STUDIA MATHEMATICA, T. XLIV. (1972)

Analytic functions and linearly ordered groups*

by

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Abstract. If $\sum_0^\infty |f(n)| < \infty$ then $\{\theta: f^\wedge(\theta) = 0\}$, where $f^\wedge(\theta) = \sum_0^\infty f(n)e^{in\theta}$, has measure 0. It is shown that if the integer group replaced by an arbitrary linearly ordered discrete group then a (weak) analogue of this result is valid.

Let Z be the additive group of integers and let $L^1(Z)$ consist of those complex valued functions f on Z for which $\|f\|_1 < \infty$ where

$$\|f\|_1 = \sum_Z |f(n)|.$$

We say that $f \in A^1(Z)$ if $f \in L^1(Z)$ and if $f(n) = 0$ for all $n < 0$. Let R be the real numbers and let $T = R/2\pi Z$. If $f \in A^1(Z)$, $f \neq 0$, and if

$$f^\wedge(\theta) = \sum_Z e^{in\theta} f(n) \qquad \theta \in T,$$

then $\{\theta: f^\wedge(\theta) = 0\}$ is a (closed) set in T of measure 0. On the contrary if $f \in L^1(Z)$, $f \neq 0$, then $\{\theta: f^\wedge(\theta) = 0\}$ can be an arbitrary closed set of measure less than 2π. A weaker version of this statement, which does not depend on a detailed description of T is the following. If $f \in A^1(Z)$, $f \neq 0$, if $g \in L^1(Z)$, and if

$$\{\theta: g^1(\theta) \neq 0\} \subset \{\theta: f^\wedge(\theta) = 0\},$$

then $g \equiv 0$. We will show that properly interpreted this statement holds for an arbitrary linearly ordered discrete group G, which need *not* be Abelian.

Let G be a group with elements $a, b, c, \ldots$; e is the identity of G. We assume that there has been distinguished on G a linear order relation "$<$" compatible with the group structure, that is:

(i) for each $a, b \in G$ exactly one of $a = b$, $a > b$ or $a < b$ holds;

(1) (ii) $a < b$ and $b < c$ implies $a < c$;

(iii) $a < b$ implies $ca < cb$ and $ac < bc$ for all $c \in G$.

* Research supported by the U.S. National Science Foundation under NSF Grant GP-6907.

We take G in the discrete topology. G is then unimodular and an invariant measure is obtained by assigning mass 1 to each point of G. $L^1(G)$ consists of all complex valued functions f on G for which $\|f\|_1 < \infty$ where

$$\|f\|_1 = \sum_G |f(a)|.$$

We say that $f \in A^1(G)$ if $f \in L^1(G)$ and if $f(a) = 0$ for all $a < e$.

Let Ω be the set of equivalence classes of irreducible unitary representations of G. For each $\omega \in \Omega$ we choose a representation $[U_\omega(\cdot), H_\omega]$ from ω. Here H_ω is a Hilbert space and $a \to U_\omega(a)$ is a homomorphism of G into an irreducible group of unitary operators on H_ω. Given $g \in L^1(G)$ we set

$$[\alpha, \beta]_g = \sum_{a \in G} g(a) \langle U_\omega(a)\, \alpha, \beta \rangle.$$

Here $\alpha, \beta \in H_\omega$, and $\langle \cdot, \cdot \rangle$ is the inner product in H_ω. It is easily verified that

$$|[\alpha, \beta]_g| \leqslant \|\alpha\|\, \|\beta\|\, \|g\|_1.$$

Since $[\alpha, \beta]_g$ is sesquilinear it follows that there is a unique bounded l inea transformation $g^\wedge(\omega)$ on H_ω such that for all $\alpha, \beta \in H_\omega$

$$[\alpha, \beta]_g = \langle g^\wedge(\omega)\, \alpha, \beta \rangle.$$

We can now state our principal result.

Theorem 1. *Let G be as in* (1), *and let $g \in L^1(G)$, $f \in A^1(G)$, $f(e) = 1$. If*

$$\{\omega\colon\, g^\wedge(\omega) \neq 0_\omega\} \subset \{\omega\colon\, f^\wedge(\omega) = 0_\omega\}$$

then $g \equiv 0$. Here 0_ω is the identically zero transformation on H_ω.

This will be a consequence of Theorem 2 below. Given $g \in L^1(G)$ we define ${}_cg(a)$ to be $g(ca)$. We further define the convolution $f*g$ of two functions f, g by

$$f*g\cdot(a) = \sum_b f(b)\, g(b^{-1}a).$$

If $f, g \in L^1(G)$ then $f*g \in L^1(G)$ and $\|f*g\|_1 \leqslant \|f\|_1\, \|g\|_1$; if $f \in L^1(G)$ and $g \in L^2(G)$ then $f*g \in L^2(G)$ and $\|f*g\|_2 \leqslant \|f\|_1\, \|g\|_2$, etc. Moreover if $f \in L^1(G)$, and $g, h \in L^2(G)$ we have $f*(g*h) = (f*g)*h$. In the present case direct verification of all these formulas is very simple indeed.

Theorem 2. *Let G be as in* (1), *let $g \in L^1(G)$, $f \in A^1(G)$, and let $f(e) = 1$. If*

(2) $$g*{}_cf \equiv 0$$

for all $c \leqslant e$ then $g \equiv 0$.

Proof. Our demonstration is an adaptation of an argument taken from Helson ([2], p. 4).

We define M to be the closed linear manifold in $L^2(G)$ generated by the functions $\{{}_cf\}_{c<e}$. We note that for b fixed the mapping $h \to h(b)$ of $L^2(G)$ into the complex numbers satisfies $|h(b)| \leqslant \|h\|_2$ and is thus a continuous linear functional on $L^2(G)$. Since, if $c < e$, ${}_cf(a) = 0$ for $a \leqslant e$ it follows that if $h \in M$ then $h(a) = 0$ for $a \leqslant e$. Now let f_M be the projection of f on M and let $k = f - f_M$. We have $k(e) = 1$ so that $k \neq 0$. Since k is orthogonal to M and since ${}_ck \in M$ if $c < e$ we see that

$$\sum_a k(a)\, \overline{k(ca)} = 0 \qquad c < e,$$

which we can rewrite as

$$k*k^\sim(c) = 0 \qquad \text{for } c > e.$$

Here $k^\sim(a)$ is defined as $\overline{k(a^{-1})}$. The identity, $k*k^\sim\cdot(c) = \overline{k*k^\sim(c^{-1})}$, implies that

$$k*k^\sim\cdot(c) = 0 \qquad \text{for } c < e.$$

Finally

$$k*k^\sim\cdot(e) = \|k\|_2^2 \neq 0.$$

Thus

$$k*k^\sim\cdot(c) = \delta(c)\,\|k\|_2^2$$

where $\delta(c)$ is 1 if $c = e$ and is 0 otherwise.

It is apparent from (2) that

$$g*k\cdot(a) = 0 \qquad \text{all } a \in G,$$

and thus that

$$\|k\|_2^2\, g(a) = g*k*k^\sim\cdot(a) = 0 \qquad \text{all } a \in G;$$

that is, $g \equiv 0$.

Now let f and g satisfy the assumptions of Theorem 1. It is simple to verify and well known that

$$({}_cf)^\wedge(\omega) = U_\omega(c^{-1}) f^\wedge(\omega),$$

and

$$(g*{}_cf)^\wedge(\omega) = g^\wedge(\omega)[{}_cf]^\wedge(\omega) = g^\wedge(\omega)\, U_\omega(c^{-1}) f^\wedge(\omega).$$

Thus our assumptions imply that for each $\omega \in \Omega$

$$(g*{}_cf)^\wedge(\omega) = 0_\omega \qquad \text{for all } c \in G.$$

This in turn, see [3; p. 360], implies that

$$g*{}_cf = 0$$

for all $c \in G$. It follows from Theorem 2 that $g = 0$.

An example of a non-commutative group which has an order satisfying (1) is the free group on n letters, $n > 1$. See [1] p. 47.

References

[1] L. Fuchs, *Partially Ordered Algebraic Systems*, New York 1963.
[2] H. Helson, *Lectures on Invariant Subspaces*, New York 1964.
[3] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol. 1., Berlin 1963.

Received June 25, 1971 (356)

STUDIA MATHEMATICA, T. XLIV. (1972)

On the function of Marcinkiewicz

by

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Abstract. Define the Marcinkiewicz integral transformation acting on locally integrable functions in R^n by

$$\mu(f)(x) = \Big(\int_0^\infty \Big|\int \Omega(y)|y|^{-n+1}\psi(t^{-1}y)f(x-y)\,dy\Big|^2 t^{-3}\,dt\Big)^{1/2},$$

where Ω is homogeneous of degree 0. Rearrangement-invariant conditions on Ω are found under which μ is bounded in L^p.

0. Introduction. The Marcinkiewicz function of a locally integrable function of one variable f is defined by

$$\mu(f)(x) = \Big(\int_0^\infty |F(x+t)+F(x-t)-2F(x)|^2 t^{-3}\,dt\Big)^{1/2},$$

where F is an indefinite integral of f. Stein has considered the following generalization to n variables

$$\mu(f)(x) = \Big(\int_0^\infty \Big|\int_{|y|\leqslant t} \Omega(y)f(x-y)\,dy\Big|^2 t^{-3}\,dt\Big)^{1/2}, \tag{1}$$

where Ω denotes a locally integrable function which is homogeneous of degree 0 and has mean value 0 on the unit sphere $S^{n-1} = \{x\colon |x| = 1\}$ with respect to Euclidean surface measure σ.

Using the boundedness in L^p of the 1-dimensional Marcinkiewicz integral transformation Stein showed that if Ω is odd μ defined by (1) is also bounded in $L^p(R^n)$ for $1 < p < \infty$ ([9], Theorem 2). The results for Calderón–Zygmund singular integrals in [4] give rise to the question whether similar results hold for the Marcinkiewicz integral (1) and general kernels.

For a homogeneous function Ω let $\|\Omega\|_p$ denote the L^p norm with respect to the measure σ on S^{n-1}. Also for a positive increasing function Φ let

$$\|\Omega\|[\Phi(L)] = \int_{S^{n-1}} \Phi\big(|\Omega(\xi)|\big)\,d\sigma(\xi).$$