sequence $\{z_n\}$ such that $f(z_n)$ tends to ζ as z_n tends to z_0. The range of f at z_0 is defined as the set of values ζ such that to each ζ there exists a sequence $\{z_n\}$ such that z_n tends to z_0 and $f(z_n)=\zeta$. It follows from the above result that