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A note on a generalized hypersingular integral

by

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Abstract. We study integral transforms related to hypersingular integrals and certain Marcinkiewicz integrals, but which have less stringent homogeneity requirements. We prove the results in the context of the Lebesgue spaces with mixed homogeneity of C. Sadosky and M. Cotlar.

Introduction. In this note, we will extend the results of [7] to transforms of the form

$$\text{p.v.}\int [f(x-z)-f(x)]\,d\mu(z),$$

where μ is a complex measure satisfying the condition $\int\limits_{|z|>\delta} |d\mu(z)| = O(\delta^{-a})$, $\delta > 0$, and f belongs to the Lebesgue (Sobolev) space L_a^p, $0 < a < 1$. An example of such a measure is $d\mu(z) = \dfrac{\Omega(z')}{|z|^{n+a+i\gamma}}\,dz$, where $z \in R^n$, $|z'| = 1$ and Ω is integrable over $|z'| = 1$.

That such an extension might be possible is clearly indicated by results in the paper [3] of E. H. Ostrow and E. M. Stein which correspond essentially to the case $n = 1$ and $a = 1$. The method can easily be adapted to the Lebesgue spaces with mixed homogeneity introduced in [5], and we shall present the results in this form. We will also derive a theorem on certain Marcinkiewicz-type integrals as a corollary of the method.

Preliminaries. Let $x = (x_1, \ldots, x_n)$ denote a point in n-dimensional Euclidean space R^n and $|x| = (\sum\limits_{i=1}^{n} x_i^2)^{1/2}$. Given a fixed vector $a = (a_1, \ldots, a_n)$ of rational numbers $a_i \geqslant 1$, $a_1 = 1$, we call $k(x)$ quasi-homogeneous of degree σ if $k(\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n) = \lambda^\sigma k(x)$ for $\lambda > 0$. In particular, letting m be the smallest integer divisible by $2a_i$, $i = 1, \ldots, n$, the function $\varrho(x) = [x] = (\sum x_i^{m/a_i})^{1/m}$ is quasi-homogeneous of degree 1. It is easy to see that ϱ is a metric. Moreover, for $\varrho(x) \neq 0$, the point $\left(\dfrac{x_1}{\varrho^{a_1}}, \ldots, \dfrac{x_n}{\varrho^{a_n}}\right) = (y_1, \ldots, y_n)$ belongs to $\sum = \varrho^{-1}(1)$ and $dx = \varrho^{|a|-1}d\varrho\, d\sigma(y)$, where $|a| = a_1 + \ldots + a_n$ and $d\sigma(y)$ is the element of area on Σ. (See [2], [4].)

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Following [5], we say $f \in L^p_a$, $1 \leqslant p \leqslant \infty$, if $f^\wedge(x) = (1+[x]^m)^{-a/m}\varphi^\wedge(x)$ for some $\varphi \in L^p$, and we put $\|f\|_{p,a} = \|\varphi\|_p$. For simplicity, we consider only $0 < a < 1$ throughout the paper. It then turns out (see [5]) that $f = \mathscr{J}^a\varphi = G_a * \varphi$ where $G_a \in L^1$ and

$$|D^\gamma G_a(x)| \leqslant c\,|x|^{-r} \quad \text{if } a_i r > |a| + a_i\gamma_i - a$$

and

$$|D^\gamma G_a(x)| \leqslant c\,[x]^{-|a|+a-a\cdot\gamma} \quad \text{if } [x] \leqslant 1.$$

Of course the case $a_1 = \ldots = a_n = 1$, $m = 2$ and $[x] = |x|$ corresponds to the ordinary homogeneous Bessel potentials.

We list here a few simple specific facts about $[x]$, G_a and μ which we will refer to later. c denotes a positive constant depending on n and a.

(1) (a) If $[x] \geqslant 1$ then $[x] \leqslant c|x|$. (b) If $[x] \leqslant 1$ then $[x] \geqslant c|x|$.

For (a), note $[x] \geqslant 1$ clearly implies $|x| \geqslant c$. Hence if $|x_i| \leqslant 1$, $x_i^{m/a_i} \leqslant 1 \leqslant c|x|^m$. If $|x_i| \geqslant 1$, $x_i^{m/a_i} \leqslant x_i^m \leqslant |x|^m$ and (a) follows by adding over i. (b) follows from $1 \geqslant [x] \geqslant |x_i|^{1/a_i} \geqslant |x_i|$.

(2) If $[x]\,[z] \leqslant 1$ then $|(x\cdot z)| \leqslant c[x]\,[z]$.

For $|(x\cdot z)| = \left|\sum \frac{x_i}{[x]^{a_i}}[x]^{a_i} z_i\right| \leqslant c(\sum [x]^{2a_i} z_i^2)^{1/2}$. But $1 \geqslant [x]\,[z] \geqslant [x]\,|z_i|^{1/a_i} \geqslant [x]^{a_i}|z_i|$. Squaring and adding, we obtain (2).

(3) For $0 < a < 1$, $|G_a(x)| \leqslant c[x]^{a-|a|}$ and $\left|\frac{\partial}{\partial x_i} G_a(x)\right| \leqslant c[x]^{a-|a|-a_i}$.

This follows from the two estimates on G_a cited above. For $[x] \leqslant 1$ it is exactly the second estimate, and for $[x] \geqslant 1$, it follows from the first by choosing r large and using (1a).

(4) For $0 < a < 1$ and $[x] > 2[z]$, $|G_a(x+z) - G_a(x)| \leqslant c[z]\,[x]^{a-|a|-1}$.

For

$$|G_a(x+z) - G_a(x)| = \left|\sum \frac{z_i}{[z]^{a_i}}[z]^{a_i}\frac{\partial}{\partial x_i}G_a(x+\theta z)\right| \leqslant c(\sum [z]^{2a_i}[x]^{2a-2|a|-2a_i})^{1/2}$$

by (3), since $[x+\theta z] \geqslant [x]-[z] \geqslant [x]/2$. But since $[z]\,[x]^{-1} < 1$, and $a_i \geqslant 1$, $[z]^{2a_i}[x]^{-2a_i} \leqslant [z]^2[x]^{-2}$ for all i, and (4) follows.

(5) If μ is any complex measure on R_n whose total variation satisfies $\int_{[z]>\delta} |d\mu(z)| = O(\delta^{-a})$, $\delta > 0$, then (cf. Lemma 1 of [3]).

(a) $$\int_{[z]>\delta} [z]^\beta |d\mu(z)| = O(\delta^{\beta-a}) \text{ for } \beta < a$$

and

(b) $$\int_{[z]<\delta} [z]^\beta |d\mu(z)| = O(\delta^{\beta-a}) \text{ for } \beta > a.$$

To show (b) for example, note that

$$\int_{[z]<\delta} [z]^\beta |d\mu(z)| = \sum_{k=0}^{\infty} \int_{2^{-k-1}\delta \leqslant [z] < 2^{-k}\delta} [z]^\beta |d\mu(z)| \leqslant c\sum_{k=0}^{\infty} (2^{-k}\delta)^\beta \int_{[z]\geqslant 2^{-k-1}\delta} |d\mu(z)| \leqslant c\sum_{k=0}^{\infty} (2^{-k}\delta)^\beta (2^{-k-1}\delta)^{-a} = O(\delta^{\beta-a})$$

if $\beta - a > 0$.

§ 1. In this section we prove the following theorem.

THEOREM 1. *For $f \in L^p_a$, $1 \leqslant p < \infty$, $0 < a < 1$, let $f_\varepsilon^\sim(x) = \int_{[z]>\varepsilon} [f(x-z) - f(x)]\,d\mu(z)$, where μ is a complex measure satisfying $\int_{[z]>\delta} |d\mu(z)| = O(\delta^{-a})$, $\delta > 0$. Then for $1 < p < \infty$, $\|f_\varepsilon^\sim\|_p \leqslant c\|f\|_{p,a}$ and $f_\varepsilon^\sim$ converges in L^p norm as $\varepsilon \to 0$. Moreover, $|\{x\colon |f_\varepsilon^\sim(x)| > s\}| \leqslant \frac{c}{s}\|f\|_{1,a}$ for $s > 0$. The constants c are independent of ε and f.*

The method of proving this is so well-known that we will only give the main points. We first suppose $f \in L^2_a$ so that

$$f_\varepsilon^{\sim\wedge}(x) = f^\wedge(x) B_\varepsilon(x); \quad B_\varepsilon(x) = \int_{[z]>\varepsilon} [e^{i(x\cdot z)} - 1]\,d\mu(z).$$

If $\varepsilon[x] \geqslant 1$,

$$|B_\varepsilon(x)| \leqslant 2\int_{[z]>[x]^{-1}} |d\mu(z)| = O([x]^a).$$

If $\varepsilon[x] < 1$,

$$|B_\varepsilon(x)| \leqslant \int_{[z]<[x]^{-1}} |(x\cdot z)|\,|d\mu(z)| + \int_{[z]>[x]^{-1}} 2\,|d\mu(z)| = O([x]^a)$$

by (2) and (5). Hence $|f_\varepsilon^{\sim\wedge}(x)| \leqslant c(1+[x]^m)^{a/m}|f^\wedge(x)| \leqslant c|\varphi^\wedge(x)|$, and $\|f_\varepsilon^\sim\|_2 \leqslant c\|f\|_{2,a}$ by Plancherel's formula.

The next step in the proof is to establish the weak-type statement of the theorem. For this we may suppose $f = \mathscr{J}^a\varphi$ with $\varphi \geqslant 0$, and recall (see e.g. [1]) that given $s > 0$ there are non-overlapping rectangles $\{I_k\}$ with dimensions $d_k^{a_1}, \ldots, d_k^{a_n}$ so that $\sum |I_k| \leqslant cs^{-1}\|\varphi\|_1$, and a decomposition $\varphi = \psi + \theta$ with $\|\psi\|_1 \leqslant \|\varphi\|_1$, $\psi \leqslant cs$ almost everywhere, and $\int_{I_k} \theta\,dx = 0$. Since $\psi \in L^2$ with $\|\psi\|_2^2 \leqslant cs\|\varphi\|_1$, it follows immediately from our L^2 result [1]

[1] We use L^2_a here only because we have already proved that case. If we had *a priori* knowledge for some other L^p_a we could use that information instead. We will need this fact later.

that $\{x\colon |(\mathcal{J}^\alpha\varphi)_\varepsilon^\sim(x)|>s\}$ has measure at most $cs^{-1}\|\varphi\|_1$. Hence, letting I_k^* denote the rectangle concentric with I_k whose dimensions are $(2\lambda d_k)^{\alpha_1}, \ldots, (2\lambda d_k)^{\alpha_n}$ for a large fixed λ independent of k, it is enough to assume $\varphi=\theta$ and prove $\big|\{x\notin\bigcup_k I_k^*\colon |f_\varepsilon^\sim(x)|>s\}\big|\leqslant cs\|\theta\|_1$, or that

(1.1) $$\int_{(\cup I_k^*)'} |f_\varepsilon^\sim(x)|\,dx \leqslant c\|\theta\|_1.$$

Recall that the integral of θ over I_k is zero, and note that $[x-y_k]>\lambda d_k$ for $x\notin I_k^*$, if y_k denotes the center of I_k. Using Fubini's theorem and a change of variables it is then easy to see that (1.1) follows from the statement

$$\int_{[x]>\lambda d} dx\Big|\int_{[z]>\varepsilon}[G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+G_\alpha(x)]\,d\mu(z)\Big|\leqslant C$$

for $[y]<d$.

To prove this we argue much as in [7]. For $0<\alpha<1$, we will in fact prove the stronger result that

(1.2) $$\int_{[x]>\lambda d} dx\int |G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+G_\alpha(x)|\,|d\mu(z)|$$

is bounded for $[y]<d$. For (1.2) is majorized by

$$\int_{[x]>\lambda d}dx\int_{[z]<d}|G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+G_\alpha(x)|\,|d\mu(z)|$$
$$+\int_{[x]>\lambda d}dx\int_{[z]>d}|G_\alpha(x-y)-G_\alpha(x)|\,|d\mu(z)|$$
$$+\int_{[x]>\lambda d}dx\int_{[z]>d}|G_\alpha(x-y-z)-G_\alpha(x-z)|\,|d\mu(z)| = A+B+C.$$

To estimate A, note that for the indicated ranges of x and z, $|G_\alpha(x-z)-G_\alpha(x)|\leqslant c[x]^{\alpha-|\alpha|-1}[z]$ by (4). The same estimate holds for $|G_\alpha(x-y-z)-G_\alpha(x-y)|$ since $[y]$ is small compared to $[x]$. Hence

$$A\leqslant c\int_{[x]>\lambda d}\frac{dx}{[x]^{|\alpha|+1-\alpha}}\int_{[z]<d}[z]\,|d\mu(z)| = O(d^{\alpha-1}d^{1-\alpha})=O(1)$$

by 5 (b).

By (4) again,

$$B=\int_{[x]>\lambda d}|G_\alpha(x-y)-G_\alpha(x)|\,dx\int_{[z]>d}|d\mu(z)|$$
$$\leqslant c\int_{[x]>\lambda d}\frac{[y]}{[x]^{|\alpha|+1-\alpha}}\,dx\cdot d^{-\alpha}=O(1).$$

Next, majorize C by

$$\int dx\int_{\substack{[z]>d\\ [x-z]>2d}} + \int dx\int_{\substack{[z]>d\\ [x-z]<2d}}|G_\alpha(x-z-y)-G_\alpha(x-z)|\,|d\mu(z)| = C_1+C_2.$$

By (4),

$$C_1\leqslant c\int dx\int_{\substack{[z]>d\\ [x-z]>2d}}\frac{[y]}{[x-z]^{|\alpha|+1-\alpha}}\,|d\mu(z)|$$
$$\leqslant cd\int_{[z]>d}|d\mu(z)|\int_{[x-z]>2d}\frac{dx}{[x-z]^{|\alpha|+1-\alpha}} = O(dd^{-\alpha}d^{\alpha-1})=O(1).$$

Finally, by (3),

$$C_2\leqslant c\int dx\int_{\substack{[z]>d\\ [x-z]<2d}}\Big\{\frac{1}{[x-z-y]^{|\alpha|-\alpha}}+\frac{1}{[x-z]^{|\alpha|-\alpha}}\Big\}|d\mu(z)|$$
$$\leqslant c\int_{[z]>d}|d\mu(z)|\int_{[x]<3d}\frac{dx}{[x]^{|\alpha|-\alpha}} = O(d^{-\alpha}d^{\alpha})=O(1).$$

To prove Theorem 1 for $1<p<2$ we use the Marcinkiewicz interpolation theorem, and for $p>2$ we use duality, L^p_α and $L^{p'}_{-\alpha}$, $p^{-1}+p'^{-1}=1$, being dual spaces. (See [5], Theorem 2.) That $f_\varepsilon^\sim$ converges in L^p for $f\in L^p_\alpha$, $1<p<\infty$, follows from the norm inequality of Theorem 1 and the fact that it converges in L^p for very smooth f. See [7] for details.

2. Maximal operator. We now use a method like that in [3] to prove

THEOREM 2. *Let* $f\in L^p_\alpha$, $0<\alpha<1$, *and*

$$f^{\sim *}(x)=\sup_\varepsilon|f_\varepsilon^\sim(x)|=\sup_\varepsilon\Big|\int_{[z]>\varepsilon}[f(x-z)-f(x)]\,d\mu(z)\Big|$$

with μ *as in Theorem* 1. *Then* $\|f^{\sim *}\|_p\leqslant c\|f\|_{p,\alpha}$ *for* $1<p<\infty$ *and* $|\{x\colon f^{\sim *}(x)>s\}|\leqslant cs^{-1}\|f\|_{1,\alpha}$ *for* $s>0$. *In particular,* $f_\varepsilon^\sim(x)$ *converges pointwise almost everywhere as* $\varepsilon\to 0$ *for* $1\leqslant p<\infty$.

We will need one additional fact stated in the following form ([2]).

LEMMA. *If* $f=\mathcal{J}^\alpha\varphi$ *and* $q>|\alpha|/\alpha$ *then*

$$|f(x+z+y)-f(x+z)|\leqslant c[y]^\alpha\Big\{1+\Big(\frac{[z]}{[y]}\Big)^{|\alpha|/q}\Big\}M_q(\varphi)(x),$$

where $M_q(\varphi)(x)=\sup_{\delta>0}\big(\delta^{-|\alpha|}\int_{[s]<\delta}|\varphi(x+s)|^q ds\big)^{1/q}$.

([2]) This particular statement is convenient for present purposes, but is somewhat arbitrary.

Proof. Write

$$f(x+z+y)-f(x+z) = \int_{[s]>2([z]+[y])} + \int_{[s]<2([z]+[y])} \varphi(x-s)[G_a(s+z+y)-G_a(s+z)]ds = \mathrm{I}+\mathrm{II}.$$

Applying Hölder's inequality and changing variables, we have

$$|\mathrm{II}| \leqslant \Big(\int_{[s]<2([z]+[y])} |\varphi(x-s)|^q ds\Big)^{1/q}\Big(\int |G_a(y+s)-G_a(s)|^{q'}ds\Big)^{1/q'}$$

for $\frac{1}{q}+\frac{1}{q'}=1$. However by (3) and (4),

$$\Big(\int |G_a(y+s)-G_a(s)|^{q'}ds\Big)^{1/q'} \leqslant c\bigg(\int_{[s]>2[y]}\Big(\frac{[y]}{[s]^{|a|+1-a}}\Big)^{q'}ds\bigg)^{1/q'} + c\bigg(\int_{[s]<3[y]}\Big(\frac{1}{[s]^{|a|-a}}\Big)^{q'}ds\bigg)^{1/q'}.$$

If $(|a|-a)q'<|a|$, or if $q>\frac{|a|}{a}$, this is $O([y]^{a-\frac{|a|}{q}})$. Therefore,

$$|\mathrm{II}| \leqslant c[y]^{a-\frac{|a|}{q}}([z]+[y])^{\frac{|a|}{q}} M_q(\varphi)(x).$$

On the other hand, by 4 and Hölder's inequality,

$$|\mathrm{I}| \leqslant c\int_{[s]>2[y]} |\varphi(x-s)|\frac{[y]}{[s]^{|a|+1-a}}ds \leqslant c[y]^{\frac{a}{q'}}\bigg(\int_{[s]>2[y]}|\varphi(x-s)|^q\frac{[y]}{[s]^{|a|+1-a}}ds\bigg)^{\frac{1}{q}}$$

for any $q\geqslant 1$. A standard result about approximations to the identity (see [4], p. 76) now gives $|\mathrm{I}|\leqslant c[y]^a M_q(\varphi)(x)$, and the lemma follows.

Returning to Theorem 2, suppose $f\in L^p_a$, $1<p<\infty$, and let $f^{\sim}$ denote the limit in L^p of $f^{\sim}_\varepsilon$. (See Theorem 1.) Let $\Delta(x)$ be a smooth, non-negative, decreasing function of $[x]$ supported in $[x]<1$ with integral 1. Then $\Delta_\varepsilon(x) = \varepsilon^{-|a|}\Delta\Big(\frac{x_1}{\varepsilon^{a_1}},\dots,\frac{x_n}{\varepsilon^{a_n}}\Big)$ is supported in $[x]<\varepsilon$ and $\sup_\varepsilon |f^{\sim}*\Delta_\varepsilon|$ has L^p norm bounded by a constant times $\|f\|_{p,a}$. By [4], $\sup_\varepsilon|(g*\Delta_\varepsilon)(x)| \leqslant cM_1(g)(x)$ and M_1 is a bounded operator on L^p, $1<p<\infty$. Hence Theorem 2 for $1<p<\infty$ will follow if we show that $\sup_\varepsilon|f^{\sim}_\varepsilon - f^{\sim}*\Delta_\varepsilon|$ has L^p norm less than a constant times $\|f\|_{p,a}$. Since $f^{\sim}*\Delta_\varepsilon = (f*\Delta)^{\sim}_\varepsilon$, it is enough to show the same for $\sup_\varepsilon A_\varepsilon$ and $\sup_\varepsilon B_\varepsilon$, where

$$A_\varepsilon(x) = \int_{[z]>\varepsilon}|(f*\Delta_\varepsilon)(x-z)-f(x-z)|\,|d\mu(z)| + \int_{[z]>\varepsilon}|(f*\Delta_\varepsilon)(x)-f(x)|\,|d\mu(z)|,$$

and $B_\varepsilon(x) = \int_{[z]<\varepsilon}|(f*\Delta_\varepsilon)(x-z)-(f*\Delta_\varepsilon)(x)|\,|d\mu(z)|$.

Consider first $B_\varepsilon(x)$.

$$B_\varepsilon(x)\leqslant \int_{[z]<\varepsilon}|d\mu(z)|\int|f(x-y)-f(x)|\,|\Delta_\varepsilon(y-z)-\Delta_\varepsilon(y)|\,dy.$$

In the inner integral $[y]<2\varepsilon$ since otherwise $\Delta_\varepsilon(y-z)=\Delta_\varepsilon(y)=0$. Moreover,

$$|\Delta_\varepsilon(y-z)-\Delta_\varepsilon(y)| \leqslant c\Big(\sum_{i=1}^n [z]^{2a_i}|D_i\Delta_\varepsilon(y-\theta z)|^2\Big)^{1/2} \leqslant c\bigg(\sum_{i=1}^n\Big(\frac{[z]}{\varepsilon}\Big)^{2a_i}\varepsilon^{-2|a|}\bigg)^{1/2} \leqslant c\frac{[z]}{\varepsilon}\varepsilon^{-|a|}$$

since $\frac{[z]}{\varepsilon}<1$. Hence,

$$B_\varepsilon(x)\leqslant c\varepsilon^{-|a|-1}\int_{[z]<\varepsilon}[z]\,|d\mu(z)|\int_{[y]<2\varepsilon}|f(x-y)-f(x)|\,dy \leqslant c\varepsilon^{-|a|-a}\int_{[y]<2\varepsilon}|f(x-y)-f(x)|\,dy.$$

By the lemma,

$$\sup_\varepsilon B_\varepsilon(x)\leqslant cM_q(\varphi)(x)\cdot\sup_\varepsilon \varepsilon^{-|a|-a}\int_{[y]<2\varepsilon}[y]^a dy\leqslant cM_q(\varphi)(x).$$

The second part of $\sup_\varepsilon A_\varepsilon(x)$ is

$$\sup_\varepsilon|(f*\Delta_\varepsilon)(x)-f(x)|\int_{[z]>\varepsilon}|d\mu(z)|\leqslant c\sup_\varepsilon\varepsilon^{-a}\int|f(x-y)-f(x)|\,\Delta_\varepsilon(y)\,dy \leqslant c\sup_\varepsilon\varepsilon^{-|a|-a}\int_{[y]<\varepsilon}|f(x-y)-f(x)|\,dy,$$

which is the expression just considered.

Finally, in the first part of A_ε,

$$|(f*\Delta_\varepsilon)(x-z)-f(x-z)| = \Big|\int[f(x-z-y)-f(x-z)]\Delta_\varepsilon(y)\,dy\Big| \leqslant cM_q(\varphi)(x)\int[y]^a\bigg\{1+\Big(\frac{[z]}{[y]}\Big)^{\frac{|a|}{q}}\bigg\}\Delta_\varepsilon(y)\,dy \leqslant cM_q(\varphi)(x)\,(\varepsilon^a+\varepsilon^{a-\frac{|a|}{q}}[z]^{\frac{|a|}{q}})$$

since $a-\frac{|a|}{q}>0$. Hence the first part of $\sup_\varepsilon A_\varepsilon(x)$ is at most a constant times

$$M_q(\varphi)(x)\sup_\varepsilon\int_{[z]>\varepsilon}\{\varepsilon^a+\varepsilon^{a-\frac{|a|}{q}}[z]^{\frac{|a|}{q}}\}\,|d\mu(z)|\leqslant cM_q(\varphi)(x)$$

by 5 (a), since $a>\frac{|a|}{q}$.

We can now easily complete the proof of Theorem 2. We have shown that for $f = \mathcal{J}^a \varphi$ and $q > \frac{|a|}{a}$

$$\sup_\varepsilon |f_\varepsilon^{\sim}(x) - f^{\sim} * \Delta_\varepsilon| \leqslant c M_q(\varphi)(x).$$

Since M_1 is a bounded operator on L^p for $p > 1$, M_q is a bounded operator on L^p for $p > q$. Theorem 2 follows immediately for $p > \frac{|a|}{a}$. But it is also true for $p = 1$, that is, $|\{x: f^{\sim *}(x) > s\}| \leqslant c s^{-1} \|f\|_{1,a}$. To see this, we simply refer to the proof in Section 1, adding two comments. First, the expression (1.2) used in the proof for the "bad" part θ does not even depend on ε. Second, instead of the *a priori* L_a^2 result used in the argument for the "good" part ψ, we use the L_a^p result for $f^{\sim *}$ for any $p > \frac{|a|}{a}$. (See the footnote on p. 4.) Theorem 2 follows for all $1 \leqslant p < \infty$ by interpolation.

§ 3. In this section we show that results on some Marcinkiewicz-type integrals can be obtained as corollaries of the method of Section 1.

Let $B_t = \{x: [x] < t\}$ and let ν be a non-negative measure on R^n satisfying

$$\text{(3.1)}\qquad \begin{aligned} &\text{(i)}\ \ 0 < \nu(B_t) < \infty, \quad 0 < t < \infty \\ &\text{(ii)}\ \ \nu(B_{2t}) \leqslant c\nu(B_t), \quad c \text{ independent of } t. \end{aligned}$$

For example, let $d\nu(z) = k(z)dz$ where $k \geqslant 0$ is quasi-homogeneous of degree $\beta - |a|$, $\beta > 0$, and integrable over Σ. Then $\nu(B_t)$ is a constant times t^β. (See [6] and [8].)

Let

$$\text{(3.2)}\qquad \mathcal{F}(x, t) = \frac{1}{\nu(B_t)} \int_{B_t} |f(x-z) - f(x)|\, d\nu(z).$$

For functions $g(x, t)$ we use the norm $X^p T^2 g = \left\| \left(\int_0^\infty g^2(x, t) \frac{dt}{t} \right)^{1/2} \right\|_p$.

THEOREM 3. *Let $f \in L_a^p$, $1 \leqslant p < \infty$, $0 < a < 1$. Then*

$$X^p T^2 \{t^{-a} \mathcal{F}(x, t)\} \leqslant c \|f\|_{p,a}, \quad 1 < p < \infty,$$

and

$$|\{x: T^2\{t^{-a} \mathcal{F}(x, t)\} > s\}| \leqslant c s^{-1} \|f\|_{1,a}.$$

We will briefly prove the cases $p = 2$ and $p = 1$ of Theorem 3. Let $f \in L_a^2$. By Schwarz's inequality,

$$\mathcal{F}^2(x, t) \leqslant \frac{1}{\nu(B_t)} \int_{B_t} |f(x-z) - f(x)|^2 d\nu(z),$$

and therefore

$$\begin{aligned} \left(X^2 T^2 \{t^{-a} \mathcal{F}(x, t)\}\right)^2 &= \int dx \int_0^\infty t^{-2a-1} \mathcal{F}^2(x, t)\, dt \\ &\leqslant \int_0^\infty \frac{dt}{t^{2a+1} \nu(B_t)} \int_{B_t} d\nu(z) \int |f(x-z) - f(x)|^2 dx \\ &= \int |f^{\wedge}(x)|^2 I(x)\, dx; \end{aligned}$$

$$I(x) = \int_0^\infty \frac{dt}{t^{2a+1}\nu(B_t)} \int_{B_t} |e^{i(x\cdot z)} - 1|^2 d\nu(z).$$

For any t, $\int_{B_t} |e^{i(x\cdot z)} - 1|^2 d\nu(z) \leqslant 4\nu(B_t)$, while if $t < \frac{1}{[x]}$ then by (2)

$$\int_{[z]<t} |e^{i(x\cdot z)} - 1|^2 d\nu(z) \leqslant c[x]^2 \int_{[z]<t} [z]^2 d\nu(z) \leqslant c[x]^2 t^2 \nu(B_t).$$

Hence

$$I(x) \leqslant c[x]^2 \int_0^{1/[x]} \frac{dt}{t^{2a-1}} + c \int_{1/[x]}^\infty \frac{dt}{t^{2a+1}} = O([x]^{2a}),$$

for $0 < a < 1$. Thus $\left(X^2 T^2\{t^{-a}\mathcal{F}(x, t)\}\right)^2 \leqslant \int |f^{\wedge}(x)|^2 [x]^{2a} dx$ and the result follows.

We will use the second part of (3.1) to prove the weak-type result for $p = 1$. Let $f = \mathcal{J}^a \varphi$, $\varphi \geqslant 0$. Given $s > 0$, decompose R^n and φ as in Section 1. By the usual considerations, it is enough to show

$$\int_{(\bigcup_k I_k^*)'} T^2\{t^{-a}\mathcal{F}(x, t)\}\, dx \leqslant c\|\theta\|_1$$

for $f = \mathcal{J}^a \theta$. But

$$T^2\{t^{-a}\mathcal{F}(x, t)\} \leqslant \int d\mu(z) \left| \int \theta(y)\, [G_a(x-y-z) - G_a(x-y)]\, dy \right|,$$

where $d\mu(z) = w(z)\, d\nu(z)$, $w(z) = \left(\int_{[z]}^\infty \frac{dt}{t^{2a+1}\nu^2(B_t)} \right)^{1/2}$. It is therefore enough to show that

$$\int_{[x]>\lambda d} dx \int |G_a(x-y-z) - G_a(x-y) - G_a(x-z) + G_a(x)|\, d\mu(z)$$

is bounded for $[y] < d$, which is what we did in Section 1 for measures satisfying $\int_{[z]>\delta} |d\mu(z)| \leqslant c\delta^{-a}$. In the present case, $\mu \geqslant 0$ and

$$\int_{[z]>\delta} d\mu(z) = \sum_{k=0}^\infty \int_{2^k\delta < [z] \leqslant 2^{k+1}\delta} w(z)\, d\nu(z).$$

Since w decreases as $[z]$ increases, this is at most

$$\sum_{k=0}^{\infty}\Bigl(\int_{2^k\delta}^{\infty}\frac{dt}{t^{2\alpha+1}\nu^2(B_t)}\Bigr)^{1/2}\nu(B_{2^{k+1}\delta})\leqslant c\sum_{0}^{\infty}\nu(B_{2^k\delta})^{-1}(2^k\delta)^{-\alpha}\nu(B_{2\cdot 2^k\delta})$$

$$\leqslant c\delta^{-\alpha}\sum_{0}^{\infty}2^{-\alpha k}=O(\delta^{\alpha}).$$

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A characterization of commutators with Hilbert transforms

by

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Abstract. There are given necessary and sufficient conditions for the commutator of the Hilbert transform with an operator bounded in L^2 to be compact. Similar results are obtained for the cotangent Hilbert transform and for the Cauchy singular integral operator on a closed arc. These conditions follow from a property of commutators of linear operators with an involution.

The purpose of the present note is to give necessary and sufficient conditions for the commutator of the Hilbert transform with an operator bounded in L^2 to be compact. Similar results will be obtained for the cotangent Hilbert transform and for the Cauchy singular integral operator on a closed arc, and so on. The proofs are based on a simple property of commutators with an involution, which are presented at the begining.

1. Let $\mathfrak{X}$ be an algebra (a linear ring) with unit e over the field of complex scalars. An element $a\in\mathfrak{X}$ is said to be an *involution* if $a\neq e$ and $a^2=e$. An element $a\in\mathfrak{X}$ is said to be an *almost involution*, with respect to a proper two-sided ideal $J\subset\mathfrak{X}$, if the coset $[a]$ is an involution in the quotient algebra $\mathfrak{X}/J$, i.e. if there is a $b\in J$ such that $a^2=e+b$ (see [3] and [4]). Let us denote the commutator and the anticommutator of two elements $a,b\in\mathfrak{X}$ as follows:

$$[a,b]=ab-ba,\qquad (a,b)=ab+ba.$$

PROPOSITION 1.1. *Let a be an involution in an algebra $\mathfrak{X}$ with unit. An element $b\in\mathfrak{X}$ commutes with a if and only if there is a $b_0\in\mathfrak{X}$ such that $b=(a,b_0)$.*

Proof. Let us suppose that $b=(a,b_0)$ for an element $b_0\in\mathfrak{X}$. Then

$$[a,b]=ab-ba=a(ab_0+b_0a)-(ab_0+b_0a)a$$
$$=a^2b_0+ab_0a-ab_0a-b_0a^2=eb_0-b_0e=0.$$