- \frac{1}{a_k} (see [3], page 36). Hence \( f \in \mathcal{L}_p^k \) and Theorem 4 applies and we have

\begin{equation}
\mathcal{D}_p^k(f, x) \rightarrow \mathcal{I}(x) \quad a.s., \quad \beta > \lambda_k - 1.
\end{equation}

The "dimension" of this capacity is \( \leq k - a_k \beta - k(k - a)(k - a) \). We let \( a_k \rightarrow 0 \) and see that the "dimension" of the set where (3.10) holds is not larger than \( k - a \).

**Remark.** The problem of Riesz–Bochner summability for Fourier series below the critical index for exceptional sets remains open. For almost everywhere results in Lebesgue measure see [7] and [8].

**Remark.** In Theorem 6 it is not necessary to assume \( f \in \mathcal{L}_1 \), as in Theorem 4, provided one uses as the definition of \( \mathcal{D}_p^k(f, x) \), the Bochner integral representation (see [6]) instead of (1.3).

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**References**


A quasi-normed space is a t.v.s. with topology given by a quasi-norm.

**Definition 3.** Let \((A_t, \| \cdot \|_t)\), \(t = 0, 1\) be two quasi-normed spaces. If both are continuously embedded in a t.v.s. \(\mathcal{A}\) we shall say that \((A_0, \| \cdot \|_0, A_1, \| \cdot \|_1, \mathcal{A})\) is an interpolation triplet. When \(\mathcal{A}\), \(\| \cdot \|_0\) are clear from the context they are usually dropped, and one talks of the interpolation pair \((A_0, A_1)\).

**Definition 4.** Let \((A_0, A_1)\) be an interpolation pair, \(Q\) a QC in \(A_1\). Let \(a, a_0, a_1\) be.

Define:

\[
K(t, a; Q_0, Q_1) = \inf \{ \max \{ \| \cdot \|_0, \| \cdot \|_1 \} : a_0 + a = a, a_0 \in Q_0, a_1 \in Q_1 \}.
\]

**Definition 5.** Let \(0 < \theta < 1\). Define:

\[
\| a \|_{\theta, 0, Q_0} = \left( \int_0^\infty \frac{t^{-\theta}K(t, a; Q_0, Q_1)^{\frac{1}{\theta}}}{t} \right)^{\frac{1}{1-\theta}} < \infty
\]

when \(q < \infty\), and

\[
\| a \|_{\infty, 0, Q_0} = \sup_{t > 0} t^{-\theta}K(t, a; Q_0, Q_1).
\]

\((Q_0, Q_1)_{\lambda, \theta} = \text{the set of all elements of } Q_0 + Q_1 \text{ so that } \| \cdot \|_{\lambda, \theta} < \infty. It is easily seen that } (A_0, A_1)_{\lambda, \theta} \text{ is a quasi-normed space, with } \| \cdot \|_{\lambda, \theta} = \| \cdot \|_{\lambda, \theta}, \text{ a quasi-norm of } (A_0, A_1)_{\lambda, \theta}.

**Definition 6.** Let \(0 < \theta < 1\), \(0 < q < \infty\). Define:

\[
\| a \|_{\lambda, \theta, 0, Q_0} = \sup_{t > 0} t^{-\theta}K(t, a; Q_0, Q_1)^{\frac{1}{\lambda}}.
\]

\((Q_0, Q_1)_{\lambda, \theta} = \text{for every } a \in Q_0, a_0 \in Q_0 \text{ find } a_1 \text{ as in hypothesis, and we have}

\[
K(t, a; Q_0, Q_1) \leq C K(t, a; Q_1, Q_1)\]

and we have also

\[
Q \cap (A_0, A_1)_{\lambda, \theta} = (Q_0, Q_1)_{\lambda, \theta}.
\]

**Definition 17.** Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space \(0 < \mu \). For every measurable \(f\) we define:

\[
f_*(y) = \mu([x : f(x) > y]), \quad f^*(t) = \inf \{y : f_*(y) \leq t\}.\]

For \(0 < p < \infty\), \(0 < q < \infty\),

\[
\| f \|_{p, q} = \left( \int_X f^*(t)^{q} \sup_{a \in \mathcal{A}} dt \right)^{\frac{1}{q}}
\]

while for \(0 < p \leq \infty\),

\[
\| f \|_{p, \infty} = \sup_{a \in \mathcal{A}} \| f^*(t) \|.\]

In all cases:

\[
L(p, q) = (f : \| f \|_{p, q} < \infty).
\]

**Definition 21.** If \(f \in L(p, q), r < p, r < q, r \leq 1\), define:

If \(0 < t < \mu(X)\)

\[
f**(t) = \sup \left\{ \left( \frac{1}{\mu(B)} \int |f| \, d\mu \right)^{\frac{1}{r}} : |t| < \mu(B) \right\},\]

While if \(\mu(X) \leq t\)

\[
f**(t) = \left( \frac{1}{t} \int |f| \, d\mu \right)^{\frac{1}{r}}
\]

Define also for \(0 < p < \infty\), \(0 < q < \infty\):

\[
\| f \|_{p, q} = \left( \int |f**(t)|^{q} \sup_{a \in \mathcal{A}} dt \right)^{\frac{1}{q}}
\]
while for $0 < p < \infty$

$$\|f\|_{p_0, p} = \sup_{t \in \mathbb{R}} f^{*}(t).$$

It can be shown that $\| \|_{p_0, p} \sim \| f \|_{p_0, p}$.

**Theorem 29.** We have

$$\|L(p_0, q_0) = L(p_1, q_1)\|_{h_{p_0}, p_1} = L(p, r)$$

where $\frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1}, p_0 \neq p_1$.

We also have

$$\|L(p_0, q_0) = L(p, q)\|_{h_{p_0}, q_1} = L(p, q)$$

where $\frac{1}{q} = \frac{1}{q_1} + \frac{\theta}{q_1}$.

The connection between interpolation theory and $L(p, q)$ spaces becomes clearer when we note that $\frac{1}{t} E(t, a; q_0, q_1)$ is monotone non-increasing and the quasi-norm on the intermediate spaces $(A_{p_0}, A_{p_1})_{h_{p_0}, p_1}$ is

$$\| \| = \frac{1}{t} E(t, a; A_{p_0}, A_{p_1})_{h_{p_0}, p_1}.$$

We note also, that since

$$\int \frac{1}{t} \left[ f^{*}(t) \right]^{p_0} dt = \int f^{*}(s) \frac{ds}{s},$$

we have $L(p, p) = L^{p}.$

**Definition 28.** Let $(X, \Sigma, \mu)$ be as above. Let $a$ be a positive measurable function on $X$. Define

$$L_{a}(p, p) = \left\{ f : \left( \int |f|^{p} \mu \right)^{\frac{1}{p}} < \infty \right\}.$$

**Theorem 29.** (Peetre [5], Stein–Weiss, [8]). We have

$$\|L_{a}(p_0, p_0) = L_{a}(p_1, p_1)\|_{h_{p_0}, p_1} = L_{a}(p, p)$$

where $\frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1}.$

**Theorem 30.** Let $Q$ be a QC in $L(p_0, q_0) + L(p_1, q_1)$ where $p_0 \neq p_1$.

Then if for every $f \in Q$ and every $0 < y$ both:

$$f^{*} = \begin{cases} f & \text{if } y < |f|, \\ 0 & \text{if } |f| \leq y, \end{cases}$$

and:

$$f_{r} = \begin{cases} 0 & \text{if } y < |f|, \\ \frac{f}{y} \text{ if } |f| \leq y \end{cases}$$

belong to $Q$, then $Q$ is a MQC.

Basic in the theory of $L(p, q)$ spaces are Hardy’s inequalities:

$$\left( \int \frac{1}{t} \left[ \int f(t)^{q} \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{\frac{1}{r}} \leq q/r \left( \int \frac{1}{t} \left[ f(t)^{p} t^{q} \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{\frac{1}{r}}$$

and:

$$\left( \int \frac{1}{t} \left[ \int f(t)^{q} \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{\frac{1}{r}} \leq q/r \left( \int \frac{1}{t} \left[ f(t)^{p} t^{r} \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{\frac{1}{r}},$$

where $0 < f, 0 < r, 1 < q < \infty$.

Either one can be derived from the other by a simple change of variable.

**Using interpolation, an inequality analogous to (31), but which is sometimes stronger, can be proved.**

Consider the sublinear operator

$$T : f \rightarrow \int \frac{1}{t} \left[ \int f(s)^{q} \frac{ds}{s} \right]^{\frac{1}{q}} \frac{dt}{t}.$$

By Hölder’s inequality:

$$\left( \int \frac{1}{t} \left[ \int f(t)^{q} \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{\frac{1}{r}} \leq q/r \left( \int \frac{1}{t} \left[ f(t)^{p} t^{r} \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{\frac{1}{r}}$$

$$\left( \int \frac{1}{t} \left[ \int f(t)^{q} \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{\frac{1}{r}} \leq q/r \left( \int \frac{1}{t} \left[ f(t)^{p} t^{r} \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{\frac{1}{r}},$$

where $0 < f, 0 < r, 1 < q < \infty$.

We shall apply this inequality in the sequel. It is of interest to compare (37) with Hardy’s inequality. It is weaker than Hardy’s when $0 < r < 1,$
stronger when \( 1 < r < q \). In the range \( 0 < q < 1 \) Hardy's inequality does not hold, but (37) does.

It would be of interest to find a best possible \( C_{pr} \) in (37).

II. Integrability conditions and Fourier coefficients. The \( L(p, q) \) spaces we shall consider will be over \((0, \pi)\) with Lebesgue measure, or else over the positive integers with measure 1 assigned to each integer. In the second case the spaces will be denoted by \( L(p, q) \) and we shall write:

\[
\|(a_n)\|^*_p = \left( \sum_{n=1}^{\infty} a_n^{2r-1} \right)^{1/2r}, \quad \|(a_n)\|^*_\infty = \sup_{n} |a_n|.
\]

**Theorem 2.** Let \( g \in L^{2}(1, 1) \), \( b_n = \frac{2}{\pi} \int_{0}^{\pi} g(x) \sin nx \, dx \). Then:

\[
\left( \frac{b_n}{n} \right)^* \leq \frac{1}{1 - 6^{-1}} \left( \frac{1}{1 - 6^{-1}} \right)
\]

and therefore:

\[
\sum |b_n| n^{-1-\delta} < \infty.
\]

If \( g(x) \in L^{2}(1, 1) \), \( 0 \leq g(x) \) in a right neighbourhood of 0, then if (4) holds, then \( g \in L^{2}(1, 1) \).

**Proof.** Consider \( T : g \rightarrow \left( \frac{g}{n} \right) \). We have

\[
|b_n| \leq \frac{2}{\pi} \int_{0}^{\pi} |g(x)| \sin nx \, dx \leq \frac{2}{\pi} \int_{0}^{\pi} |g(x)| \, dx
\]

and:

\[
|b_n| \leq \frac{2}{\pi} \int_{0}^{\pi} |g(x)| \, dx.
\]

The first inequality proves

\[
T : L^{2}(1, 1) \rightarrow I(\infty, \infty)
\]

while (6) implies \( \left( \frac{b_n}{n} \right)^* \leq \frac{1}{1 - 6^{-1}} \left( \frac{1}{1 - 6^{-1}} \right) \) so that:

\[
T : L^{2}(1, 1) \rightarrow I(1, \infty).
\]

Interpolation gives

\[
T : (L^{2}(1, 1), L^{2}(1, 1))_{1,1} \rightarrow I(1, \infty), I(\infty, \infty)).
\]

that is:

\[
T : L^{2}(1, 1) \rightarrow I\left( \frac{1}{1 - \theta}, 1 \right),
\]

\[
\sum_{n=1}^{\infty} \left( \frac{b_n}{n} \right)^* n^{-\delta} \leq C_{\theta} \int_{0}^{\pi} x^{d} |g(x)| \, dx.
\]

Since \( n^{-\delta} \) is decreasing we also have

\[
\sum_{n=1}^{\infty} \left( \frac{b_n}{n} \right)^* n^{-\delta} \leq \sum_{n=1}^{\infty} \left( \frac{b_n}{n} \right)^* n^{-\delta}.
\]

For the second part of the theorem, see ([33], 3.11) and ([33], 3.13).

**Theorem 10.** Let \( f \in L^{2}(1, 1) \), \( a_n = -\frac{2}{\pi} \int_{0}^{\pi} f(x)(1 - \cos nx) \, dx \). We have then: For \( 1/2 < \theta < 1 \)

\[
\left( \frac{a_n}{n} \right)^* \leq \frac{1}{1 - 6^{-1}} \left( \frac{1}{1 - 6^{-1}} \right)
\]

while for \( 0 < \theta < 1 \)

\[
\left( \frac{a_n}{n} \right)^* \leq \frac{1}{1 - 6^{-1}} \left( \frac{1}{1 - 6^{-1}} \right).
\]

(11) and (12) imply

\[
\sum_{n=1}^{\infty} |a_n| n^{\delta} < \infty.
\]

If \( 0 \leq f(x) \) in a right neighbourhood of 0 and (13) holds, then \( f \in L^{2}(1, 1) \).

**Proof.** Consider \( T_1 : f \rightarrow \left( \frac{a_n}{n} \right) \). We have

\[
\left( \frac{a_n}{n} \right)^* \leq \frac{1}{n} \int_{0}^{\pi} |f(x)| \, dx.
\]

We also have:

\[
|a_n| \leq \frac{2}{\pi} \int_{0}^{\pi} |f(x)| \cdot 2 \sin \frac{nx}{2} \, dx,
\]

and so:

\[
\left( \frac{a_n}{n} \right)^* \leq \frac{2}{\pi} \int_{0}^{\pi} |f(x)| \cdot 2 \, dx.
\]

(15)

Therefore:

\[
T_1 : L^{2}(1, 1) \rightarrow I(1, \infty) \quad \text{and} \quad T_1 : L^{2}(1, 1) \rightarrow I(\infty, \infty).
\]
Interpolating:

\[ T_1 : L^p(1, 1) \rightarrow L \left( \frac{1}{1 - \eta}, 1 \right). \]

Writing \( 2(1 - \theta) = \eta \), we get for \( 1/2 < \theta < 1 \)

\[ \sum_{n=1}^{\infty} \left( \frac{a_n}{n^\eta} \right)^* \leq C_0 \int_0^\infty |f(x)| x^{-2\theta} \, dx. \]

Consider now \( T_2 : f \rightarrow \left\{ \frac{a_n}{n^\eta} \right\} \). We have:

\[ \frac{a_n}{n^\eta} \leq \frac{1}{n^\eta} \frac{4}{\pi} \pi \int_0^\infty |f(x)| x^{-2\theta} \, dx. \]

We also have \( |a_n| \leq \frac{1}{n^\eta} \frac{4}{\pi} \pi \int_0^\infty |f(x)| x^{-2\theta} \, dx \), so that

\[ \frac{\left( \frac{a_n}{n^\eta} \right)^*}{\left( \frac{a_n}{n^\eta} \right)^*} \leq \frac{4}{\pi} \frac{\pi}{\pi} \int_0^\infty |f(x)| x^{-2\theta} \, dx. \]

while from (18) we have

\[ T_2 : L^p(1, 1) \rightarrow L \left( \frac{1}{2}, \infty \right). \]

Interpolating:

\[ T_1 : L^\infty(1, 1) \rightarrow L \left( \frac{1}{2\theta}, 1 \right), \]

that is:

\[ \sum_{n=1}^{\infty} \left( \frac{a_n}{n^\eta} \right)^* \leq C_0 \pi \int_0^\infty |f(x)| x^{-2\theta} \, dx. \]

For \( \theta < \eta \leq 1/2 \) we have:

\[ \sum_{n=1}^{\infty} |a_n| x^{-2\theta} \leq \sum_{n=1}^{\infty} \left( \frac{a_n}{n^\eta} \right)^* \leq C_0 \pi \int_0^\infty |f(x)| x^{-2\theta} \, dx \]

while for \( 1/2 < \theta < 1 \) we have:

\[ \sum_{n=1}^{\infty} |a_n| x^{-2\theta} \leq \sum_{n=1}^{\infty} \left( \frac{a_n}{n^\eta} \right)^* \leq C_0 \pi \int_0^\infty |f(x)| x^{-2\theta} \, dx. \]

For the proof of the last part of the theorem, see ([33], 3.14).

We next consider series with positive coefficients.

**Theorem 21.** Let \( 0 \leq \lambda_n \) be the Fourier sine or cosine coefficients of \( f \).

Let

\[ \varphi(x) = \frac{1}{\pi} \int_0^x f(t) \, dt, \quad A_n = \sum_{k=n}^{\infty} \frac{\lambda_k}{k}, \]

then: \( A_n \in L(p, q) \iff \varphi(x) \in L(p', q) \). Here \( 1 < p < \infty, 0 < q \leq \infty \).

Proof. Assume \( f(x) = \sum A_n \cos \lambda x \). \( \{ A_n \} \) is a non-increasing sequence. Therefore \( \varphi(x) = \sum A_n \cos \lambda x \) implies \( g(x) = \sum A_n \cos \lambda x \).

We have:

\[ g(x) = \sum \lambda_n \pi \int_0^{x} \varphi(x) \, dx. \]

and so:

\[ \sum \lambda_n \pi \int_0^{x} \varphi(x) \, dx \leq \frac{1}{2} \sum \lambda_n \pi \int_0^{x} \varphi(x) \, dx. \]

We also have \( |a_n| \leq \frac{1}{n^\eta} \frac{4}{\pi} \pi \int_0^\infty |f(x)| x^{-2\theta} \, dx \), so that

\[ \sum \lambda_n \pi \int_0^{x} \varphi(x) \, dx \leq \frac{1}{2} \sum \lambda_n \pi \int_0^{x} \varphi(x) \, dx. \]

Since \( \varphi(x) \in L(p, q) \), both series are in \( L(p', q) \) and hence, so is

\[ \sum \lambda_n \cos \lambda x. \]

Therefore

\[ \sum \lambda_n \cos \lambda x \]

Since \( \lambda_n \sum \lambda_n \sin nx \) we have in a right neighbourhood of 0, \( \varphi(x) \in L(p', q) \). Away from 0 however \( \varphi \) is bounded, so that \( \varphi(x) \in L(p', q) \).

The proof can be reversed without many changes, and so the converse is also proved. The case of a cosine series is also easily proved.

The condition \( \varphi(x) \in L(p', q) \) can be replaced by

\[ \int_0^\infty |\varphi(x)| x^{-1} \, dx < \infty \]

which is stronger when \( p' \leq q' \), weaker when \( q \leq p' \). To see this we note that for series with monotone coefficients

\[ \int_0^\infty |g(x)| x^{-1} \, dx < \infty \quad \text{iff} \quad g(x) \in L(p', q). \]

(See [7]). Apply this to \( g(x) \) and proceed as in the proof.
The theorem of Boas (see [4], Theorem 8):
\[ f(x) = x^{-e} e^x \]
implies
\[ \sum_{n=1}^{\infty} \frac{1}{n^{p+q+1}} \sum_{k=1}^{\infty} \frac{1}{k^{p+q-1}} \lambda_k < \infty, \]
where \( 1 - q < \gamma < \frac{1}{q} \), follows from Theorem 21 and the subsequent remark. In fact the condition \( 1 < q < \infty \) imposed can be replaced by \( 1 \leq q < \infty \).

**Theorem 24.** Let \( \{\lambda_k\} \) be the Fourier sine or cosine coefficients of \( 0 < f \), \( f \neq 0 \) on \((0, \pi)\). The following conditions are equivalent:

(a) \( f \neq L(p, q) \),
(b) \( \{\lambda_k\} \in l(p', q') \),
(c) \( \sum_{k=1}^{\infty} \frac{1}{k^{p+q-1}} < \infty \),

where \( 1 < p < \infty, 1 < q < \infty \).

**Proof.** Consider \( T: f \rightarrow \{\lambda_k\} \). Clearly
\[ T: L(1, 1) \rightarrow l(1, \infty) . \]

Integration by parts yields
\[ |\lambda_k| \leq \frac{1}{n} \|f\|_{l(n)} \text{ for } f \neq 0, \]
so that \( \lambda_k \leq \frac{1}{n} \|f\|_{l(n)} \). Denoting by \( Q \) the cone of monotone non-increasing functions, we have:
\[ T: L(\infty, \infty) \cap Q \rightarrow l(1, \infty) . \]

Since \( Q \) is a Marenkiewicz cone we get:
\[ T: L(p, \infty) \cap Q \rightarrow l(p', q'), \]

We could have interpreted (26) as:
\[ T: L(\infty, \infty) \cap Q \rightarrow l(1, \infty) . \]

and interpolating this with (25) gives:
\[ T: L(p, \infty) \cap Q \rightarrow l(p', q') . \]

Interpolating again with
\[ T: L(\infty, \infty) \cap Q \rightarrow l(p', q') . \]

we have:
\[ T: L(\infty, \infty) \cap Q \rightarrow l(p', q') . \]

Taking \( r' = \delta q \) we have:
\[ T: L(\delta, q) \cap Q \rightarrow l(\delta q, q) . \]

Therefore:
\[ \left( \sum_{k=1}^{\infty} \frac{1}{k^{p+q-1}} \right)^{1/2} \leq C_{\delta q} \left( \int_{\delta}^{x} \left( f(x) \right)^{p+q-1} dx \right)^{1/2} . \]

We have shown that (a) implies both (b) and (c). The proof of the converse makes use of Theorem 21. There is a difference here between the case of sine series and that of cosine series. This is so since if \( g = \sum b_n \sin \pi x \) is monotone non-increasing, then \( 0 \leq b_n \). Invoking Theorem 21, we note that (b) implies \( (A_n) \in l(p', q') \) by 1.37 while (c) implies it by Hardy's inequality 1.31. We therefore have:
\[ \varphi(x) = \frac{1}{x^d} \int_{\delta}^{x} g \ast L(p, q) . \]

Since \( 0 \leq g \) is non-increasing, we have \( 0 \leq \varphi \leq \varphi \) so that \( g \ast L(p, q) \), and the proof for the sine-series case is finished.

For the cosine series we define:
\[ a_n = \begin{cases} a_n & \text{if } a_n \geq 0, \\ 0 & \text{if } a_n < 0, \end{cases} \]
\[ b_n = \begin{cases} b_n & \text{if } a_n \geq 0, \\ -a_n & \text{if } a_n < 0, \end{cases} \]

so that, denoting
\[ A_n = \sum_{k=1}^{\infty} a_k, \quad B_n = \sum_{k=1}^{\infty} b_k . \]

Both (b) and (c) imply:
\[ \{A_n\} \in l(p', q') \quad \text{and} \quad \{B_n\} \in l(p', q') . \]

This, by Theorem 21, implies
\[ \varphi_a(x) = \frac{1}{x^d} \left( \sum a_n \cos nx \right) dt \ast L(p, q) \]

and
\[ \varphi_b(x) = \frac{1}{x^d} \left( \sum b_n \cos nx \right) dt \ast L(p, q) . \]

Therefore \( \varphi(x) = \frac{1}{x^d} \int_{\delta}^{x} f \ast \varphi \ast L(p, q) . \)
Since \(0 \leq f\) is non-increasing, \(0 \leq f \leq \varphi\) and so \(f \in L(p, q)\). This concludes the proof of the theorem. An alternative proof can be worked out by properly generalizing Zygmund’s proof of Theorem XII.6.8 in [10].

We next consider quasi-monotone functions. Askey and Boas in [1] give two results for this class of functions. We show how these results follow from results for monotone functions via a standard decomposition. The procedure is analogous to one we introduced in [6] for quasi-monotone sequences.

**Definition 37.** \(f(x)\) is quasi-monotone on \((0, a)\) if \(0 \leq f(x) \leq \varphi\) for some \(0 < \beta < a\).

**Theorem 38.** Let \(f\) be quasi-monotone, \(0 < \beta\) such that \(\varphi \perp f(x) \perp\) \(f(x)\).

Define:

\[
    f_1(x) = \int_0^x f(t) \frac{dt}{t}, \quad f_2(x) = f + \beta f_1.
\]

The \(f_i\) are non-negative and non-increasing.

**Proof.** The statement for \(f_2\) is trivial. To see \(f_2\) note that if \(1 < a, 0 < a < \pi\), then \(f(a) \leq f(x) \leq f(a)\). Let \(0 < x < x_1 < \pi\). Then

\[
    f_1(x_1) - f_2(x_1) = \int_{x_1}^x f(t) \frac{dt}{t} \geq 0.
\]

Since finally \(f_2(x_1) = 0\), \(f_2 \geq 0\).

**Theorem 39.** Let \(f\) be quasi-monotone. Then \(f \in L(p, q)\) iff

\[
    \int_0^\infty \frac{[f(t)]^p \omega^q dt}{t} < \infty
\]

where \(1 < p, q < \infty\), \(1 < q < \infty\).

**Proof.** Let \(f \in L(p, q)\). Define \(f_1, f_2\) as in (39). By L37 \(f_1 \in L(p, q)\) and so \(f_2 \in L(p, q)\). Since \(f_2 \perp\), it is

\[
    \int_0^\infty \frac{[f(t)]^p \omega^q dt}{t} < \infty.
\]

By Minkowski’s inequality we have also (41). The converse is proved similarly, using L31 rather than L37.

**Theorem 40.** Let \(f\) be quasi-monotone, \(\lambda_n\) its Fourier sine or cosine coefficients. The following statements are equivalent:

(a) \(f \in L(p, q)\),
(b) \(\lambda_n \in L(p', q')\),
(c) \(\sum |\lambda_n| \varphi^{q' - 1} < \infty\).

**Proof.** Define \(f_1\) as in (39). Then \(\lambda_n = f_1(n) - \beta f_1(n)\). \(f_1 \in L(p, q)\), \(0 < f_1 \perp\), so that by Theorem 24, \(\{f_1(n)\}\) satisfy both (b) and (c), so that \(\lambda_n\) do too.

Do the converse in the cosine case:

\[
    \frac{\pi}{2} f_1(n) = \int_0^\pi \cos n \int f(t) \frac{dt}{t} \frac{dx}{x} = \int_0^\pi \sin t \int f(t) \frac{dt}{t} \sin t dt = \frac{1}{n} \int_0^\pi \sin n \int f(t) \frac{dt}{t} \sin t dt = \frac{\pi}{2} D_n(f(t)) dt = \frac{1}{n} \int_0^\pi \sin n \int \sin(n + 1/2) t \int f(t) dt \sin t dt = A_n + B_n.
\]

Clearly \(|A_n| \leq \frac{\pi}{2} \lambda_0 + \ldots + \lambda_n \leq C \lambda_n\), so that if \(\lambda_n\) satisfies either (b) or (c), so does \(A_n\). It is easy to check that \(|B_n| \leq \frac{\pi}{2} \lambda_n\), so that \(B_n\) satisfies both (b) and (c). We have shown therefore that if \(\lambda_n\) satisfies either (b) or (c), so does \(f_1(n)\), and since \(f_2(n) = \lambda_n + \beta f_1(n)\) we have that \(f_2(n)\) satisfies (b) or (c) too. Now \(0 < f_2 \perp\), so that by Theorem 24, \(f_2 \in L(p, q)\) and so \(f = f_1 - \beta f_1 \in L(p, q)\). The proof is complete.

We end presenting the dual of Theorem 21.

**Theorem 43.** Let \(0 < f \in L. \lambda_n\) the Fourier sine or cosine coefficients of \(f\).

Denote:

\[
    \psi(x) = \int_0^x f(t) \frac{dt}{t}, \quad C_n = \frac{1}{n} (\lambda_1 + \ldots + \lambda_n).
\]

Then \(\psi(x) \in L(p, q)\) iff \(C_n \in L(p', q')\), where \(1 < p < \infty, 0 < q < \infty\).

**Proof.** Do the sine case. We have

\[
    \int_0^\infty \frac{[f(t)]^p \omega^q dt}{t} = \int_0^\infty \frac{[\psi(t)]^p \omega^q dt}{t} = \int_0^\infty \frac{[\psi(t)]^p \omega^q dt}{t} = \int_0^\infty \frac{[\psi(t)]^p \omega^q dt}{t} = \int_0^\infty \frac{[\psi(t)]^p \omega^q dt}{t} = \int_0^\infty \frac{[\psi(t)]^p \omega^q dt}{t}.
\]

Clearly \(|A_n| \leq \frac{\pi}{2} \lambda_n\), so that \(C_n \in L(p', q')\) iff \(\psi(x) \in L(p', q')\). But \(0 < \psi \perp\), so that \(\psi(x) \in L(p', q')\) iff \(\psi \in L(p, q)\) and the proof is complete.
Singular integrals and spherical convergence

by

VICTOR L. SHAPIRO*
(Riverside, Calif.)

Dedicated to Professor Antoni Zygmund on the occasion of his 60th year of mathematical publication

Abstract. With $K(x)$ designating a spherical harmonic kernel of Calderón–Zygmund type and letting $f(x)$ be in $L^p$ on the $N$-torus, this paper studies the connection between the convergence of the singular integral $\mathcal{I}(f(x)) = \int f(x-y)K(y)dy$ and the spherical convergence of the multiple trigonometric series $\sum_{m \in \mathbb{Z}^N} f^{(m)}(m)\xi(m)\mathbf{e}^{in\cdot\xi}$.

1. Introduction. Let $f(x)$ be a real-valued $2\pi$-periodic function in $L^1(-\pi, \pi)$, and for $m$ an integer, set $f^{(m)}(m) = (2\pi)^{-1} \int f(x)e^{-imx}dx$. Also, let $K(x) = x^{-1}$, and let $K^*(x)$ designate its principal-valued Fourier transform. In particular, $K^*(0) = 0$, and $K^*(m) = -i(\text{sgn} m)/2$. Suppose that at a fixed point $x_0$, there exists a positive constant $A$ such that for $m \neq 0$

$$f^{(m)}K^*(m)e^{imx} + f^{(-m)}K^*(-m)e^{-imx} \geq -A|m|.$$

Then Hardy and Littlewood showed in [3] that a necessary and sufficient condition that

$$\lim_{m \to \infty} \sum_{m \in \mathbb{Z}^N} f^{(m)}K^*(m)e^{imx} = a,$$

where $a$ is finite-valued, is that

$$\lim_{x \to \infty} (2\pi)^{-1} \int_{|x|<\infty} f(z^0 - x)K(x)dx \to a \quad \text{as} \quad x \to 0.$$

Motivated by our recent paper [4], we intend to show here that a similar situation prevails in Euclidean $N$-space, $\mathbb{R}^N$, $N \geq 2$, when $K(x)$ is a spherical harmonic kernel of the Calderón–Zygmund type.

From now on, $x = (x_1, \ldots, x_N)$, $(x, y) = x_1y_1 + \cdots + x_Ny_N$, $T_N = [-\pi < x < \pi]$, and

$$f^{(m)}(m) = (2\pi)^{-N} \int f(x)e^{-im\cdot\xi}dx,$$

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