On the function of Marcinkiewicz

by

T. WALSH (Princeton N. J.)

Abstract. Define the Marcinkiewicz integral transformation acting on locally integrable functions in $\mathbb{R}^n$ by

$$\mu(f)(x) = \left( \int \int |\Omega(y)||y|^{-n+1} \phi(t^{-1}y)f(x-y)dydt |t|^{n-2}dt \right)^{1/2},$$

where $\Omega$ is homogeneous of degree 0. Rearrange-invariant conditions on $\Omega$ are found under which $\mu$ is bounded in $L^p$.

0. Introduction. The Marcinkiewicz function of a locally integrable function of one variable $f$ is defined by

$$\mu(f)(x) = \left( \int_0^\infty \left( \int_0^\infty |\Omega(x+t)+\Omega(x-T)^{n-1}dt \right)^{1/2},$$

where $F$ is an indefinite integral of $f$. Stein has considered the following generalization to $n$ variables

$$\mu(f)(x) = \left( \int_0^\infty \int_0^\infty |\Omega(y)||y|^{-n+1} \phi(t^{-1}y)dydt |t|^{n-2}dt \right)^{1/2},$$

where $\Omega$ denotes a locally integrable function which is homogeneous of degree 0 and has mean value 0 on the unit sphere $S^{n-1} = \{x: |x| = 1\}$ with respect to Euclidean surface measure $\sigma$.

Using the boundedness in $L^p$ of the 1-dimensional Marcinkiewicz integral transformation Stein showed that if $\Omega$ is odd $\mu$ defined by (1) is also bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ (cf. Theorem 2). The results for Calderon-Zygmund singular integrals in [4] give rise to the question whether similar results hold for the Marcinkiewicz integral (1) and general kernels.

For a homogeneous function $\Omega$ let $||\Omega||_p$ denote the $L^p$ norm with respect to the measure $\sigma$ on $S^{n-1}$. Also for a positive increasing function $\Phi$ let

$$||\Omega||_p[\Phi(L)] = \int_{S^{n-1}} \Phi(||\Omega(\xi)||)d\sigma(\xi).$$
C will denote a positive constant not necessarily the same at each occurrence.

The main results of the present paper dealing with somewhat more general types of integrals can then be stated as follows.

Proposition 1. Suppose \( \Omega \) is homogeneous of degree 0 as well as locally integrable in \( \mathbb{R}^n \) and \( \psi \) is a measurable function on the positive half-line satisfying

\[
\int_0^\infty |\psi(u)| \, du = |\psi|_1 < \infty,
\]

\[
\int_1^\infty \left\{ \int_1^\infty |\psi(u)| \, du \right\}^2 \, dt \leq B^2,
\]

\[
\int_0^\infty \left\{ \int_0^\infty |\psi(u + t) - \psi(u)| \, du \right\}^2 \, dt \leq B^2.
\]

Define \( \mu \) by

\[
\mu(f)(x) = \left( \int_1^\infty \left( \int_0^\infty \Omega(y) |y|^{-n+1} \psi(t-y) f(x-y) \, dy \right) \, dt \right)^1/2.
\]

If also

\[
\int_0^\infty |\psi(u)| \, du = 0
\]

then

\[
\|\mu(f)\|_p \leq C \|\Omega\|_1 (|\psi|_1 + B) \|f\|_p.
\]

If instead of (6)

\[
\int_{S^{n-1}} \Omega(\xi) \, d\sigma(\xi) = 0
\]

then

\[
\|\mu(f)\|_p \leq C \|\Omega\|_1 (|\psi|_1 + B) \|f\|_p,
\]

where

\[
N(\Omega)^2 = \sup_{|\xi| = 1} \left( \int_{|\eta| < |\xi|} \Omega(\eta) \, d\sigma(\eta) \right)^2 \, dt.
\]

On the other hand if (8) holds, \( \int_0^\infty |\psi(u)| \, du \neq 0 \) and \( N(\Omega) = \infty \) then \( \mu \) is not bounded in \( L^2 \).

This will be proved by considering Fourier transforms. The proof of the next proposition is by use of the 1-dimensional results and Bessel transforms similarly as in the case of singular integrals [4] and interpolation.

Proposition 2. Suppose \( \Omega \) is homogeneous of degree 0, integrable and of mean value 0 on \( S^{n-1} \). Suppose also that \( \psi \) is integrable in the interval (0, \( \infty \)) and satisfies the following conditions stronger than (3), (4)

\[
\int_1^\infty \left( \int_1^\infty |\psi(tu)| \, dt \right)^2 \, du \leq B,
\]

\[
\int_1^\infty \left( \int_1^\infty |\psi(tu)| \, dt \right)^2 \, du \leq B,
\]

\[
\int_1^\infty \left( \int_1^\infty |\psi((u-1) - \psi(tu)) \, dt \right)^2 \, du \leq B.
\]

Let \( \Omega_0, \Omega_1 \) be the even and odd parts respectively of \( \Omega \). Then for \( \mu \) defined by (5) and \( r = \min(p, p') \) (1/p + 1/p' = 1)

\[
\|\mu(f)\|_p \leq C_r(\Omega) (pp')^r (|\psi|_1 + B) \|f\|_p,
\]

where

\[
C_r(\Omega) \leq C(1 + \|\Omega\|_L (\log^2 L)^r (\log^2 \log^2 L)^{3n/2}) + \|\Omega\|_1).
\]

There appears to be no reason to expect this result to be in any sense best possible. In the case \( p = 2 \), however, the last part of Proposition 1 can be strengthened by a similar (but simpler) argument as in Weiss and Zygmund [12].

Proposition 3. For any increasing positive function \( \varphi \) such that \( \varphi(2t) \leq C_0(t) \) and \( \int_0^\infty \varphi(t) \, dt = \infty \) there is an integrable function \( \Omega \) which is homogeneous of degree 0 has mean value 0 on \( S^{n-1} \) and satisfies

\[
\sup_{|t| = 1} \int_{|\xi| < |\eta|} \Omega(\eta) \, d\sigma(\eta) \leq \varphi(t)
\]

and a continuous function of compact support \( f \) such that the Marcinkiewicz integrals (1) is infinite for a.e. \( x \) in the support of \( f \).

1. Proof of Proposition 1. Let \( H \) denote the Hilbert space of measurable complex valued functions on the positive half line \( (0, \infty) \) which are square integrable with respect to multiplication invariant measure \( \, dt \) and for \( f \in H \) let \( T^2 f \) denote the norm

\[
T^2 f = \left( \int_0^\infty \left( \int_0^\infty |f(tu)| \, dt \right)^2 \, du \right)^1/2.
\]

If \( f \) is a function on \( \mathbb{R}^n \) and \( t > 0 \) \( f_t, f_t' \) are defined by

\[
f_t(x) = t^{-n} f(tx), \quad f_t'(x) = f(tx).
\]

For \( f \in L^1 \) the formula for the Fourier transform \( \hat{f} \) is taken to be

\[
\hat{f}(x) = \int e^{-2\pi i f(y)} \, dy.
\]
For the definition of Lorentz (quasi-) norms, in particular, the weak \( L^1 \) quasi-norm \( \| \cdot \|_{\infty} \) see [2]. For \( x \in \mathbb{R}^n \) let \( n' = |x|^{-1}x \).

The following generalization of results in [1] will be needed for \( n = 1 \).

**Lemma 1.** Suppose \( \varphi \) is a measurable function on \( \mathbb{R}^n \) such that

\[
\int |\varphi(x)| \, dx = \| \varphi \|_1 < \infty,
\]

\[
\int \varphi(x) \, dx = 0,
\]

\[
\int_{|x| \leq 1} \left( \int \varphi(x) \, dx \right)^2 \, dt \leq B,
\]

\[
\int \left( \int_{|x| \leq 1} \varphi(x) \, dx \right)^2 \, dt \leq B^2.
\]

For \( f \in L^1 + L^\infty \) set \( \tilde{v}(f)(x) = \varphi_n \varphi f(x) \). Then

\[
\| \tilde{v}(f) \|_{\infty} \leq C(\| \varphi \|_1 + B) \| f \|_1.
\]

and \( \tilde{v}(f) = \tilde{v}_f \), where \( \tilde{v} \) is the H-valued function such that \( \tilde{v}(x)(t) = \varphi(x) \).

**Proof.** Note that if \( f \in L^1 + L^\infty \) then \( \| \varphi_n \varphi f \| \leq C(\| f \|_1 + \| f \|_\infty) \). It follows that for a.e. \( x \) \( \varphi_n \varphi f(x) \) is well defined for a.e. \( t \). Note that for \( f \in L^1 \)

\[
\tilde{v}(f)(x) = \varphi_n \varphi f(x) = \varphi_n \varphi f(x) = \int e^{-inx} \varphi(y) \, dy + \int \varphi(y) \, dy + \int e^{-inx} \varphi(y) \, dy = \int I_1(x) \, dy
\]

say.

\[
\int I_1(x) \, dy \leq \| \varphi \|_1 + B.
\]

By (16)

\[
I_2(x) = -\int_{|x| \leq 1} \varphi(y) \, dy
\]

hence

\[
\int I_1(x) \, dy + |I_2(x)| \leq \| \varphi \|_1 + B.
\]

By a change of the variable of integration

\[
\int I_1(x) \, dy \leq \int \int |\varphi(y)| \, dy \, dt \leq \| \varphi \|_1 \int |\varphi(y)| \, dy.
\]

Now by Minkowski’s inequality for integrals the last integral is at most equal to

\[
\int |\varphi(y)| \, dy \leq \| \varphi \|_1.
\]

Also by (17)

\[
\int \left( \int |I_1(x) \varphi(y) \, dy \right)^2 \, dt \leq B^2
\]

(20), (21), (22) altogether yield

\[
\int \left( \int |\varphi(x) \, dy \right)^2 \, dt \leq B^2
\]

Furthermore

\[
\tilde{v}(x) = -\int \exp \left(-int \cdot (y + nt \cdot x) \right) \varphi(y) \, dy
\]

\[
= (1/2) \int e^{-inx} \varphi(y) - \varphi(y - nt \cdot x) \, dy.
\]

Hence by (18)

\[
\int \left( \int |\varphi(x) \, dy \right)^2 \, dt \leq B^2
\]

(23) and (24) add up to

\[
\int \varphi(x) \, dy \leq C(\| \varphi \|_1 + B)
\]

for \( |x| = 1 \). However, by the invariance of the measure \( dt/\lambda \) (and \( \tilde{v}(0) = 0 \))

(25) is valid for all \( x \). Hence by Plancherel’s theorem

\[
\| \tilde{v}(f) \|^2 \leq \int \| \varphi_n \varphi f \|^2 \, dt \leq \int \int |\varphi(y)\|^2 \, dy \, \| f \|_1
\]

\[
\leq C(\| \varphi \|_1 + B) \| f \|_1.
\]

This completes the proof of (19).

The fact that the H-valued function \( \varphi(x) \) has the Fourier transform \( \tilde{v} f \) with \( \tilde{v}(x)(t) = \varphi(x) \) follows by the usual continuity argument.

The following lemma will not be needed until the proof of Proposition 2.

**Lemma 2.** If in place of (17), (18) the stronger conditions

\[
\int |\varphi(t(x) + y') - \varphi(t(x)) \, dy' \, dt \leq B
\]

for \( |y'| = 1 \). Then

\[
\int |\varphi(t(x) + y') - \varphi(t(x)) \, dy' \, dt \leq B
\]

for \( |y'| = 1 \). Then

\[
\int |\varphi(t(x) + y') - \varphi(t(x)) \, dy' \, dt \leq B
\]

for \( |y'| = 1 \). Then

\[
\int |\varphi(t(x) + y') - \varphi(t(x)) \, dy' \, dt \leq B
\]
are satisfied then
\begin{equation}
\|T^p\varphi\|_\text{H}^p \leq C(\|\varphi\|_1 + B)\|f\|_1, \quad 1 < p < \infty,
\end{equation}
\begin{equation}
\|T^p\varphi\|_p \leq C_p\|\varphi\|_1\|f\|_1, \quad 1 < p < \infty.
\end{equation}

Proof. By Minkowski’s inequality for integrals
\begin{equation}
\int \left( \int |\varphi(x)|^p dx \right)^{\frac{1}{p}} dt \leq \int \left( \int |\varphi(x)|^p dx \right)^{\frac{1}{p}} dt \left( \int |\varphi(x)|^p dx \right)^{\frac{1}{p}} dt \leq \int \left( \int \varphi(x)^{p-1} dx \right)^{\frac{1}{2}} dt \leq B
\end{equation}
i.e., \varphi satisfies (17) and similarly (37), (38) and (15) imply (18). Thus by Lemma 1 (30) holds in case \( p = 2 \). By (11), Theorem 2 (29) and (30) now follow from
\begin{equation}
\int \left( \int |\varphi(x-y) - \varphi(x)|^p dx \right)^{\frac{1}{p}} dt \leq OB
\end{equation}
which is valid since the left hand side equals
\begin{equation}
\int \left( \int |\varphi(x-y) - \varphi(x)|^p dx \right)^{\frac{1}{p}} dt \leq \int \left( \int |\varphi(x-y) - \varphi(x)|^{p-1} dx \right)^{\frac{1}{2}} dt + 2 \int \left( \int \varphi(x)^{p-1} dx \right)^{\frac{1}{2}} dt.
\end{equation}
Hence (31) is implied by (26), (28).

As in the proof of Lemma 1 Proposition 1 will follow from the following
Lemma 3. Under the assumptions of Proposition 1 let
\begin{equation}
F_\eta(x) = \int e^{-ix\eta} \varphi(y) dy, \quad a = \int \varphi(x) dx,
\end{equation}
\begin{equation}
Q(x) = \Omega(x) |x|^{-\alpha+1} \varphi(x|z|).
\end{equation}
Then for all \( x \in \mathbb{R}^n \)
\begin{equation}
\int |\hat{\varphi}(x) - aP_\alpha(x)|^p dx \leq C(\|\varphi\|_1, \|\varphi\|_1 + B).
\end{equation}
(Note that \( aP_\alpha(x) = 0 \) for \( |x| = 1 \).

Proof. Observe that \( \int \varphi(x) dx = a \int \varphi(x) dx \). Hence in any case \( \varphi \) satisfies (15), (16), (17) of Lemma 1 with \( \|\varphi\|_1 = \|\varphi\|_1 \) and \( B \) replaced by \( \|\Omega\|_1 + B \). As before it suffices to prove (32) in case \( |x| = 1 \), so assume this from now on. By the proof of Lemma 1.
\begin{equation}
\int |\varphi(x)|^p dx \leq C(\|\varphi\|_1, \|\varphi\|_1 + B).
\end{equation}

Furthermore if \( \tilde{\varphi} \) denotes the Fourier transform of \( \varphi \) extended to all of \( \mathbb{R}^n \) by \( \varphi(x) = 0 \) for \( x < 0 \)
\begin{equation}
\tilde{\varphi}(x) = \int \varphi(y) \sum_{s=1}^n e^{isy} \varphi(s) dy = \int \varphi(y) \hat{\varphi}(x-y) dy
\end{equation}
\begin{equation}
= \int \varphi(y) \tilde{\varphi}(x-y) dy
\end{equation}
\begin{equation}
= \tilde{\varphi}(0) \int \varphi(y) \tilde{\varphi}(x-y) dy
\end{equation}
\begin{equation}
= \int \varphi(y) \tilde{\varphi}(x-y) dy = aP_\alpha(x) + I_1(x) + I_2(x),
\end{equation}

By (33) the proof of (32) will be complete if it can be shown that
\begin{equation}
\int |I_1(x) + I_2(x)|^p dx \leq C(\|\varphi\|_1 + B).
\end{equation}
Observe that it is sufficient to consider the case of real \( \varphi \). Then \( \tilde{\varphi}(-x) \) is the complex conjugate of \( \tilde{\varphi}(x) \). Hence by means of Minkowski’s inequality for integrals (34) is implied by
\begin{equation}
\int |\varphi(x)|^p dx \leq C(\|\varphi\|_1 + B)^2.
\end{equation}

Note that
\begin{equation}
\hat{\varphi}(x) = \tilde{\varphi}(0) = \int e^{-ix\eta} \varphi(y) dy + \int e^{-ix\eta} \varphi(y) dy
\end{equation}
\begin{equation}
= \int \varphi(y) dy,
\end{equation}

Hence (35) follows exactly as in the first part of the proof of Lemma 1. (36) was shown in the proof of (34) (which did not use (28)).

Returning to the last part of Proposition 1 note that \( P_\alpha \) is continuous. It follows that if \( T^p P_\alpha(x) \) is unbounded then it is arbitrarily large on sets of positive measure. Hence \( \mu \) cannot be bounded in \( U^\gamma \).

Remark. By Minkowski’s inequality for integrals
\begin{equation}
\|\varphi\|_1 \leq \sup_{|\eta| = 1} \int |\varphi(\eta)| \left( \int |\varphi(y)| dx \right)^2 dy
\end{equation}
\begin{equation}
= \sup_{|\eta| = 1} \int |\varphi(\eta)| |\varphi(\eta)|^2 dy.
\end{equation}
By Young's inequality (see e.g., [3] p. 91, and [7] p. 275) and the integrability of \(|\xi|^{-n}\) for \(\beta < 1\),

\[
\mathcal{N}(\Omega) \leq C \left( 1 + \int_{\mathbb{R}^n} |Q| |\Omega|^{1/2} |\Omega|^{1/2} \text{d} s(\eta) \right).
\]

2. Proof of Proposition 2. This proposition will be proved by interpolation between Proposition 1 and the following

**Lemma 4.** Suppose \(\Omega \subset \mathbb{R}^n\), i.e., \(\Omega(\pm \varepsilon) = \Omega(0)\), has mean value 0 on \(\mathbb{R}^{n-1}\) and the function \(\psi \in C_c^\infty(\mathbb{R})\) vanishes in \((\infty, 0)\), satisfies \((3), (10), (11), (12)\). Then for \(1 < p < \infty\)

\[
\|\mu(f)\|_p \leq C \|\psi\|_r [1 + \|\Omega\| (\|\log \|\Omega\| + 1)]]).
\]

**Proof.** If \(G\) is any locally integrable function on \(\mathbb{R}^n\) which is homogeneous of degree 0 and \(g\) is an integrable function on the interval \((0, \infty)\) it will be convenient to use the notation \(M(\psi, g)(f)(x, t)\) for the convolution

\[
\Gamma^{-1} \int \sigma(y) \left| t^{-n-1} g(t^{-1}|y|) f(x-y) \right| \text{d} y
\]

so that, e.g.,

\[
\mu(f)(x) = \Gamma^T (M(\psi, \psi)(f))(x, r^2).
\]

Let \(\lambda\) be an infinitely differentiable function supported in the interval \((1, 2)\) and such that \(\int \lambda(s) \text{d}s = 1\). Then for \(a\) as in Lemma 3 and \(\varphi_1 = \varphi - a\)

\[
M(\lambda) \varphi = M(\lambda, \varphi_1) + a M(\lambda, \varphi).
\]

Moreover, if \(\varphi_1(z) = 0\) for \(|z| < 2r\) and \(\varphi_1\) satisfies \((10), (11), (12)\) with \(B\) replaced by \(B + C\) where \(C\) depends on the choice of \(\lambda\). Note that

\[
M(\lambda, \varphi_1)(f)(x, t) = \int_{\mathbb{R}^{n-1}} \Omega(\theta(y), \theta(\theta(y)), \theta(\theta(y))) f(x, t; y) \text{d} s(y),
\]

where

\[
\theta(y) = \Gamma^{-1} \int \varphi_1(\theta^{-1} t) f(x-y) \text{d} s.
\]

By Lemma 2 for \(n = 1\) applied to lines parallel to \(y\) it follows as in \([4]\) that

\[
\|\Gamma \theta \| \leq C \|\psi\| \left( 1 + \|\varphi_1\| + B \right) \|f\|_p,
\]

\(1 < p < \infty\).

Hence by Minkowski's inequality for integrals (and double norms) \((37)\) implies

\[
\|\Gamma \theta \| \leq C \|\psi\| \left( 1 + \|\varphi_1\| + B \right) \|f\|_p.
\]

As in \([4]\) the proof that \(\Gamma \theta \) is bounded in \(L^p\) proceeds by reducing it to the case of odd kernels by means of Riesz transforms. Let

\[
A(x) = \Omega(x) |x|^{-n+1} \lambda(|x|) \text{ so that } M(\Omega, \lambda)(f)(x, t) = A_1 f(x).
\]

If a function \(g\) has the property that \((1+|z|)^{-n} g(x)\) is integrable then its \(n\)-vector valued Riesz transform \(R_g = (R_{g_1}, \ldots, R_{g_n})\) is defined by

\[
R_g(f) = \frac{1}{c_n} \lim_{t \to 0} \int y |y|^{-n-1} g(x-y) \text{d} y
\]

for an appropriate constant \(c_n\). It is well known that for \(f \in D' (1 < p < \infty)\)

\[
f = - \sum_{i=1}^n R_i B_i f.
\]

Assume that \(Rf\) is integrable (which will be shown below). Then by \((11), \text{Corollary 5 and Remark 8})\)

\[
A_1 R_i B_i f = R_i A_1 f
\]

hence

\[
A_1 f = - \sum_{i=1}^n R_i A_1 B_i f = - R A_1 B f = -(R A_1) f
\]

where the pairing used in the definition of the convolution is the inner product of two \(n\)-vectors.

Since the Riesz kernel is of class \(C^\infty\) away from 0 and \(A\) has compact support contained in \(\{x : |x| \geq 1\}\) it follows that \(RA\) is of class \(C^\infty\) in \(\{x : |x| < 1\}\). Further since \(\Omega\) and \(\sigma\) has mean value 0 for \(|x| > 2\)

\[
\|\partial^n R A(x)\| \leq C \|\Omega\| \|\lambda\| |x|^{-n-1}
\]

for \(|x| > 4\).

In what follows derivatives will be taken in the distributional sense. Let \(r = |x|\) then

\[
\partial^r R A(x) = r^{-1} [\partial^r |x| R A(x)] + r^{-1} A_1 r^{-1} \text{d} s(\lambda)
\]

hence

\[
\partial^r R A(x) = r^{-1} - B(\partial^r |x| R A(x)) + B(\partial^r |x| R A(x)) |x|^{-n-1}
\]

It follows that

\[
\partial^r R A(x) = r^{-1} A_1 |x| \lambda(r)
\]

for certain constants \(a(r)\) which also depend on \(a\).
Note that all \( A_i \) are supported in the interval (1, 2) and of class \( C^\infty \) in particular, integrable hence by (21), Chapter 1, Theorem 2)

\[
\int_{\gamma \subset \mathcal{C}} |R A_i| \leq C (1 + \|\Omega\|([L \log^+ L])),
\]

where \( C \) depends on \( \lambda \). Now (41) implies that for \( l = 0, 1, 2 \), 1/2 \( r \leq r \leq 4 \)

\[
|\partial / \partial r| r^{n-1} R A_i (r, x) | \leq C \sum_{i=0}^{l} |R A_i| (r, x).
\]

Define

\[
\mathcal{L}_i (s) = \sup_{0 < r < \infty} |\partial / \partial r| r^{n-1} R A_i (r, x),
\]

so that

\[
\mathcal{L}_i (s) \leq |\partial / \partial r| r^{n-1} R A (r, x)| \leq C (1 + \|\Omega\|([L \log^+ L])).
\]

Hence by (42),

\[
\int_{0 < r < \infty} |\partial / \partial r| r^{n-1} R A (r, x) | \leq C (1 + \|\Omega\|([L \log^+ L])).
\]

Define

\[
\mathcal{L}_i (s) = \sup_{0 < r < \infty} |\partial / \partial r| r^{n-1} R A (r, x),
\]

Then from (40) and the discussion preceding it and (44) it follows that

\[
\int_{0 < r < \infty} \mathcal{L}_i (s) \leq C (1 + \|\Omega\|([L \log^+ L])).
\]

By a change to polar coordinates

\[
(R A_i)^* B_i (s) = \int_{0 < t < \infty} \int_{0 < r < \infty} (R A_i) (t^{-1} r^{-1}) B_i (s - r y) \partial R (y'),
\]

where for \( |y'| = 1 \) \( R (y', r) = r^{n-1} R A (r, y') \). Note that, for each \( y', \) \( R (y', r) \) is odd since \( A \) is even. It is easily verified that definition (45) implies that for any \( |y'| = 1 \) \( \mathcal{L}_i (s) \leq \mathcal{L}_i (y') \) and \( \mathcal{L}_i (y', r) \) satisfies (10), (11), (12) with \( B = \mathcal{L}_i (y') \) in (0, \( \infty \)) and hence on the whole real line. Since \( \mathcal{L}_i (y', r) \) is odd it also has mean value 0. Thus by Lemma 2 applied to parallel lines and Minkowski's inequality for integrals

\[
\| \mathbb{T} \mathcal{M} (\Omega, \mathcal{L}_i (y')) \|_p \leq C \mathbb{P} \int \mathcal{L}_i (y') \| d\mathcal{L}_i (y') \| B_i.
\]

By (46) and the norm inequality for the Riesz transform this implies that

\[
\| \mathbb{T} \mathcal{M} (\Omega, \mathcal{L}_i (y')) \|_p \leq C (p q)^{-1} (1 + \|\Omega\|([L \log^+ L])) \| B_i \|_p.
\]

Along with (38) this completes the proof of Lemma 4.

To conclude the proof of Proposition 2 it suffices to establish (13) for the odd and even parts of \( \Omega \) separately. If \( \Omega \) is odd (13) follows in familiar fashion from Lemma 2 and Minkowski's inequality for integrals (cf. (9), Theorem 2).

So suppose \( \Omega \) is even and further that \( \Omega \in [L \log^+ L]^{1/2} \) where \( 1 \leq q \leq 2 \).

For any complex number \( z \) define \( \mathcal{L}_z \) by

\[
\mathcal{L}_z (s) = \mathcal{L}_z (s) = C (1 + \log^+ \|\Omega(s)\|^{1/2} + \mathcal{L}_z (s) \| d\mathcal{L}_z (s)),
\]

where \( \omega_i \) is the surface area of \( \mathcal{S}^{n-1} \). Notice that

\[
\| \sup_{0 < r < \infty} |\partial / \partial r| r^{n-1} R A (r, x) | \leq C \int \mathcal{L}_z (s) \| d\mathcal{L}_z (s)).
\]

Hence if \( f \in L^1 \cap L^\infty \) then for any \( s \) and \( t \) \( \mathcal{M} (\Omega, \mathcal{L}_z (s)) \) is a holomorphic function of \( s \) in the region \( \mathcal{R} < 2 \). In particular \( \mathcal{M} (\Omega, \mathcal{L}_z (s)) \) is holomorphic in \( 0 < \mathcal{R} < 1 \) continuous and bounded in its closure. Also \( \Omega \) has mean value 0 on \( \mathcal{S}^{n-1} \). Hence by Lemma 4 for \( \mathcal{R} = 0 \),

\[
\| \mathcal{T} \mathcal{M} (\Omega, \mathcal{L}_z (s)) \|_p \leq C (1 + \|\Omega\|([L \log^+ L])) \| B_i \|_p.
\]

Also by Proposition 1 and the remark at the end of Section 1, for \( \mathcal{R} = 1 \)

\[
\| \mathbb{T} \mathcal{M} (\Omega, \mathcal{L}_z (s)) \|_p \leq C (1 + \|\Omega\|([L \log^+ L])) \| B_i \|_p.
\]

But

\[
\| \sup_{0 < r < \infty} |\partial / \partial r| r^{n-1} R A (r, x) | \leq C \|\Omega\|([L \log^+ L]) \| B_i \|_p.
\]

(48) \( \| \mathbb{T} \mathcal{M} (\Omega, \mathcal{L}_z (s)) \|_p \leq C (1 + \|\Omega\|([L \log^+ L])) \| B_i \|_p. \)

(49) \( \| \mathbb{T} \mathcal{M} (\Omega, \mathcal{L}_z (s)) \|_p \leq C (1 + \|\Omega\|([L \log^+ L])) \| B_i \|_p. \)

(50) \( \| \mathbb{T} \mathcal{M} (\Omega, \mathcal{L}_z (s)) \|_p \leq C (1 + \|\Omega\|([L \log^+ L])) \| B_i \|_p. \)

Applying this inequality to \( \mathcal{A} \mathcal{L} \) for \( \mathcal{A} > 0 \) yields

\[
\| \mathbb{T} \mathcal{M} (\Omega, \mathcal{L}_z (s)) \|_p \leq C (1 + \|\Omega\|([L \log^+ L])) \| B_i \|_p.
\]

The rest of the proof is analogous to an extrapolation argument of Yano (12). Let \( \lambda = |y'| + 1 \) and \( \mathcal{L}_s (s) = \mathcal{L} (s) \) if \( \log^+ \|\Omega\| < 2 \mathcal{R} = 0 \)
otherwise. For any integer \( j > j_0 \) let \( \Omega_j(x) = \Omega(x) \) if \( 2^{j-1} \leq x \leq 2^j \), otherwise. Also define \( \Omega_j = \Omega_j - \omega \sum_{j_0}^j \Omega_j(x') \delta(x') \). Then \( \Omega = \sum_{j=j_0}^\infty \Omega_j \).

Apply now (50) to \( \Omega_j \) with \( 1/q = 1/p + 1/j \), \( A = j^p \) and add to obtain

\[
\|T^* M(\Omega, \psi)(f)\|_p \\
\leq C\|\psi\|_1 + C \sum_{j=j_0}^\infty \left( \int_{2^{j-1}}^{2^j} \Omega_j(x') \|\psi\| \|\Omega_j(x')\|^{1/p} \|\Omega_j(x')\|^1 \right) dx' \\
\leq C \|\psi\|_1 + C \int \left( \|\Omega(x')\| \|\psi(x')\| \|\Omega(x')\|^1 \right) dx'.
\]

This finishes the proof of (13) for \( p < 2 \). If \( 2 < p < \infty \) observe that the preceding argument (for even \( \Omega \)) can be applied to the adjoint of the Hilbert space valued operator \( M(\Omega, \psi) \) which takes \( g \) to \( M^*(\Omega, \psi) 

\[ M^*(\Omega, \psi)(g)(x) = \int \Omega(y) \psi^{-1}(y) g(x-y) dy, \]

Hence the validity of (13) for \( 2 < p < \infty \) follows by duality.

3. Proof of Proposition 3. Suppose first \( n = 2 \). By the reasoning of [12] it suffices to display a continuous periodic function of one variable \( g \) and a suitable \( \Omega \) such that the truncated integral

\[ \int_0^1 |M(\Omega)(f)(x, \theta)| d\theta \]

which is independent of \( y \) is finite for \( x, \theta \) real \( \theta \leq \theta \) and \( \theta \). For \( a + iy = e^{\theta}, \) real \( \Omega(a, y) \) will be of the form \( \omega \sum_{n=1}^\infty \delta_n \chi_n \) with \( \chi_n \) the characteristic function of the interval \( [0, \delta] \) and let

\[ \chi_n(\theta) = \delta^{-1} \sum_{j=0}^{\delta n} (-1)^j \chi_n(\theta - j\pi/2). \]

For \( a + iy = e^{\theta}, \) real \( \Omega(a, y) \) will be of the form \( \omega \sum_{n=1}^\infty \delta_n \chi_n(\theta) \) with \( \delta_n > 0, \) \( v = 1, 2, \ldots, \sum \delta_n < \infty \). Now

\[ M(\Omega, f)(x, \theta) = \sum_{n} a_n \delta_n \exp(i\theta_n x) \chi_n(\theta_n, \theta), \]

where

\[
J(n, \lambda, \theta) = \int_0^\lambda \int_0^\lambda [\sin(n \cos \theta) - \sin(n \sin \theta)] d\theta d\lambda
\]

\[
= \frac{2}{n\lambda} \int_0^\lambda \left[ \sin(n \cos \theta) \cos \theta - \sin(n \sin \theta) \sin \theta \right] d\theta.
\]

The proof of Proposition 3 will be based on

**Lemma 5.** \( J(n, \lambda, \theta) \) satisfies

\[
J(n, \lambda, \theta) \leq Cn^{1/2} \lambda^{1/2} \cos \theta \quad \text{for} \quad 0 < \theta < \pi/2,
\]

\[
J(n, \lambda, \theta) < -1 \quad \text{for} \quad \pi/2 < \theta < \lambda/4
\]

\[
J(n, \lambda, \theta) \leq C(\lambda/4)^{1/2} \cos \theta
\]

Proof. Suppose \( n \lambda < 4 \). To prove (31) note that \( |w^{-1} \sin n u^2 - n| \leq C \) hence

\[
\left| \frac{\sin(n \cos \theta)}{\cos \theta} - \frac{\sin(n \sin \theta)}{\sin \theta} \right| \leq Cn^{1/2} \lambda^{1/2}
\]

and so (31) follows by integration over the interval \( 0, \lambda \).

If \( 4 \leq n \lambda 

\[
\int_0^\lambda \left| \frac{\sin(n \cos \theta)}{\cos \theta} \right| d\theta \leq n \lambda \lambda^{1/2} \cos \theta \leq V 2\lambda
\]

Also it is well known that \( u^{-1} \sin u > 1 - u^{-1} / 2 \)

\[
\int_0^\lambda \left| \frac{\sin \theta}{\sin \theta} \right| d\theta > n \lambda - (1/2) \int_0^\lambda \lambda^{1/2} \sin \theta \cos \theta d\lambda = n \lambda - (1/2) \lambda^{3/2} \lambda^{1/2}
\]

and therefore \( J(n, \lambda, \theta) < -2(1 - 1/2 - \sqrt{2}/4) \). This proves (32).

In any case

\[
\int_0^\lambda \left| \frac{\sin(n \cos \theta)}{\cos \theta} \right| d\theta \leq V 2\lambda
\]

and

\[
\left| \frac{\sin(n \sin \theta)}{\sin \theta} \right| \leq \int_0^\lambda \left| \frac{\sin(n \theta)}{\sin \theta} \right| d\theta \leq \int_0^\lambda \left| \frac{\sin(n \theta)}{\sin \theta} \right| d\theta
\]

Hence (33) follows.
Let now $d_{i}(t) = \sum_{j=1}^{n} d_{ij} \lambda_{j}$. Then

$$\int_{0}^{1} \frac{d}{dt} \left( f(x, t) \right) dt = -\infty \quad \text{a.e.}$$

if

$$\sum_{j=1}^{n} a_{j}^{2} \int_{0}^{1} d_{ij}(t) dt = -\infty$$

(see e.g., [14] Vol. 1, p. 203 which is well known to extend to Hilbert space valued functions).

However by Lemma 5

$$\int_{0}^{1} \left( \sum_{j=1}^{n} \left( \int_{\lambda_{j}} \frac{d}{dt} \left( f(x, t) \right) dt \right)^{2} \right)^{1/2}$$

$$\geq \frac{1}{\sum_{j=1}^{n} \left( \int_{\lambda_{j}} \frac{d}{dt} \left( f(x, t) \right) dt \right)^{2}} - C \sum_{j=1}^{n} \delta \left( \int_{\lambda_{j}} \frac{d}{dt} \left( f(x, t) \right) dt \right)^{2}$$

$$\geq \frac{1}{\sum_{j=1}^{n} \left( \int_{\lambda_{j}} \frac{d}{dt} \left( f(x, t) \right) dt \right)^{2}} - C \sum_{j=1}^{n} \delta \left( \int_{\lambda_{j}} \frac{d}{dt} \left( f(x, t) \right) dt \right)^{2} - C \sum_{j=1}^{n} \delta.$$

Let now $h_{i} = e^{-1}$, $\delta = \phi(h_{i}) - \phi(h_{i+1})$ for $i = 1, 2, \ldots$ so that

$$\sum_{i=1}^{n} \delta = \phi(e^{-1}) < \infty \quad \text{and for } t < e^{-1}$$

$$\sum_{i=1}^{n} \delta = \phi(e^{-1}) < \infty \quad \text{and for } t < e^{-1}$$

Thus for $n_{2}$ sufficiently large

$$\int_{0}^{1} \left( \sum_{j=1}^{n} \left( \int_{\lambda_{j}} \frac{d}{dt} \left( f(x, t) \right) dt \right)^{2} \right)^{1/2} \geq C \int_{0}^{1} \left( \int_{\lambda_{j}} \frac{d}{dt} \left( f(x, t) \right) dt \right)^{2}.$$