$-\frac{\alpha_2}{k}$ (see [2], page 36). Hence $f \in \mathscr{L}^q_{\alpha_1}$ and Theorem 4 applies and we have

$$\sigma_R^\beta(f, x) \to \bar{f}(x) \qquad c_{k,p} \qquad \text{a.e.}, \ \beta > \lambda_0 - 1. \tag{3.10}$$

The "dimension" of this capacity is $\leqslant k-\alpha_1 q = k(k-\alpha)/(k-\alpha_2)$. We let $\alpha_2 \to 0$ and see that the "dimension" of the set where (3.10) holds is not larger than $k-\alpha$.

Remark. The problem of Riesz–Bochner summability for Fourier series below the critical index for exceptional sets remains open. For almost everywhere results in Lebesgue measure see [7] and [8].

Remark. In Theorem 6 it is not necessary to assume $f \in L^1$, as in Theorem 4, provided one uses as the definition of $\sigma_R^\lambda(f, x)$, the Bochner integral representation (see [6]) instead of (1.2).

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Some remarks on interpolation of operators and Fourier coefficients

by

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To my teacher, Antoni Zygmund

Abstract. The weak interpolation theory is applied in this note to problems on Fourier coefficients of some special function classes which, in general, do not form linear spaces.

Introduction. The connection between the weak interpolation theory and theorems on Fourier coefficients is well known. In fact the theory of $L(p, q)$ spaces, the cornerstone of the weak interpolation theory, was motivated by the classical theorems of Hardy, Littlewood and Paley on Fourier coefficients.

Recently, we have shown [7], that the theory of weak interpolation can be generalized in a way which permits application also to problems on Fourier coefficients of special classes of functions.

In the first part of this note some results on interpolation are presented in brief, to make the exposition reasonably complete. We then present some applications of the theory to problems on Fourier coefficients. The use of interpolation and $L(p, q)$ notation make the statements and proofs of the theorems more conceptual. In most cases the theorems are also strengthened. We have therefore included theorems proved elsewhere, but by a different technique.

I. Interpolation.

DEFINITION 1. Let $\mathscr{A}$ be a vector space. A subset Q of $\mathscr{A}$ is called a *quasi-cone* (QC) iff $Q+Q \subset Q$. It is a *cone* if also $\lambda Q \subset Q$ for all $0 < \lambda$.

DEFINITION 2. Let $\mathscr{A}$ be a vector space. A quasi-norm on $\mathscr{A}$ is a function $\|\ \|$: $\mathscr{A} \to R^+$ satisfying:

(a) $\|a\| = 0$ iff $a = 0$.

(b) For all $\lambda \in C$, $a \in \mathscr{A}$, $\|\lambda a\| = |\lambda|\,\|a\|$.

(c) A number $k = k(\mathscr{A})$ exists so that

$$\|a_1+a_2\| \leqslant k(\|a_1\|+\|a_2\|) \qquad \text{all } a_1, a_2 \in \mathscr{A}.$$

A quasi-normed space is a t.v.s. whose topology is given by a quasi-norm.

DEFINITION 3. Let $(A_i, \| \|_i)$, $i = 0, 1$ be two quasi-normed spaces. If both are continuously embedded in a t.v.s. $\mathscr{A}$ we shall say that $(A_0, \| \|_0; A_1, \| \|_1; \mathscr{A})$ is an *interpolation triplet*. When $\mathscr{A}$, $\| \|_i$ are clear from the context they are usually dropped, and one talks of the *interpolation pair* (A_0, A_1).

DEFINITION 4. Let (A_0, A_1) be an interpolation pair, Q_i a QC in A_i. Let $a \in Q_0 + Q_1$. Define:

$$K(t, a; Q_0, Q_1) = \operatorname{Inf}\{\max_{i=0,1} t^i \|a_i\|_i : a_0 + a_1 = a, a_i \in Q_i\}. \tag{5}$$

DEFINITION 6. Let $0 < \theta < 1$. $0 < q \leqslant \infty$. Define:

$$\|a\|_{\theta,q,Q_i} = \left(\int_0^\infty [t^{-\theta} K(t, a; Q_0, Q_1)]^q \frac{dt}{t}\right)^{1/q} < \infty \tag{7}$$

when $q < \infty$, and

$$\|a\|_{\theta,\infty,Q_i} = \operatorname{Sup}_{0<t} t^{-\theta} K(t, a; Q_0, Q_1). \tag{8}$$

$(Q_0, Q_1)_{\theta,q}$ is now defined as the set of all elements of $Q_0 + Q_1$ so that $\| \|_{\theta,q,Q_i} < \infty$. It is easily seen that $(A_0, A_1)_{\theta,q}$ is a quasi-normed space, with $\| \|_{\theta,q} = \| \|_{\theta,q,A_i}$ serving as a quasi-norm. $(Q_0, Q_1)_{\theta,q}$ is a quasi-cone in $(A_0, A_1)_{\theta,q}$.

DEFINITION 9. Let (A_0, A_1), (B_0, B_1) be two interpolation pairs. Q_i quasi-cones in A_i, R_i in B_i. An operator $T: Q_0 + Q_1 \to R_0 + R_1$ will be called a *quasi-linear operator* from (Q_0, Q_1) to (R_0, R_1) iff K_i exist so that for every $a_0 \in Q_0$, $a_1 \in Q_1$ we can find $b_i \in R_i$ so that

$$T(a_0 + a_1) = b_0 + b_1 \tag{10}$$

$\|b_i\|_{B_i} \leqslant K_i \|a_i\|_{A_i}$.

THEOREM 11. *Let T be a quasi-linear operator from (Q_0, Q_1) to (R_0, R_1). Then for $0 < \theta < 1$, $0 < q \leqslant \infty$, T maps $(Q_0, Q_1)_{\theta,q}$ into $(R_0, R_1)_{\theta,q}$ and*

$$\|Ta\|_{\theta,q} \leqslant K_0^{1-\theta} K_1^\theta \|a\|_{\theta,q}. \tag{12}$$

DEFINITION 13. Let (A_0, A_1) be an interpolation pair. Q a quasi-cone in $A_0 + A_1$. Q will be called a *Marcinkiewicz quasi-cone* (MQC) iff, for $Q_i = Q \cap A_i$ we have

$$(Q_0, Q_1)_{\theta,q} = Q \cap (A_0, A_1)_{\theta,q} \tag{14}$$

for all $0 < \theta < 1$, $0 < q \leqslant \infty$.

THEOREM 15. *Let (A_0, A_1) be an interpolation pair. $E_i = (A_0, A_1)_{\theta_i,q_i}$ where $0 < \theta_i < 1$, $0 < q_i \leqslant \infty$ and $\theta_0 \neq \theta_1$. Then if Q is a MQC in $A_0 + A_1$, then $Q \cap (E_0 + E_1)$ is a MQC in $E_0 + E_1$.*

The following theorem gives a natural sufficient condition for a QC to be a MQC. It generalizes in the context of weak interpolation theory a theorem of Baouendi and Goulauic [2], which in turn formalizes an idea of Stein [8].

THEOREM 16. *Let Q be a QC in $A_0 + A_1$. Then if for every $a_0 \in A_0$, $a_1 \in A_1$ satisfying $a_0 + a_1 \in Q$, we can find $a_i' \in Q \cap A_i$ such that $a_0' + a_1' = a_0 + a_1$ and $\|a_i'\|_i \leqslant M \|a_i\|_i$, then Q is a MQC in (A_0, A_1).*

Proof. Let $a \in Q$. Clearly $K(t, a; A_0, A_1) \leqslant K(t, a; Q_0, Q_1)$ where $Q_i = Q \cap A_i$. Therefore

$$(Q_0, Q_1)_{\theta,q} \subset (A_0, A_1)_{\theta,q} \cap Q.$$

On the other hand if $a = a_0 + a_1$ $a_i \in A_i$, find a_i' as in hypothesis, and we have

$$K(t, a; Q_0, Q_1) \leqslant MK(t, a; A_0, A_1)$$

and we have also

$$Q \cap (A_0, A_1)_{\theta,q} \subset (Q_0, Q_1)_{\theta,q}.$$

DEFINITION 17. Let (X, Σ, μ) be a σ-finite measure space $0 \leqslant \mu$. For every measurable f we define:

$$f_*(y) = \mu\{x / |f(x)| > y\}, \qquad f^*(t) = \operatorname{Inf}\{y / f_*(y) \leqslant t\}. \tag{18}$$

For $0 < p < \infty$, $0 < q < \infty$

$$\|f\|_{p,q}^* = \left(\int_0^\infty [f^*(t)]^q t^{q/p} \frac{dt}{t}\right)^{1/q} \tag{19}$$

while for $0 < p \leqslant \infty$,

$$\|f\|_{p,\infty}^* = \operatorname{Sup}_{0<t} t^{1/p} f^*(t). \tag{20}$$

In all cases: $L(p, q) = \{f: \|f\|_{p,q}^* < \infty\}$.

DEFINITION 21. If $f \in L(p, q)$, $r < p$, $r \leqslant q$, $r \leqslant 1$, define:
If $0 < t < \mu(X)$

$$f^{**}(t) = \operatorname{Sup}\left\{\left(\frac{1}{\mu(E)} \int_E |f|^r d\mu\right)^{1/r} / t < \mu(E)\right\}. \tag{22}$$

While if $\mu(X) \leqslant t$

$$f^{**}(t) = \left(\frac{1}{t} \int_X |f|^r d\mu\right)^{1/r}. \tag{23}$$

Define also for $0 < p < \infty$, $0 < q < \infty$:

$$\|f\|_{p,q} = \left(\int_0^\infty [f^{**}(t)]^q t^{q/p} \frac{dt}{t}\right)^{1/q}, \tag{24}$$

while for $0 < p \leqslant \infty$

$$\|f\|_{p,\infty} = \sup_{0<t} t^{1/p} f^{**}(t). \tag{25}$$

It can be shown that $\| \ \|^*_{p,q} \sim \| \ \|_{p,q}$.

THEOREM 26. *We have*

$$(L(p_0, q_0), L(p_1, q_1))_{\theta,r} = L(p, r) \quad \textit{where}\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ p_0 \neq p_1.$$

We also have

$$(L(p, q_0), L(p, q_1))_{\theta,q} = L(p, q) \quad \text{where}\ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

The connection between interpolation theory and $L(p, q)$ spaces becomes clearer when we note that $\frac{1}{t} K(t, a; Q_0, Q_1)$ is monotone non-increasing and the quasi-norm on the intermediate spaces $(A_0, A_1)_{\theta,q}$ is $\left\| \frac{1}{t} K(t, a; A_0, A_1) \right\|^*_{\frac{1}{1-\theta}, q}$.

We note also, that since

$$\int_0^\infty [f^*(t)]^p\, dt = \int_X |f(x)|^p\, d\mu. \tag{27}$$

We have $L(p, p) = L^p$.

DEFINITION 28. Let (X, Σ, μ) be as above. Let a be a positive measurable function on X. Define

$$L_a(p, p) = \left\{ f\colon \left(\int |af|^p\, d\mu \right)^{1/p} < \infty \right\}.$$

THEOREM 29. (Peetre [5], Stein–Weiss, [8]). *We have*

$$(L_{a_0}(p_0, p_0), L_{a_1}(p_1, p_1))_{\theta,p} = L_{a_0^{1-\theta} a_1^{\theta}}(p, p)$$

where $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$.

THEOREM 30. *Let Q be a QC in $L(p_0, q_0) + L(p_1, q_1)$ where $p_0 \neq p_1$. Then if for every $f \in Q$ and every $0 < y$ both:*

$$f^y = \begin{cases} f & \textit{if } y < |f|, \\ 0 & \textit{if } |f| \leqslant y, \end{cases}$$

and:

$$f_y = \begin{cases} 0 & \textit{if } y < |f|, \\ f & \textit{if } |f| \leqslant y \end{cases}$$

belong to Q, then Q is a MQC.

Basic in the theory of $L(p, q)$ spaces are Hardy's inequalities:

$$\left(\int_0^\infty \left[\int_t^\infty f(s) \frac{ds}{s} \right]^q t^r \frac{dt}{t} \right)^{1/q} \leqslant q/r \left(\int_0^\infty [f(t)]^q t^r \frac{dt}{t} \right)^{1/q} \tag{31}$$

and:

$$\left(\int_0^\infty \left[\int_0^t f(s) \frac{ds}{s} \right]^q t^{-r} \frac{dt}{t} \right)^{1/q} \leqslant q/r \left(\int_0^\infty [f(t)]^q t^{-r} \frac{dt}{t} \right)^{1/q}, \tag{32}$$

where $0 \leqslant f$, $0 < r$, $1 \leqslant q < \infty$.

Either one can be derived from the other by a simple change of variable.

Using interpolation, an inequality analogous to (31), but which is sometimes stronger, can be proved.

Consider the sublinear operator

$$T\colon f \to \int_t^\infty |f(s)| \frac{ds}{s}. \tag{33}$$

By Hölder's inequality:

$$Tf(t) \leqslant \left(\frac{1}{p'-1} \right)^{1/p'} \|f\|^*_{p,p} t^{-1/p} \tag{34}$$

$\left(\frac{1}{p} + \frac{1}{p'} = 1\right.$, throughout this paper).

(34) is valid for any $1 \leqslant p < \infty$. We can express (34) by

$$T\colon L(p, p) \to L(p, \infty), \quad 1 \leqslant p < \infty. \tag{35}$$

Interpolating between two different p's we get

$$T\colon L(k, q) \to L(k, q), \quad \text{where } 1 < k < \infty,\ 0 < q \leqslant \infty. \tag{36}$$

Denoting $q/k = r$ we have shown:

$$\int_0^\infty \left(\int_t^\infty |f(s)| \frac{dt}{t} \right)^q t^r \frac{ds}{s} \Bigg)^{1/q} \leqslant C_{q,r} \left(\int_0^\infty [f^*(t)]^q t^r \frac{dt}{t} \right)^{1/q} \tag{37}$$

where $0 < q < \infty$, $0 < r < q$.

We shall apply this inequality in the sequel. It is of interest to compare (37) with Hardy's inequality. It is weaker than Hardy's when $0 < r < 1$, stronger when $1 < r < q$. In the range $0 < q < 1$ Hardy's inequality does not hold, but (37) does.

It would be of interest to find a best possible $C_{q,r}$ in (37).

II. Integrability conditions and Fourier coefficients. The $L(p, q)$ spaces we shall consider will be over $(0, \pi)$ with Lebesgue measure, or else over the positive integers with measure 1 assigned to each integer. In the second case the spaces will be denoted by $l(p, q)$ and we shall write:

(1) $$\|\{a_n\}\|^*_{p,q} = \Big(\sum_{n=1}^{\infty} a_n^{*q} n^{q/p-1}\Big)^{1/q}, \qquad \|\{a_n\}\|^*_{p,\infty} = \operatorname{Sup}_n n^{1/p} a_n^*.$$

THEOREM 2. *Let* $g \in L_{x^\theta}(1,1)$, $b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin nx\,dx$. *Then*:

(3) $$\Big\{\frac{b_n}{n}\Big\} \in l\Big(\frac{1}{1-\theta}, 1\Big)$$

and therefore:

(4) $$\sum |b_n| n^{-1-\theta} < \infty.$$

If $g(x) \in L_x(1,1)$, $0 \leqslant g(x)$ *in a right neighbourhood of* 0, *then if* (4) *holds, then* $g \in L_{x^\theta}(1,1)$.

Proof. Consider $T\colon g \to \Big\{\frac{b_n}{n}\Big\}$. We have

(5) $$|b_n| \leqslant \frac{2}{\pi}\int_0^\pi |g(x)\sin nx|\,dx \leqslant \frac{2}{\pi} n \int_0^\pi x|g(x)|\,dx$$

and:

(6) $$|b_n| \leqslant \frac{2}{\pi}\int_0^\pi |g(x)|\,dx.$$

The first inequality proves

(7) $$T\colon L_x(1,1) \to l(\infty, \infty)$$

while (6) implies $\Big(\frac{b_n}{n}\Big)^* \leqslant \frac{1}{n}\,\frac{2}{\pi}\int_0^\pi |g(x)|\,dx$ so that:

(8) $$T\colon L(1,1) \to l(1, \infty).$$

Interpolation gives

$$T\colon \big(L(1,1),\, L_x(1,1)\big)_{\theta,1} \to \big(l(1,\infty),\, l(\infty,\infty)\big)_{\theta,1}$$

that is:

$$T\colon L_{x^\theta}(1,1) \to l\Big(\frac{1}{1-\theta}, 1\Big),$$

(9) $$\sum_1^\infty \Big(\frac{b_n}{n}\Big)^* n^{-\theta} \leqslant C_\theta \int_0^\pi x^\theta |g(x)|\,dx.$$

Since $n^{-\theta}$ is decreasing we also have

$$\sum_1^\infty |b_n| n^{-1-\theta} \leqslant \sum_1^\infty \Big(\frac{b_n}{n}\Big)^* n^{-\theta}.$$

For the second part of the theorem, see ([3], 3.11) and ([3], 3.13).

THEOREM 10. *Let* $f \in L_{x^{2-2\theta}}(1,1)$, $a_n = -\frac{2}{\pi}\int_0^\pi f(x)(1-\cos nx)\,dx$. *We have then*: *For* $1/2 < \theta < 1$

(11) $$\Big\{\frac{a_n}{n}\Big\} \in l\Big(\frac{1}{2\theta-1}, 1\Big),$$

while for $0 < \theta < 1$

(12) $$\Big\{\frac{a_n}{n^2}\Big\} \in l\Big(\frac{1}{2\theta}, 1\Big).$$

(11) and (12) imply

(13) $$\sum_1^\infty |a_n| n^{2\theta-3} < \infty.$$

If $0 \leqslant f(x)$ in a right neighbourhood of 0 and (13) holds, then $f \in L_{x^{2-2\theta}}(1,1)$.

Proof. Consider $T_1\colon f \to \Big\{\frac{a_n}{n}\Big\}$. We have $\frac{|a_n|}{n} \leqslant \frac{1}{n}\,\frac{4}{\pi}\int_0^\pi |f(x)|\,dx$ so that also

(14) $$\Big(\frac{a_n}{n}\Big)^* \leqslant \frac{1}{n}\,\frac{4}{\pi}\int_0^\pi |f(x)|\,dx.$$

We also have: $|a_n| \leqslant \frac{2}{\pi}\int_0^\pi |f(x)|\cdot 2\sin^2\frac{nx}{2}\,dx$, and so:

(15) $$\frac{|a_n|}{n} \leqslant \frac{2}{\pi}\int_0^\pi |f(x)|\cdot x\,dx.$$

Therefore:

$$T_1\colon L(1,1) \to l(1,\infty) \quad \text{and} \quad T_1\colon L_x(1,1) \to l(\infty,\infty).$$

Interpolating:

$$T_1\colon L_{x^{\eta}}(1,1)\to l\left(\frac{1}{1-\eta},1\right). \tag{16}$$

Writing $2(1-\theta)=\eta$, we get for $1/2<\theta<1$

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{n}\right)^{*} n^{2\theta-2}\leqslant C_\theta\int_0^{\pi}|f(x)|x^{2-2\theta}\,dx. \tag{17}$$

Consider now $T_2\colon f\to\left\{\frac{a_n}{n^2}\right\}$. We have: $\frac{a_n}{n^2}\leqslant\frac{1}{n^2}\frac{4}{\pi}\int_0^{\pi}|f(x)|\,dx$, so that

$$\left(\frac{a_n}{n^2}\right)^{*}\leqslant\frac{1}{n^2}\frac{4}{\pi}\int_0^{\pi}|f(x)|\,dx. \tag{18}$$

We also have $|a_n|\leqslant\frac{1}{\pi}n^2\int_0^{\pi}x^2|f(x)|\,dx$, so that

$$T_2\colon L_{x^2}(1,1)\to l(\infty,\infty)$$

while from (18) we have

$$T_2\colon L(1,1)\to l\left(\frac{1}{2},\infty\right).$$

Interpolating:

$$T_2\colon L_{x^{2(1-\theta)}}(1,1)\to l\left(\frac{1}{2\theta},1\right), \tag{19}$$

that is:

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{n^2}\right)^{*} n^{2\theta-1}\leqslant C_\theta\int_0^{\pi}|f(x)|x^{2-2\theta}\,dx. \tag{20}$$

For $0<\theta\leqslant 1/2$ we have:

$$\sum_{n=1}^{\infty}|a_n|n^{2\theta-3}\leqslant\sum_{n=1}^{\infty}\left(\frac{a_n}{n^2}\right)^{*}n^{2\theta-1}\leqslant C_\theta\int_0^{\pi}|f(x)|x^{2-2\theta}\,dx$$

while for $1/2<\theta<1$ we have:

$$\sum_{n=1}^{\infty}|a_n|n^{2\theta-3}\leqslant\sum\left(\frac{a_n}{n}\right)^{*}n^{2\theta-2}\leqslant C_\theta\int_0^{\pi}|f(x)|x^{2-2\theta}\,dx.$$

For the proof of the last part of the theorem, see ([3], 3.14).

We next consider series with positive coefficients.

Theorem 21. *Let $0\leqslant\lambda_n$ be the Fourier sine or cosine coefficients of f. Let*

$$\varphi(x)=\frac{1}{x}\int_0^{x}f,\qquad \Lambda_n=\sum_{k=n}^{\infty}\frac{\lambda_k}{k},$$

then: $\Lambda_n\in l(p,q)$ iff $\varphi(x)\in L(p',q)$. Here $1<p<\infty$, $0<q\leqslant\infty$.

Proof. Assume $f(x)=\sum\lambda_n\cos nx$. $\{\Lambda_n\}$ is a non-increasing sequence. Therefore $\{\Lambda_n\}\in l(p,q)$ implies $g(x)=\sum\Lambda_n\cos nx\in L(p',q)$. We have:

$$g(x)=\sum\frac{\lambda_n}{n}\left(D_n(x)-1/2\right)$$

and so:

$$\sum\frac{\lambda_n}{n}\cdot\frac{\sin(n+1/2)x}{2\sin x/2}\in L(p',q),$$

$$\frac{\cos x/2}{2\sin x/2}\sum\frac{\lambda_n}{n}\sin nx+1/2\sum\frac{\lambda_n}{n}\cos nx\in L(p',q),$$

$$\sum\frac{\lambda_n}{n}\cos nx=\sum\Lambda_n\cos nx-\sum\Lambda_{n+1}\cos nx.$$

Since $\{\Lambda_n\}\in l(p,q)$, both series are in $L(p',q)$ and hence, so is $\sum\frac{\lambda_n}{n}\cos nx$. Therefore

$$\frac{\cos x/2}{2\sin x/2}\sum\frac{\lambda_n}{n}\sin nx\in L(p',q). \tag{22}$$

Since $x\varphi(x)=\sum\frac{\lambda_n}{n}\sin nx$ we have in a right neighbourhood of 0, $\varphi\in L(p',q)$. Away from 0 however φ is bounded, so that $\varphi\in L(p',q)$.

The proof can be reversed without many changes, and so the converse is also proved. The case of a sine series is also easily proved.

The condition $\varphi\in L(p',q)$ can be replaced by

$$\int_0^{\pi}|\varphi(x)|^q x^{q/p'-1}\,dx<\infty \tag{23}$$

which is stronger when $p'\leqslant q$, weaker when $q\leqslant p'$. To see this we note that for series with monotone coefficients

$$\int_0^{\pi}|g|^q x^{q/p'-1}\,dx<\infty\quad\text{ iff }\ g\in L(p',q).$$

(See [7]). Apply this to $g(x)$ and proceed as in the proof.

The theorem of Boas (see [4], Theorem 8):

$$f(x)x^{-\gamma}\in L^q \quad \text{implies} \quad \sum_{n=1}^{\infty} n^{q\gamma+q-2}\Big(\sum_{k=n}^{\infty} k^{-1}\lambda_k\Big)^q < \infty,$$

where $\frac{1-q}{q} < \gamma < \frac{1}{q}$, follows from Theorem 21 and the subsequent remark. In fact the condition $1 < q < \infty$ imposed there can be replaced by $1 \leqslant q < \infty$.

THEOREM 24. *Let $\{\lambda_n\}$ be the Fourier sine or cosine coefficients of $0 \leqslant f$, $f\downarrow$ on $(0, \pi)$. The following conditions are equivalent:*

(a) $f\in L(p, q)$,

(b) $\{\lambda_n\}\in l(p', q)$,

(c) $\sum|\lambda_n|^q n^{q/p'-1} < \infty$,

where $1 < p < \infty$, $1 \leqslant q \leqslant \infty$.

Proof. Consider $T: f\to\{\lambda_n\}$. Clearly

$$T: L(1,1)\to l(\infty, \infty). \tag{25}$$

Integration by parts yields

$$|\lambda_n| \leqslant \frac{c}{n}\|f\|_{\infty,\infty} \quad \text{for } f\downarrow , \tag{26}$$

so that $\lambda_n^* \leqslant \frac{c}{n}\|f\|_{\infty,\infty}$. Denoting by Q the cone of monotone non-increasing functions, we have:

$$T: L(\infty, \infty)\cap Q\to l(1, \infty). \tag{27}$$

Since Q is a Marcinkiewicz cone we get:

$$T: L(p, q)\cap Q\to l(p', q), \tag{28}$$

We could have interpreted (26) as:

$$T: L(\infty, \infty)\cap Q\to l_n(\infty, \infty) \tag{29}$$

and interpolating this with (25) gives:

$$T: L(p, \infty)\cap Q\to l_{n^{1/p'}}(\infty, \infty). \tag{30}$$

Interpolating again with

$$T: L(r, r')\cap Q\to l(r', r')$$

we have :

$$T: \big(L(p, \infty), L(r, r')\big)_{\theta,q}\cap Q\to \big(l_{n^{1/p'}}(\infty, \infty), l(r', r')\big)_{\theta,q}.$$

Taking $r' = \theta q$ we have:

$$T: L(s, q)\cap Q\to l_{n^{1/s'-1/q}}(q, q). \tag{31}$$

Therefore:

$$\Big(\sum_{n=1}^{\infty}|\lambda_n|^q n^{q/s'-1}\Big)^{1/q} \leqslant C_{s,q}\Big(\int_0^{\pi}[f(x)]^q x^{q/s-1}dx\Big)^{1/q}. \tag{32}$$

We have shown that (a) implies both (b) and (c). The proof of the converses makes use of Theorem 21. There is a difference here between the case of sine series and that of cosine series. This is so since if $g = \sum b_n \sin nx$ is monotone non-increasing, then $0 \leqslant b_n$. Invoking Theorem 21, we note that (b) implies $\{A_n\}\in l(p', q)$ by I.37 while (c) implies it by Hardy's inequality I.31. We therefore have:

$$\varphi(x) = \frac{1}{x}\int_0^x g\in L(p, q).$$

Since $0 \leqslant g$ is non-increasing, we have $0 \leqslant g \leqslant \varphi$ so that $g\in L(p, q)$, and the proof for the sine-series case is finished.

For the cosine series we define:

$$\alpha_n = \begin{cases} a_n & \text{if } a_n \geqslant 0, \\ 0 & \text{if } a_n \leqslant 0, \end{cases} \qquad \beta_n = \begin{cases} 0 & \text{if } a_n \geqslant 0, \\ -a_n & \text{if } a_n \leqslant 0, \end{cases} \tag{33}$$

so that, denoting

$$A_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{k}, \qquad B_n = \sum_{k=n}^{\infty}\frac{\beta_k}{k}. \tag{34}$$

Both (b) and (c) imply:

$$\{A_n\}\in l(p', q) \quad \text{and} \quad \{B_n\}\in l(p', q). \tag{35}$$

This, by Theorem 21, implies

$$\varphi_\alpha(x) = \frac{1}{x}\int_0^x (\Sigma\alpha_n\cos nt)\,dt\in L(p, q)$$

(36) and

$$\varphi_\beta(x) = \frac{1}{x}\int_0^x (\Sigma\beta_n\cos nt)\,dt\in L(p, q).$$

Therefore $\varphi(x) = \frac{1}{x}\int_0^x f = \varphi_\alpha - \varphi_\beta\in L(p, q)$.

Since $0 \leqslant f$ is non-increasing, $0 \leqslant f \leqslant \varphi$ and so $f \in L(p, q)$. This concludes the proof of the theorem. An alternative proof can be worked out by properly generalizing Zygmund's proof of Theorem XII 6.8 in [10].

We next consider quasi-monotone functions. Askey and Boas in [1] give two results for this class of functions. We show how these results follow from results for monotone functions via a standard decomposition. The procedure is analogous to one we introduced in [6] for quasi-monotone sequences.

DEFINITION 37. $f(x)$ is *quasi-monotone* on $(0, \pi)$ iff $0 \leqslant x^{-\beta} f(x) \downarrow$, for some $0 < \beta$.

THEOREM 38. *Let f be quasi-monotone, $0 < \beta$ such that $x^{-\beta} f(x) \downarrow$. Define*:

$$(39) \qquad f_1(x) = \int_x^\pi f(t)\frac{dt}{t}, \qquad f_2(x) = f + \beta f_1.$$

The f_i are non-negative and non-increasing.

Proof. The statement for f_1 is trivial. To see $f_2 \downarrow$ note that if $1 < a$, $0 < ax < \pi$, then $f(ax) \leqslant a^\beta f(x)$. Let $0 < x_0 < x_1 < \pi$. Then

$$f_2(x_0) - f_2(x_1) = f(x_0) - f(x_1) + \beta \int_{x_0}^{x_1} f(t)\frac{dt}{t}$$

$$\geqslant f(x_0) - f(x_1) + \beta f(x_1) \int_{x_0}^{x_1} \left(\frac{t}{x_1}\right)^\beta \frac{dt}{t} \geqslant 0.$$

Since finally $f_2(\pi) = 0$, $f_2 \geqslant 0$.

THEOREM 40. *Let f be quasi-monotone. Then $f \in L(p, q)$ iff*

$$(41) \qquad \int_0^\pi [f(t)]^q t^{q/p} \frac{dt}{t} < \infty$$

where $1 < p < \infty$, $1 \leqslant q \leqslant \infty$.

Proof. Let $f \in L(p, q)$. Define f_1, f_2 as in (39). By I.37 $f_1 \in L(p, q)$ and so $f_2 \in L(p, q)$. Since $f_i \downarrow$, it is $\int_0^\pi [f_i(t)]^q t^{q/p} \frac{dt}{t} < \infty$.

By Minkowski's inequality we have also (41). The converse is proved similarly, using I.31 rather than I.37.

THEOREM 42. *Let f be quasi-monotone, λ_n its Fourier sine or cosine coefficients, The following statements are equivalent*:

(a) $f \in L(p, q)$,

(b) $\{\lambda_n\} \in l(p', q)$,

(c) $\Sigma |\lambda_n|^q n^{q/p'-1} < \infty$.

Proof. Define f_i as in (39). Then $\lambda_n = \hat{f}_2(n) - \beta \hat{f}_1(n)$. $f_i \in L(p, q)$, $0 \leqslant f_i \downarrow$, so that by Theorem 24, $\{\hat{f}_i(n)\}$ satisfy both (b) and (c), so that λ_n do too.

Do the converse in the cosine case:

$$\frac{\pi}{2}\hat{f}_1(n) = \int_0^\pi \cos nx \int_x^\pi f(t)\frac{dt}{t}\,dx = \frac{1}{n}\int_0^\pi \frac{\sin nt}{t} f(t)\,dt$$

$$= \frac{1}{n}\int_0^\pi D_n(t) f(t)\,dt + \frac{1}{n}\int_0^\pi \left[\frac{\sin nt}{t} - \frac{\sin(n+1/2)t}{2\sin t/2}\right] f(t)\,dt$$

$$= A_n + B_n.$$

Clearly $|A_n| \leqslant \frac{C}{n}\left|\frac{1}{2}\lambda_0 + \ldots + \lambda_n\right| \leqslant C\lambda_n^{**}$, so that if λ_n satisfies either (b) or (c), so does A_n. It is easy to check that $|B_n| \leqslant \frac{C}{n}\|f\|_{1,1}^*$, so that B_n satisfies both (b) and (c). We have shown therefore that if $\{\lambda_n\}$ satisfies either (b) or (c), so does $\{\hat{f}_1(n)\}$, and since $\hat{f}_2(n) = \lambda_n + \beta \hat{f}_1(n)$ we have that $\{f_2(n)\}$ satisfies (b) or (c) too. Now $0 \leqslant f_i \downarrow$, so that by Theorem 24, $f_i \in L(p, q)$ and so $f = f_2 - \beta f_1 \in L(p, q)$. The proof is complete.

We end presenting the dual of Theorem 21.

THEOREM 43. *Let $0 \leqslant f \in L$. λ_n the Fourier sine or cosine coefficients of f. Denote*:

$$(44) \qquad \psi(x) = \int_x^\pi f(t)\frac{dt}{t}, \qquad C_n = \frac{1}{n}(\lambda_1 + \ldots + \lambda_n).$$

Then $\psi(x) \in L(p, q)$ iff $C_n \in l(p', q)$, where $1 < p < \infty$, $0 < q \leqslant \infty$.

Proof. Do the sine case. We have

$$\frac{\pi}{2}\hat{\psi}(n)$$

$$= \int_0^\pi \sin nx \int_x^\pi f(t)\frac{dt}{t}\,dx = \frac{1}{n}\int_0^\pi \frac{1-\cos nt}{t} f(t)\,dt$$

$$= \frac{1}{n}\int_0^\pi \tilde{D}_n(t) f(t)\,dt + \frac{1}{n}\int_0^\pi \left[\frac{1-\cos nt}{t} - \frac{\cos t/2 - \cos(n+1/2)t}{2\sin t/2}\right] f(t)\,dt$$

$$= C_n + E_n.$$

Clearly $|E_n| \leqslant \frac{C}{n}\|f\|_{1,1}^*$, so that $C_n \in l(p', q)$ iff $\hat{\psi}(n) \in l(p', q)$. But $0 \leqslant \psi \downarrow$, so that $\hat{\psi}(n) \in l(p', q)$ iff $\psi \in L(p, q)$ and the proof is complete.

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Singular integrals and spherical convergence

by

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Dedicated to Professor Antoni Zygmund on the occasion of his 50th year of mathematical publication

Abstract. With $K(x)$ designating a spherical harmonic kernel of Calderón–Zygmund type and letting $f(x)$ be in L^1 on the N-torus, this paper studies the connection between the convergence of the singular integral $\int f(x-y)K(y)\,dy$ and the spherical convergence of the multiple trigonometric series $\sum f^{\wedge}(m)K^{\wedge}(m)e^{i(m,x)}$.

1. Introduction. Let $f(x)$ be a real-valued 2π-periodic function in $L^1[-\pi,\pi)$, and for m an integer, set $f^{\wedge}(m) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)e^{-imx}dx$. Also, let $K(x) = x^{-1}$, and let $K^{\wedge}(x)$ designate its principal-valued Fourier transform. In particular, $K^{\wedge}(0) = 0$, and $K^{\wedge}(m) = -i(\operatorname{sgn} m)/2$. Suppose that at a fixed point x^0, there exists a positive constant A such that for $m \neq 0$

$$f^{\wedge}(m)K^{\wedge}(m)e^{imx^0} + f^{\wedge}(-m)K^{\wedge}(-m)e^{imx^0} \geqslant -A/|m|. \tag{1.1}$$

Then Hardy and Littlewood showed in [3] that a necessary and sufficient condition that

$$\lim_{R\to\infty} \sum_{|m|\leqslant R} f^{\wedge}(m)K^{\wedge}(m)e^{imx^0} = a, \tag{1.2}$$

where a is finite-valued, is that

$$\Big[\lim_{R\to\infty}(2\pi)^{-1}\int_{\varepsilon\leqslant|x|\leqslant R} f(x^0-x)K(x)\,dx\Big] \to a \quad \text{as } \varepsilon\to 0. \tag{1.3}$$

Motivated by our recent paper [4], we intend to show here that a similar situation prevails in Euclidean N-space, E_N, $N\geqslant 2$, when $K(x)$ is a spherical harmonic kernel of the Calderón–Zygmund type.

From now on. $x = (x_1, \ldots, x_N)$, $(x, y) = x_1y_1 + \ldots + x_Ny_N$, $T_N = \{x\colon -\pi\leqslant x_j<\pi,\ j = 1, \ldots, N\}$ and

$$f^{\wedge}(m) = (2\pi)^{-N}\int_{T_N} f(x)e^{-i(m,x)}dx \tag{1.4}$$

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