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Received June 25, 1971 (356)

STUDIA MATHEMATICA, T. XLIV. (1972)

On the function of Marcinkiewicz

by

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Abstract. Define the Marcinkiewicz integral transformation acting on locally integrable functions in R^n by

$$\mu(f)(x) = \Big(\int_0^\infty \Big|\int \Omega(y)|y|^{-n+1}\psi(t^{-1}y)f(x-y)\,dy\Big|^2 t^{-3}\,dt\Big)^{1/2},$$

where Ω is homogeneous of degree 0. Rearrangement-invariant conditions on Ω are found under which μ is bounded in L^p.

0. Introduction. The Marcinkiewicz function of a locally integrable function of one variable f is defined by

$$\mu(f)(x) = \Big(\int_0^\infty |F(x+t)+F(x-t)-2F(x)|^2 t^{-3}\,dt\Big)^{1/2},$$

where F is an indefinite integral of f. Stein has considered the following generalization to n variables

$$\mu(f)(x) = \Big(\int_0^\infty \Big|\int_{|y|\leqslant t} \Omega(y)f(x-y)\,dy\Big|^2 t^{-3}\,dt\Big)^{1/2}, \tag{1}$$

where Ω denotes a locally integrable function which is homogeneous of degree 0 and has mean value 0 on the unit sphere $S^{n-1} = \{x\colon |x| = 1\}$ with respect to Euclidean surface measure σ.

Using the boundedness in L^p of the 1-dimensional Marcinkiewicz integral transformation Stein showed that if Ω is odd μ defined by (1) is also bounded in $L^p(R^n)$ for $1<p<\infty$ ([9], Theorem 2). The results for Calderón–Zygmund singular integrals in [4] give rise to the question whether similar results hold for the Marcinkiewicz integral (1) and general kernels.

For a homogeneous function Ω let $\|\Omega\|_p$ denote the L^p norm with respect to the measure σ on S^{n-1}. Also for a positive increasing function Φ let

$$\|\Omega\|[\Phi(L)] = \int_{S^{n-1}} \Phi\big(|\Omega(\xi)|\big)\,d\sigma(\xi).$$

C will denote a positive constant not necessarily the same at each occurrence.

The main results of the present paper dealing with somewhat more general types of integrals can then be stated as follows.

PROPOSITION 1. *Suppose Ω is homogeneous of degree 0 as well as locally integrable in R^n and ψ is a measurable function on the positive half-line satisfying*

$$\int_0^\infty |\psi(u)|\,du = \|\psi\|_1 < \infty, \tag{2}$$

$$\int_1^\infty \Big(\int_t^\infty |\psi(u)|\,du\Big)^2 dt/t \leqslant B^2, \tag{3}$$

$$\int_0^1 \Big(\int_0^\infty |\psi(u+t)-\psi(u)|\,du\Big)^2 dt/t \leqslant B^2. \tag{4}$$

Define μ by

$$\mu(f)(x) = \Big(\int_0^\infty \Big|\int_{R^n} \Omega(y)|y|^{-n+1}\psi(t^{-1}y)f(x-y)\,dy\Big|^2 t^{-3}\,dt\Big)^{1/2}. \tag{5}$$

If also

$$\int_0^\infty \psi(u)\,du = 0 \tag{6}$$

then

$$\|\mu(f)\|_2 \leqslant C\|\Omega\|_1(\|\psi\|_1+B)\|f\|_2. \tag{7}$$

If instead of (6)

$$\int_{S^{n-1}} \Omega(\xi)\,d\sigma(\xi) = 0 \tag{8}$$

then

$$\|\mu(f)\|_2 \leqslant C\big(\|\Omega\|_1+N(\Omega)\big)(\|\psi\|_1+B)\|f\|_2, \tag{9}$$

where

$$N(\Omega)^2 = \sup_{|\xi|=1}\int_0^1 \Big|\int_{|\xi\cdot\eta|\leqslant t} \Omega(\eta)\,d\sigma(\eta)\Big|^2 dt/t.$$

On the other hand if (8) *holds, $\int_0^\infty \psi(u)\,du \neq 0$ and $N(\Omega) = \infty$ then μ is not bounded in L^2.*

This will be proved by considering Fourier transforms. The proof of the next proposition is by use of the 1-dimensional result and Riesz transforms similarly as in the case of singular integrals [4] and interpolation.

PROPOSITION 2. *Suppose Ω is homogeneous of degree 0, integrable and of mean value 0 on S^{n-1}. Suppose also that ψ is integrable in the interval $(0, \infty)$ and satisfies the following conditions stronger than* (3), (4)

$$\int_1^\infty \Big(\int_1^\infty |\psi(tu)|^2 t\,dt\Big)^{1/2} du \leqslant B, \tag{10}$$

$$\int_0^1 \Big(\int_0^1 |\psi(tu)|^2 t\,dt\Big)^{1/2} du \leqslant B, \tag{11}$$

$$\int_2^\infty \Big(\int_0^1 |\psi(t(u-1))-\psi(tu)|^2 t\,dt\Big)^{1/2} du \leqslant B. \tag{12}$$

Let Ω_0, Ω_1 be the even and odd parts respectively of Ω. Then for μ defined by (5) *and $r = \min(p, p')$ $(1/p+1/p' = 1)$*

$$\|\mu(f)\|_p \leqslant C_r(\Omega)(pp')^2(\|\psi\|_1+B)\|f\|_p, \tag{13}$$

where

$$C_r(\Omega) \leqslant C\big(1+\|\Omega_0\|[L(\log^+L)^{1/r}(\log^+\log^+L)^{2(1-2/r')}]+\|\Omega_1\|_1\big).$$

There appears to be no reason to expect this result to be in any sense best possible. In the case $p = 2$, however, the last part of Proposition 1 can be strengthened by a similar (but simpler) argument as in Weiss and Zygmund [12].

PROPOSITION 3. *For any increasing positive function φ such that $\varphi(2t) \leqslant C\varphi(t)$ and $\int_0^1 \varphi(t)^2 dt/t = \infty$ there is an integrable function Ω which is homogeneous of degree 0 has mean value 0 on S^{n-1} and satisfies*

$$\sup_{|\xi|=1}\int_{|\xi\cdot\eta|\leqslant t} |\Omega(\eta)|\,d\sigma(\eta) \leqslant \varphi(t) \tag{14}$$

and a continuous function of compact support f such that the Marcinkiewicz integral (1) *is infinite for a.e. x in the support of f.*

1. Proof of Proposition 1. Let H denote the Hilbert space of measurable complex valued functions on the positive half line $(0, \infty)$ which are square integrable with respect to multiplication invariant measure dt/t and for $f \in H$ let $T^2 f$ denote the norm

$$T^2 f = \Big(\int_0^\infty f(t)^2\,dt/t\Big)^{1/2}.$$

If f is a function on R^n and $t > 0$ f_t, f^t are defined by

$$f_t(x) = t^{-n}f(t^{-1}x), \qquad f^t(x) = f(tx).$$

For $f \in L^1$ the formula for the Fourier transform $\hat{f}$ is taken to be

$$\hat{f}(x) = \int e^{-ix\cdot y} f(y)\,dy.$$

For the definition of Lorentz (quasi-) norms, in particular, the weak L^1 quasi-norm $\| \ \|_{1\infty}$ see [2]. For $x \in R^n$ let $x' = |x|^{-1}x$.

The following generalization of results in [1] will be needed for $n = 1$.

Lemma 1. *Suppose φ is a measurable function on R^n such that*

$$\int |\varphi(x)|\,dx = \|\varphi\|_1 < \infty, \tag{15}$$

$$\int \varphi(x)\,dx = 0, \tag{16}$$

$$\int_1^\infty \Big(\int_{|x|\geq t} |\varphi(x)|\,dx \Big)^2 dt/t \leq B^2, \tag{17}$$

$$\int_0^1 \Big(\int |\varphi(x-ty') - \varphi(x)|\,dx \Big)^2 dt/t \leq B^2 \quad \text{for } |y'| = 1. \tag{18}$$

For $f \in L^1 + L^\infty$ set $\nu(f)(x)(t) = \varphi_t * f(x)$. Then

$$\|T^2 \nu(f)\|_2 \leq C(\|\varphi\|_1 + B)\|f\|_2 \tag{19}$$

and $\widehat{\nu(f)} = \hat{\nu}\hat{f}$, where $\hat{\nu}$ is the H-valued function such that $\hat{\nu}(x)(t) = \hat{\varphi}(tx)$.

Proof. Note that if $f \in L^1 + L^\infty$ then $\|\varphi_t * f\| \leq C\|f\|[L^1 + L^\infty]$. It follows that for a.e. x $\varphi_t * f(x)$ is well defined for a.e. t. Note that for $f \in L^2$ $(\varphi_t * f)^\wedge = \hat{\varphi}^t \hat{f}$. Suppose $|x| = 1$. Then

$$\hat{\varphi}^t(x) = \hat{\varphi}(tx) = \int e^{-itx\cdot y}\varphi(y)\,dy = \int_{|y|\leq t^{-1}} (e^{-itx\cdot y} - 1)\varphi(y)\,dy +$$
$$+ \int_{|y|\leq t^{-1}} \varphi(y)\,dy + \int_{|y|>t^{-1}} e^{-itx\cdot y}\varphi(y)\,dy = \sum_{j=1}^3 I_j(tx)$$

say.

$$|I_1(tx)| \leq t \int_{|y|\leq t^{-1}} |y|\,|\varphi(y)|\,dy.$$

By (16)

$$I_2(tx) = -\int_{|y|>t^{-1}} \varphi(y)\,dy$$

hence

$$|I_2(tx)| + |I_3(tx)| \leq 2 \int_{|y|\geq t^{-1}} |\varphi(y)|\,dy.$$

By a change of the variable of integration

$$\Big(\int_0^1 |I_1(tx)|^2 dt/t \Big)^{1/2} \leq \Big(\int_1^\infty \Big(t^{-1} \int_{|y|\leq t} |y|\,|\varphi(y)|\,dy \Big)^2 dt/t \Big)^{1/2}. \tag{20}$$

Now by Minkowski's inequality for integrals the last integral is at most equal to

$$\int |y|\,|\varphi(y)| \Big(\int_{|y|}^\infty t^{-3} dt \Big)^{1/2} dy \leq (1/\sqrt{2}) \int |\varphi(y)|\,dy \leq \|\varphi\|_1. \tag{21}$$

Also by (17)

$$\int_0^1 |I_2(tx) + I_3(tx)|^2 dt/t \leq \varphi \int_1^\infty \Big(\int_{|y|\geq t} |\varphi(y)|\,dy \Big)^2 dt/t \leq 4B^2 \tag{22}$$

(20), (21), (22) altogether yield

$$\Big(\int_0^1 |\hat{\varphi}(tx)|^2 dt/t \Big)^{1/2} \leq C(\|\varphi\|_1 + B). \tag{23}$$

Furthermore

$$\hat{\varphi}(tx) = -\int \exp[-itx\cdot(y + \pi t^{-1}x)]\varphi(y)\,dy$$
$$= (1/2)\int e^{-itx\cdot y}[\varphi(y) - \varphi(y - \pi t^{-1}x)]\,dy.$$

Hence by (18)

$$\int_1^\infty |\hat{\varphi}(tx)|^2 dt/t \leq (1/4) \int_0^\pi \Big(\int |\varphi(y) - \varphi(y - tx)|\,dy \Big)^2 dt/t \tag{24}$$
$$\leq (\log \pi)\|\varphi\|_1^2 + (1/4) \int_0^1 \Big(\int |\varphi(y) - \varphi(y - tx)|\,dy \Big)^2 dt/t$$
$$\leq C(\|\varphi\|_1 + B)^2$$

(23) and (24) add up to

$$\int_0^\infty |\hat{\varphi}(tx)|^2 dt/t \leq C(\|\varphi\|_1 + B)^2 \tag{25}$$

for $|x| = 1$. However, by the invariance of the measure dt/t (and $\hat{\varphi}(0) = 0$) (25) is valid for all x. Hence by Plancherel's theorem

$$\|T^2\nu(t)\|_2^2 = \int_0^\infty \|\varphi_t * f\|_2^2 dt/t = \int_0^\infty \int |\hat{\varphi}(tx)|^2 |\hat{f}(x)|^2 dx\,dt/t$$
$$\leq C(\|\varphi\|_1 + B)^2 \|f\|_2^2.$$

This completes the proof of (19).

The fact that the H-valued function $\nu(f)$ has the Fourier transform $\hat{\nu}\hat{f}$ with $\hat{\nu}(x)(t) = \hat{\varphi}(tx)$ follows by the usual continuity argument.

The following lemma will not be needed until the proof of Proposition 2.

Lemma 2. *If in place of* (17), (18) *the stronger conditions*

$$\int_{|x|\geq 1} \Big(\int_1^\infty |\varphi(tx)|^2 t^{2n-1} dt \Big)^{1/2} dx \leq B, \tag{26}$$

$$\int_{|x|\leq 1} \Big(\int_0^1 |\varphi(tx)|^2 t^{2n-1} dt \Big)^{1/2} dx \leq B, \tag{27}$$

$$\int_{|x|\geq 2} \Big(\int_0^1 |\varphi(t(x-y')) - \varphi(tx)|^2 t^{2n-1} dt \Big)^{1/2} dx \leq B \quad \textit{for } |y'| = 1 \tag{28}$$

are satisfied then

$$\|T^2\nu(f)\|_{1\infty} \leqslant C(\|\varphi\|_1+B)\|f\|_1, \tag{29}$$

$$\|T^2\nu(t)\|_p \leqslant Cpp'(\|\varphi\|_1+B)\|f\|_p, \quad 1<p<\infty. \tag{30}$$

Proof. By Minkowski's inequality for integrals

$$\Big(\int_1^\infty \Big(\int_{|x|\geqslant t} |\varphi(x)|\,dx\Big)^2 dt/t\Big)^{1/2} = \Big(\int_1^\infty \Big(\int_{|x|\geqslant 1} |\varphi(tx)|\,t^n\,dx\Big)^2 dt/t\Big)^{1/2}$$
$$\leqslant \int_{|x|\geqslant 1} \Big(\int_1^\infty |\varphi(tx)|^2 t^{2n-1}\,dt\Big)^{1/2} dx \leqslant B$$

i.e., φ satisfies (17) and similarly (27), (28) and (15) imply (18). Thus by Lemma 1 (30) holds in case $p=2$. By ([1], Theorem 2) (29) and (30) now follow from

$$\int_{|x|\geqslant 2|y|} \Big(\int_0^\infty |\varphi_t(x-y)-\varphi_t(x)|^2\,dt/t\Big)^{1/2} dx \leqslant CB \tag{31}$$

which is valid since the left hand side equals

$$\int_{|x|\geqslant 2} \Big(\int_0^\infty \big|\varphi\big(t(x-y')\big)-\varphi(tx)\big|^2 t^{2n-1}\,dt\Big)^{1/2} dx$$
$$\leqslant \int_{|x|\geqslant 2} \Big(\int_0^1 \big|\varphi\big(t(x-y')\big)-\varphi(tx)\big|^2 t^{2n-1}\,dt\Big)^{1/2} dx +$$
$$+2\int_{|x|\geqslant 1} \Big(\int_1^\infty |\varphi(tx)|^2 t^{2n-1}\,dt\Big)^{1/2} dx.$$

Hence (31) is implied by (26), (28).

As in the proof of Lemma 1 Proposition 1 will follow from the following

LEMMA 3. *Under the assumptions of Proposition 1 let*

$$F_0(x) = \int_{|x'\cdot y'|\leqslant |x|^{-1}} \Omega(y')\,d\sigma(y'), \qquad a = \int_0^\infty \psi(s)\,ds,$$
$$\varphi(x) = \Omega(x)\,|x|^{-n+1}\psi(|x|).$$

Then for all $x\in R^n$

$$\Big(\int_0^\infty |\hat\varphi(tx)-aF_0(tx)|^2\,dt/t\Big)^{1/2} \leqslant C\|\Omega\|_1(\|\psi\|_1+B). \tag{32}$$

(Note that $aF_0(x)=0$ for $|x|\leqslant 1$).

Proof. Observe that $\int\varphi(x)\,dx = a\int_{S^{n-1}}\Omega(x')\,d\sigma(x')$. Hence in any case φ satisfies (15), (16), (17) of Lemma 1 with $\|\varphi\|_1 = \|\Omega\|_1\|\psi\|_1$ and B replaced by $\|\Omega\|_1 B$. As before it suffices to prove (32) in case $|x|=1$, so assume this from now on. By the proof of Lemma 1

$$\int_0^1 |\varphi(tx)|^2\,dt/t \leqslant C\|\Omega\|_1(\|\varphi\|_1+B). \tag{33}$$

Furthermore if $\hat\psi$ denotes the Fourier transform of ψ extended to all of R by $\psi(s)=0$ for $s<0$

$$\hat\varphi(tx) = \int_{S^{n-1}} \Omega(y')\int_0^\infty e^{-istx\cdot y'}\psi(s)\,ds\,d\sigma(y')$$
$$= \int_{S^{n-1}} \Omega(y')\,\hat\psi(tx\cdot y')\,d\sigma(y')$$
$$= \hat\psi(0)\int_{|x\cdot y'|\leqslant 1/t} \Omega(y')\,d\sigma(y') + \int_{|x\cdot y'|\leqslant 1/t} \Omega(y')[\hat\psi(tx\cdot y')-\hat\psi(0)]\,d\sigma(y')$$
$$+\int_{|x\cdot y'|>1/t} \Omega(y')\,\hat\psi(tx\cdot y')\,d\sigma(y') = aF_0(tx)+I_1(tx)+I_2(tx),$$

say.

By (33) the proof of (32) will be complete if it can be shown that

$$\Big(\int_1^\infty |I_1(tx)+I_2(tx)|^2\,dt/t\Big)^{1/2} \leqslant C\|\Omega\|_1(\|\psi\|_1+B). \tag{34}$$

Observe that it is sufficient to consider the case of real ψ. Then $\hat\psi(-u)$ is the complex conjugate of $\hat\psi(u)$. Hence by means of Minkowski's inequality for integrals (34) is implied by

$$\int_1^1 |\hat\psi(u)-\hat\psi(0)|^2\,du/u \leqslant C(\|\psi\|_1+B)^2, \tag{35}$$

$$\int_1^\infty |\hat\psi(u)|^2\,du/u \leqslant C(\|\psi\|_1+B)^2. \tag{36}$$

Note that

$$\hat\psi(u)-\hat\psi(0) = \int_0^{1/u}(e^{-iuv}-1)\,\psi(v)\,dv + \int_{1/u}^\infty e^{-iuv}\psi(v)\,dv - \int_{1/u}^\infty \psi(v)\,dv.$$

Hence (35) follows exactly as in the first part of the proof of Lemma 1. (36) was shown in the proof of (24) (which did not use (16)).

Returning to the last part of Proposition 1 note that F_0 is continuous. It follows that if $T^2F_0(\cdot x)$ is unbounded then it is arbitrarily large on sets of positive measure. Hence μ cannot be bounded in L^2.

Remark. By Minkowski's inequality for integrals

$$N(\Omega) \leqslant \sup_{|\xi|=1}\int_{S^{n-1}} |\Omega(\eta)|\Big(\int_{|\xi\cdot\eta|}^1 dt/t\Big)^{1/2} d\sigma(\eta)$$
$$= \sup_{|\xi|=1}\int_{S^{n-1}} |\Omega(\eta)|(\log|\xi\cdot\eta|^{-1})^{1/2}\,d\sigma(\eta).$$

By Young's inequality (see e.g., [3] p. 91 and [7] p. 275) and the integrability of $|\xi\cdot\eta|^{-\beta}$ for $\beta<1$

$$N(\Omega)\leqslant C\Big(1+\int_{S^{n-1}}|\Omega(\eta)|\,(\log^+|\Omega(\eta)|)^{1/2}d\sigma(\eta)\Big).$$

2. Proof of Proposition 2. This proposition will be proved by interpolation between Proposition 1 and the following

LEMMA 4. *Suppose Ω is even, i.e., $\Omega(x)=\Omega(-x)$, has mean value 0 on S^{n-1} and the function ψ on R vanishes in $(-\infty,0)$ and satisfies* (2), (10), (11), (12). *Then for $1<p<\infty$*

$$\|\mu(f)\|_p\leqslant Cpp'\big[B\|\Omega\|_1+pp'\|\psi\|_1(1+\|\Omega\|[L\log^+L])\big].$$

Proof. If G is any locally integrable function on R^n which is homogeneous of degree 0 and g is an integrable function on the interval $(0,\infty)$ it will be convenient to use the notation $M(G,g)(f)(x,t)$ for the convolution

$$t^{-1}\int\sigma(y)|y|^{-n+1}g(t^{-1}|y|)f(x-y)\,dy$$

so that, e.g.,

$$\mu(f)(x)=T^2\big(M(\Omega,\psi)(f)(x,\cdot)\big).$$

Let λ be an infinitely differentiable function supported in the interval (1, 2) and such that $\int\lambda(s)\,ds=1$. Then for a as in Lemma 3 and $\psi_1=\psi-a\lambda$

$$M(\Omega,\psi)=M(\Omega,\psi_1)+aM(\Omega,\lambda).$$

Moreover $\int_1^\infty\psi_1(s)\,ds=0$, $\|\psi_1\|\leqslant2\|\psi\|$ and ψ_1 satisfies (10), (11), (12) with B replaced by $B+C$ where C depends on the choice of λ. Note that

$$M(\Omega,\psi_1)(f)(x,t)=\int_{S^{n-1}}\Omega(y')\tilde f(x,t;y')\,d\sigma(y'),\tag{37}$$

where

$$\tilde f(x,t;y')=t^{-1}\int_0^\infty\psi_1(t^{-1}s)f(x-sy')\,ds.$$

By Lemma 2 for $n=1$ applied to lines parallel to y' it follows as in [4] that

$$\|T^2\tilde f(\cdot,\cdot;y')\|_p\leqslant C(pp')(1+\|\psi\|_1+B)\|f\|_p,\qquad 1<p<\infty.$$

Hence by Minkowski's inequality for integrals (and double norms) (37) implies

$$\|T^2M(\Omega,\psi_1)(t)\|_p\leqslant Cpp'\|\Omega\|_1(1+\|\psi\|+B)\|f\|_p.\tag{38}$$

As in [4] the proof that $T^2M(\Omega,\lambda)$ is bounded in L^p proceeds by reducing it to the case of odd kernels by means of Riesz transforms. Let

$\Lambda(x)=\Omega(x)|x|^{-n+1}\lambda(|x|)$ so that $M(\Omega,\lambda)(f)(x,t)=\Lambda_t*f(x)$. If a function g has the property that $(1+|x|)^{-n}g(x)$ is integrable then its n-vector valued Riesz transform $Rg=(R_1g,\ldots,R_ng)$ is defined by

$$Rg(x)=c_n\lim_{\varepsilon\to0}\int_{|y|>\varepsilon}y|y|^{-n-1}g(x-y)\,dy$$

for an appropriate constant c_n. It is well known that for $f\in L^p$ $(1<p<\infty)$ $f=-\sum_{j=1}^nR_j(R_j,f)$.

Assume that $R\Lambda$ is integrable (which will be shown below). Then by ([11], Corollary 5 and Remark 8)

$$\Lambda_t*R_j(R_jf)=R_j\Lambda_t*R_jf$$

hence

$$\Lambda_t*f=-\sum R_j\Lambda_t*R_jf=-R\Lambda_t*Rf=-(R\Lambda)_t*Rf,\tag{39}$$

where the pairing used in the definition of the convolution is the inner product of two n-vectors.

Since the Riesz kernel is of class C^∞ away from 0 and Λ has compact support contained in $\{x\colon|x|\geqslant1\}$ it follows that $R\Lambda$ is of class C^∞ in $\{x\colon|x|<1\}$. Further since Ω and so Λ has mean value 0 for $|x|>2$

$$D^aR\Lambda(x)=\int_{|y|\leqslant2}\big(D^aR(x-y)-D^aR(x)\big)\Lambda(y)\,dy$$

$(D^a=(\partial/\partial x_1)^{a_1}\cdot\ldots\cdot(\partial/\partial x_n)^{a_n}$ for $a=(a_1,\ldots,a_n)$, $|a|=\sum_{j=1}^na_j)$. It follows that

$$|D^aR\Lambda(x)|\leqslant C_a\|\Omega\|_1|x^{-n-|a|-1}\qquad\text{for }|x|\geqslant4.\tag{40}$$

In what follows derivatives will be taken in the distribution sense. Let $r=|x|$ then

$$\partial/\partial r\,R\Lambda(x)=r^{-1}(\partial/\partial s)(R\Lambda)^s(x)|_{s=1}=r^{-1}R(\partial/\partial s\Lambda^s|_{s=1})(x)$$

hence

$$(\partial/\partial r)^2R\Lambda(x)=r^{-2}\big[-R(\partial/\partial s\Lambda^s|_{s=1})(x)+R(\partial^2/\partial s_1\partial s_2\Lambda^{s_1s_2}|_{s_1=s_2=1})(x)\big].$$

It follows that

$$(\partial/\partial r)^jR\Lambda(x)=r^{-j}R(\Lambda_j)(x)\qquad j=0,1,2\tag{41}$$

where $\Lambda_j(x)=\Omega(x)\lambda_j(r)$ and

$$\lambda_j(r)=\sum_{k=0}^ja_k^{(j)}r^{-n-1-k}\lambda^{(k)}(r)$$

for certain constants $a_k^{(j)}$ which also depend on n.

Note that all λ_j are supported in the interval $(1, 2)$ and of class C^∞, in particular, integrable hence by ([2], Chapter 1, Theorem 2)

$$\int_{1/2\leqslant|x|\leqslant 4} |R\Lambda_j(x)| \leqslant C(1+\|\Omega\|[L\log^+L]), \tag{42}$$

where C depends on λ. Now (41) implies that for $l = 0, 1, 2$, $1/2 \leqslant r \leqslant 4$

$$|(\partial/\partial r)^l r^{n-1} R\Lambda(rx')| \leqslant C \sum_{j=0}^{l} |R\Lambda_j(rx')|. \tag{43}$$

Define

$$\Omega_1^*(x) = \sup_{1/2\leqslant r\leqslant 4} |(\partial/\partial r) r^{n-1} R\Lambda(rx')|$$

so that

$$\Omega_1^*(x) \leqslant \left|\partial/\partial r\, r^{n-1} R\Lambda(rx')|_{r=1/2}\right| + \int_{1/2}^{4} |(\partial/\partial r)^2 r^{n-1} R\Lambda(rx')|\, dr.$$

Hence by (42), (43)

$$\int_{S^{n-1}} |\Omega_1^*(x')|\, d\sigma(x') \leqslant C(1+\|\Omega\|[L\log^+L]). \tag{44}$$

Define

$$\Omega^*(x) = \sup_{r>0} (1+r)^{2+j} |(\partial/\partial r)^j (r^{n-1} R\Lambda(rx'))|. \tag{45}$$

Then from (40) and the discussion preceding it and (44) it follows that

$$\int_{S^{n-1}} \Omega^*(x')\, d\sigma(x') \leqslant C(1+\|\Omega\|[L\log^+L]). \tag{46}$$

By a change to polar coordinates

$$(R\Lambda)_t * Rf(x) = t^{-n} \int_{S^{n-1}} \int_0^\infty (R\Lambda)(t^{-1}ry') \cdot Rf(x-ry') r^{n-1}\, dr\, d\sigma(y')$$

$$= (1/2) \int_{S^{n-1}} t^{-1} \int_{-\infty}^{\infty} \Psi(y', t^{-1}r) \cdot Rf(x-ry')\, dr\, d\sigma(y'),$$

where for $|y'| = 1$ $\Psi(y', r) = |r|^{n-1} R\Lambda(ry')$. Note that, for each y', $\Psi(y', \cdot)$ is odd since Λ is even. It is easily verified that definition (45) implies that for any $y' \in S^{n-1}$ $\|\Psi(y', \cdot)\|_1 \leqslant \Omega^*(y')$ and $\Psi(y', \cdot)$ satisfies (10), (11), (12) with $B = C\Omega^*(y')$ in $(0, \infty)$ and hence on the whole real line. Since $\Psi(y', \cdot)$ is odd it also has mean value 0. Thus by Lemma 2 applied to parallel lines and Minkowski's inequality for integrals

$$\|T^2 M(\Omega, \lambda)(f)\|_p \leqslant Cpp' \int \Omega^*(x')\, d\sigma(x') \|Rf\|_p.$$

By (46) and the norm inequality for the Riesz transform this implies that

$$\|T^2 M(\Omega, \lambda)(f)\|_p \leqslant C(pp')^2 (1+\|\Omega\|[L\log^+L]) \|f\|_p. \tag{47}$$

Along with (38) this completes the proof of Lemma 4.

To conclude the proof of Proposition 2 it suffices to establish (13) for the odd and even parts of Ω separately. If Ω is odd (13) follows in familiar fashion from Lemma 2 and Minkowski's inequality for integrals (cf. [9], Theorem 2).

So suppose Ω is even and further that $\Omega \in L(\log^+L)^{1/q}$ where $1 < q \leqslant 2$. For any complex number z define Ω_z', Ω_z by

$$\Omega_z'(x) = \Omega(x)[1+\log^+|\Omega(x)|]^{z/2-1/q'},$$

$$\Omega_z = \Omega_z' - \omega_n^{-1} \int_{S^{n-1}} \Omega_z'(x')\, d\sigma(x'),$$

where ω_n is the surface area of S^{n-1}. Notice that

$$\|\sup_{\mathrm{Re} z\leqslant 2} |\Omega_z|\|_1 \leqslant C \int_{S^{n-1}} |\Omega(x')| \left(1+\log^+|\Omega(x')|\right)^{1/q} d\sigma(x').$$

Hence if $f \in L^1 \cap L^\infty$ then for any x and t $M(\Omega_z, \psi)(f)(x, t)$ is a holomorphic function of z in the region $\mathrm{Re} z < 2$. In particular $M(\Omega_z, \psi)(f)(x, t)$ is holomorphic in $0 < \mathrm{Re} z < 1$ and continuous and bounded in its closure. Also Ω_z has mean value 0 on S^{n-1}. Hence by Lemma 4 for $\mathrm{Re} z = 0$, $1 < s < 2$

$$\|T^2 M(\Omega_z, \psi)(f)\|_s \leqslant C(s-1)^{-} (\|\psi\|_1 + B) \cdot (1+\|\Omega_z\|[L\log^+L]) \|f\|_s. \tag{48}$$

Also by Proposition 1 and the remark at the end of Section 1, for $\mathrm{Re} z = 1$

$$\|T^2 M(\Omega_z, \psi)(f)\|_2 \leqslant C(\|\psi\|_1 + B)\left(1+\|\Omega_z\|[L(\log^+L)^{1/2}]\right)\|f\|_2. \tag{49}$$

But

$$\sup_{\mathrm{Re} z=0} \|\Omega_z\|[L\log^+L] + \sup_{\mathrm{Re} z=1} \|\Omega_z\|[L(\log^+L)^{1/2}] \leqslant C\left(1+\|\Omega\|[L(\log^+L)^{1/q}]\right).$$

(Note that $t(1+\log^+t)^{1/q}$ is convex so that Jensen's inequality applies to the second term in the definition of Ω_z.)

Suppose now $q < p \leqslant 2$ and set $1/s = 1/2\,(1/p - 1/q')/(1/2 - 1/q')$ so that $(1-2/q')1/s + (2/q')\,1/2 = 1/p$. Note that $\Omega_{2/q'} = \Omega$. Hence by a theorem of Stein for analytic families of operators (which extends to Banach space valued functions, see, e.g. [2])

$$\|T^2 M(\Omega, \psi)(f)\|_p \leqslant C(1/q - 1/p)^{-2(1-2/q')}(\|\psi\|_1 + B)\left(1+\|\Omega\|[L(\log^+L)^{1/q}]\right)\|f\|_p.$$

Applying this inequality to $A\Omega$ for $A > 0$ yields

$$\|T^2 M(\Omega, \psi)(f)\|_p \leqslant C(1/q - 1/p)^{-2(1-2/q')}(\|\psi\|_1 + B) \times \times \left(A^{-1} + \|\Omega\|[L(\log^+AL)^{1/q}]\right)\|f\|_p. \tag{50}$$

The rest of the proof is analogous to an extrapolation argument of Yano [12]. Let $j_0 = [p'] + 1$ and $\Omega_{j_0}'(x) = \Omega(x)$ if $\log^+|\Omega(x)| < 2^{j_0}$, $= 0$

otherwise. For any integer $j > j_0$ let $\Omega_j'(x) = \Omega(x)$ if $2^{j-1} \leqslant \log^+|\Omega(x)| < 2^j$, $= 0$ otherwise. Also define $\Omega_j = \Omega_j' - \omega_n^{-1}\int_{S^{n-1}} \Omega_j'(x')\,d\sigma(x')$. Then $\Omega = \sum_{j=j_0}^{\infty} \Omega_j$.

Apply now (50) to Ω_j with $1/q = 1/p + 1/j$, $A = j^2$ and add to obtain

$$\|T^2 M(\Omega, \psi)(f)\|_p$$

$$\leqslant C(\|\psi\|_1 + B) \sum_{j=j_0}^{\infty} j^{2(1-2/p')}\Big[j^{-2} + \int_{S^{n-1}} |\Omega_j(x')|\big(\log^+|\Omega_j(x')|\big)^{1/p} d\sigma(x')\Big]$$

$$\leqslant Cp^{12}(\|\psi\|_1 + B)\Big[1 + $$

$$+ \int |\Omega(x')|\big(\log^+|\Omega(x')|\big)^{1/p}\big(\log^+\log^+|\Omega(x')|\big)^{2(1-2/p')} d\sigma(x')\Big].$$

This finishes the proof of (13) for $1 < p \leqslant 2$.

If $2 < p < \infty$ observe that the preceding argument (for even Ω) can be applied to the adjoint of the Hilbert space valued operator $M(\Omega, \psi)$ which takes $g \in L^{p'}(H)$ into

$$M^*(\Omega, \psi)(g)(x) = \int \Omega(y)|y|^{-n+1}\psi(t^{-1}|y|)\,g(x-y, t)\,dt/t\,dy.$$

Hence the validity of (13) for $2 < p < \infty$ follows by duality.

3. Proof of Proposition 3. Suppose first $n = 2$. By the reasoning of [12] it suffices to display a continuous periodic function of one variable g and a suitable Ω such that the truncated integral

$$\int_0^1 |M(\Omega)(f)(x, y;\ t)|^2\,dt/t$$

which is independent of y is infinite for a.e. x. $g(x)$ will be of the form $\sum_1^{\infty} a_k \exp(i n_k x)$ where $n_1, n_2, \ldots$ is a lacunary sequence of positive integers for definiteness suppose $n_{k+1}/n_k \geqslant 2$ and $\sum_1^{\infty} |a_k| < \infty$.

As in [12] for $0 < h < 1/4$ let χ_h be the characteristic function of the interval $[0, h)$ and let

$$\chi_h'(\theta) = h^{-1}\sum_{j=-\infty}^{\infty} (-1)^j \chi_h(\theta - j\pi/2).$$

For $x + iy = e^{i\theta}$, θ real $\Omega(x, y)$ will be of the form $\omega(\theta) = \sum_{\nu=1}^{\infty} \delta_\nu \chi_{h_\nu}'(\theta)$ with $\delta_\nu > 0$, $\nu = 1, 2, \ldots$, $\sum \delta_\nu < \infty$. Now

$$M(\Omega, f)(x, t) = \sum_{k,\nu=1}^{\infty} a_k \delta_\nu \exp(i n_k x)\,J(n_k, h_\nu, t),$$

where

$$J(n, h, t) = (th)^{-1}\int_0^t \int_0^h [e^{ins\cos\theta} + e^{-ins\cos\theta} - e^{ins\sin\theta} - e^{-ins\sin\theta}]\,d\theta\,ds$$

$$= 2/(nht)\int_0^h \Big[\frac{\sin(n\cos\theta t)}{\cos\theta} - \frac{\sin(n\sin\theta t)}{\sin\theta}\Big] d\theta.$$

The proof of Proposition 3 will be based on

LEMMA 5. *$J(n, h, t)$ satisfies*

(51) $$|J(n, h, t)| \leqslant Cn^2t^2 \quad \textit{for } 0 < t \leqslant 4/n,$$

(52) $$J(n, h, t) < -1 \quad \textit{for } 4/n \leqslant t \leqslant 1/(nh),$$

(53) $$|J(n, h, t)| \leqslant C/(nht).$$

Proof. Suppose $nt \leqslant 4$. To prove (51) note that $|u^{-1}\sin nut - nt| \leqslant Cn^3u^2t^3$ hence

$$\left|\frac{\sin(n\cos\theta t)}{\cos\theta} - \frac{\sin(n\sin\theta t)}{\sin\theta}\right| \leqslant Cn^3t^3$$

and so (51) follows by integration over the interval $(0, h)$.

If $4/n \leqslant t \leqslant 1/(nh)$ observe that

$$\left|\int_0^h \frac{S\sin(n\cos\theta t)}{\cos\theta}\,d\theta\right| \leqslant h/(\cos\pi/4) = \sqrt{2}h.$$

Also it is well known that $u^{-1}\sin u \geqslant 1 - u^2/6$ hence

$$\int_0^h \frac{S\sin n\sin\theta t}{\sin\theta}\,d\theta > nht - (1/6)\int_0^h n^3\theta^2t^3\,d\theta = nht - (1/18)n^3h^3t^3$$

and therefore $J(n, h, t) < -2(1 - 1/18 - \sqrt{2}/4)$. This proves (52).

In any case

$$\left|\int_0^h \frac{\sin(n\cos\theta t)}{\cos\theta}\,d\theta\right| \leqslant \sqrt{2}/4$$

and

$$\left|\int_0^h \frac{\sin n\sin\theta t}{\sin\theta}\,d\theta\right| = \left|\int_0^{\sin h} \frac{\sin nut}{u}\sqrt{1-u^2}\,du\right|$$

$$\leqslant \left|\int_0^{nt\sin h} u^{-1}\sin u\,du\right| + C\int_0^h u\,du \leqslant C.$$

Hence (53) follows.

Let now $d_k(t) = \sum_{v=1}^{\infty} \delta_v J(n_k, h_v, t)$. Then

$$\int_0^1 |M(\Omega)(f)(x,t)|^2 dt/t = \infty \quad \text{a.e.} \tag{54}$$

if

$$\sum_{k=1}^{\infty} |a_k|^2 \int_0^1 d_k(t)^2 dt/t = \infty \tag{55}$$

(see e.g., [14] Vol. 1 p. 203 which is well known to extend to Hilbert space valued functions).

However by Lemma 5

$$\Big(\int_0^1 d_k(t)^2 dt/t\Big)^{1/2} \geqslant \Big(\int_0^1 \Big(\sum_{4/n_k \leqslant t \leqslant 1/(n_k h_v)} \delta\Big)^2 dt/t\Big)^{1/2} -$$

$$- C\sum \delta \Big[n_k^2 \Big(\int_0^{1/n_k} t^3 dt\Big)^{1/2} + (n_k h_v)^{-1} \Big(\int_{1/(n_k h_v)}^1 t^{-3} dt\Big)^{1/2}\Big]$$

$$\geqslant \Big(\int_{1/n_k}^1 \Big(\sum_{h_v \leqslant 1/(n_k t)} \delta\Big)^2 dt/t\Big)^{1/2} - C\sum\delta \geqslant \Big(\int_{1/n_k}^1 \Big(\sum_{h_v \leqslant t} \delta\Big)^2 dt/t\Big)^{1/2} - C\sum \delta.$$

Let now $h_v = e^{-v-1}$, $\delta = \varphi(h_v) - \varphi(h_{v+1})$ for $v = 1, 2, \ldots$ so that $\sum_1^{\infty} \delta = \varphi(e^{-2}) < \infty$ and for $t < e^{-1}$

$$\sum_{hv \leqslant t} \delta = \varphi\big(\exp([\log t])\big) \geqslant C\varphi(t).$$

Thus for n_k sufficiently large

$$\Big(\int_0^1 d_k(t)^2 dt/t\Big)^{1/2} \geqslant C\Big(\int_{1/n_k}^1 \varphi(t)^2 dt/t\Big)^{1/2}.$$

Let now $\{a_k\}$ be an arbitrary sequence of nonvanishing complex numbers such that $\Sigma |a_k| < \infty$ and let the lacunary sequence of integers $\{n_k\}$ increase at such a rate that

$$\int_{1/n_k}^1 \varphi(t)^2 dt/t \geqslant |a_k|^{-2}.$$

Then (55) and hence (54) are satisfied.

Moreover Ω satisfies (14). To see this note that if $0 \leqslant t \leqslant \pi/4$ then

$$\int_{\theta_0}^{\theta_0+t} |\omega(\theta)|\, d\theta \leqslant \varphi(t).$$

For if $\theta_0 = 0$ then $\int_0^t |\omega(\theta)|\, d\theta \leqslant \sum_{h_v \leqslant t} \delta_v \leqslant \varphi(t)$. Note that ω is non-negative and nondecreasing in $[0, \pi/4]$. It follows that for fixed t $\int_{\theta_0}^{\theta_0+t} |\omega(\theta)|\, d\theta$ reaches its maximum value at $\theta_0 = 0$ (as well as at $\pi/2, \pi, 3\pi/2$). It follows that for any $0 \leqslant t \leqslant 1$

$$\Big(\int_{\theta_0-t}^{\theta_0+t} + \int_{\theta_0+\pi-t}^{\theta_0+\pi+t}\Big) |\omega(\theta)|\, d\theta \leqslant 4\varphi(t).$$

Hence

$$\int_{|\cos(\theta-\theta_0)| \leqslant t} |\omega(\theta)|\, d\theta \leqslant \int_{|\theta-\theta_0 \pm \pi/2| \leqslant t\pi/2} |\omega(\theta)|\, d\theta \leqslant 4\varphi(\pi t/2).$$

As in [12] a required pair of functions Ω, f can be constructed in n dimensions by means of the kernel $\Omega(\xi) = \omega(\theta)$ where θ is the longitudinal angle defined by $\xi_1 + i\xi_2 = e^{i\theta}$.

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Received June 30, 1971 (353)