The Parseval's identity

$$\|x^\wedge\|_2 = \|x\|_2,$$

holds for any pair of corresponding $x^\wedge$, $x(\cdot)$, *so that this mapping is isometric.*

3.2.3. *Denote* $s_n(\cdot) = \sum_1^n x_i \int_a^t \varphi_i(\tau)d\tau$, $n = 1, 2, \ldots$ *If* $x^\wedge \in l^2(X)$ *is regular in* $l^2(X)$ *and* $x(\cdot) = \sum_1^\infty x_i \int_a^t \varphi_i(\tau)d\tau$ *then*

$$(*) \qquad \|s_n - x\|_2 \to 0 \quad \text{as } n \to \infty.$$

If $x \in V^2(X)$, $x_i = \int_T \varphi_i(\tau)dx$ *and the relation* (*) *is satisfied, then the sequence* $x^\wedge = \{x_i\}$ *of the Fourier coefficients of* $x(\cdot)$ *is regular and the series* $\sum_1^\infty \int_T \varphi_i(\tau)dx \int_a^t \varphi_i(\tau)d\tau$ *is perfectly convergent in* $V^2(X)$.

The statement follows immediately from 3.2, 3.2.1 and the definition of the regularity in $l^2(X)$ of a sequence $\{x_i\}$.

COROLLARY. *Assume that the orthonormal system* $\{\varphi_i\}$ *is complete in* L^2. *If any sequence in* $l^2(X)$ *is regular, in particular if* $l^2(X)$ *is separable, then the expansion* 3.1 (*) *any functions in* $V^2(X)$ *is convergent to* $x(\cdot)$ *in* $V^2(X)$ *and the convergence is perfect.*

If for any $x \in V^2(X)$ *the expansion* 3.1 (*) *converges to* $x(\cdot)$, *then any sequence in* $l^2(X)$ *is regular; in particular if* X *is separable then* $l^2(X)$ *is separable.*

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On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov*

by

S. K. PICHORIDES (Chicago)

Abstract. Let f be a real 2π-periodic function and $\tilde{f}$ its conjugate. Then; (i) The least value of the constant A_p in M. Riesz's theorem ($\|\tilde{f}\|_p \leqslant A_p\|f\|_p$, $p > 1$, $f \in L^p$) is $\tan(\pi/2p)$ if $1 < p \leqslant 2$ (and hence $\cot(\pi/2p)$ if $p > 2$). (ii) The only possible values of the constant A in Zygmund's theorem ($\|\tilde{f}\|_1 \leqslant A(1/2\pi)(\int_{-\pi}^{\pi}|f|\log^+|f|) + B$, $f \in L\log^+ L$) are those $> 2/\pi$. (iii) For non-negative functions the least value of the constant B_p in Kolmogorov's theorem ($\|\tilde{f}\|_p \leqslant B_p\|f\|_1$, $p < 1$, $f \in L^1$) is $(\cos(p\pi/2))^{-1/p}$. (iv) The constant A_p in (i) is also best possible for real non-periodic functions in R^1 (instead of the conjugate function it is now considered the Hilbert transform). The proof of these results makes use of a refinement of the inequality on which A. Calderòn's proof of the theorem of M. Riesz is based (see A. Zygmund; Trigonometric Series, Ch. VII, section 2, Cambridge Un. Press, 1968).

1. Introduction. The purpose of this paper is to examine the constants appearing in the theorems of M. Riesz, Zygmund and Kolmogorov ([4], Chapter VII, Section 2). In Section 2 we examine the case of real functions which are non-negative and 2π-periodic and we obtain sharp estimates of these constants. It turns out that the above mentioned theorems can be considered as instances of the same inequality (see Theorem 2.4 and the remarks following it). In Section 3 we consider real 2π-periodic functions of variable sign. Although the results are not as complete as in Section 2, we are able to prove that the least value of the constant A_p in M. Riesz's theorem ($\|\tilde{f}\|_p \leqslant A_p\|f\|_p$, $p > 1$, $f \in L^p$) is $\tan(\pi/2p)$ if $1 < p \leqslant 2$ (and hence $\cot(\pi/2p)$ if $p \geqslant 2$). The proof of this result (theorem 3.7) is based on a refinement of the device, due to A. Calderòn, used in [4] for the proof of M. Riesz's theorem (only the Theorems 2.4 and 2.12 (c) from Section 2 are needed for the proof). T. Gokhberg and N. Krupnik have obtained the same result for special values of p ($p = 2^n$, $n = 1, 2, \ldots$). They have also proved that $A_p \geqslant \cot(\pi/2p)$ for all $p \geqslant 2$ and conjectured that this estimate is best possible. (see [1]). In Section 4 we discuss some related results concerning non-periodic functions.

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2. Non-negative functions. In this section we shall examine the constants appearing in the theorems of M. Riesz, Zygmund and Kolmogorov in the case of real functions which are non-negative and 2π-periodic. If f is an integrable 2π-periodic function, we let $\tilde{f}$ denote its conjugate:

$$\tilde{f}(x) = \text{P.V.}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\cot\frac{t}{2}\,dt.$$

Our first lemma is a refinement of the inequality on which the proof of M. Riesz's theorem in [4] is based.

LEMMA 2.1. *Let* $|x| \leqslant \pi/2$, $0 < \gamma < \pi/2$.

(a) *If* $0 < p \leqslant 2$, $p \neq 1$, *and if we put*

$$A(p,\gamma) = \frac{\tan^{p-1}\gamma}{\tan(p-1)\gamma}, \quad B(p,\gamma) = \frac{\sin^{p-1}\gamma}{\sin(p-1)\gamma}, \quad C(p) = -\frac{1}{\cos(p\pi/2)},$$

then

(2.2) $$C(p)(\cos^p x - \cos px) \leqslant |\sin x|^p \leqslant A(p,\gamma)\cos^p x - B(p,\gamma)\cos px$$

and

(2.3) $$\frac{2}{\pi}(\cos x\log\cos x + x\sin x) \leqslant |\sin x| \leqslant \frac{1}{\gamma}(\cos x\log\cos x + x\sin x) + \left(\frac{1}{\gamma}\log\frac{1}{\cos\gamma}\right)\cos x.$$

If $p < 1$, *then* (2.2) *holds also for* $\gamma = \pi/2$ $(A(p,\pi/2) = 0)$.

(b) *If* $p = 2$, *then* (2.2) *reduces to equality. If* $p < 2$, *then the right-hand side of* (2.2) *reduces to equality if and only if* $|x| = \gamma$. *The right-hand side of* (2.3) *reduces to equality if and only if* $|x| = \gamma$.

Proof. Since the expressions appearing in (2.2) and (2.3) are continuous and even functions of x, we may assume $0 < x < \pi/2$. It is very easy to verify that both sides of (2.2) reduce to equality if $p = 2$, and hence we may also assume $p < 2$.

The proof of (2.2) will be based on a careful examination of the function $F(x) = (\sin^p x - a\cos px)/\cos^p x$, a real. The proof of (2.3) and the last statement of (b) is based on a similar examination of the function $G(x) = (\sin x - a\cos x)/(\cos x\log\cos x + x\sin x)$, a real, and it will be omitted.

We may also observe that (2.3) is the limiting case of (2.2) as $p \to 1$. This will become apparent from the remarks following Theorem 2.4.

We observe now that

$$F'(x) = p\frac{\sin^{p-1}x}{\cos^{p+1}x}g(x), \quad \text{where } g(x) = 1 + a\frac{\sin(p-1)x}{\sin^{p-1}x}.$$

It follows that

$$g'(x) = a(p-1)\frac{\sin(2-p)x}{\sin^p x}$$

is of constant sign, and hence g is strictly monotonic if $a \neq 0$.

If $a = -C(p)$, then $a(p-1) < 0$ and g is strictly decreasing. It follows that $g(x) > g\left(\frac{\pi-}{2}\right) = 0$. We conclude that $F'(x) > 0$, and hence $F(x) > F(0+) = -a$. The left-hand side of (2.2) follows.

If $a = -B(p,\gamma)$, then g (being strictly monotonic) vanishes only for $x = \gamma$, and consequently γ is the only zero of $F'(x)$. Since $(\sin(p-1)x)/\sin^{p-1}x = o(x)$ as $x \to 0$, $g(x)$, and hence $F'(x)$, is positive for small values of x. It follows that $F(\gamma) = A(p,\gamma)$ is the maximum of F, and it can be attained only for $x = \gamma$. The right-hand side of (2.2) and (b) follow. ∎

We prove now the promised inequality, (2.5), which contains the theorems of M. Riesz, Zygmund and Kolmogorov, It is obvious that (2.5) implies the theorems of M. Riesz and Kolmogorov (we have only to observe that $(p-1)A(p,\gamma) > 0$ and $(p-1)B(p,\gamma) > 0$, and then omit the non-positive of the two terms appearing in the right-hand side of the second inequality in (2.5)). We shall prove later that (2.6), and hence Zygmund's theorem, is the limiting case of (2.5) as $p \to 1$.

THEOREM 2.4. *Let* $f \geqslant 0$, $0 < p \leqslant 2$, $p \neq 1$, $0 < \gamma < \pi/2$ (*if* $p < 1$, *then we can also take* $\gamma = \pi/2$). *Then*

(2.5) $$C(p)(\|f\|_p^p - \|f\|_1^p) \leqslant \|\tilde{f}\|_p^p \leqslant A(p,\gamma)\|f\|_p^p - B(p,\gamma)\|f\|_1^p$$

for all $f \in L^p \cap L^1$, *and*

(2.6) $$\frac{2}{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\log f\right) \leqslant \|\tilde{f}\|_1 \leqslant \frac{1}{\gamma}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\log f\right) + \frac{1}{\gamma}\log\frac{1}{\cos\gamma}$$

for all $f \in L\log^+ L$ *such that* $\|f\|_1 = 1$.

Proof. (Following [4], Chapter VII, Section 2). Let $u(z) = u(re^{ix})$ and $v(z) = v(re^{ix})$, $0 \leqslant r < 1$, be the Poisson and conjugate Poisson integral of f respectively. We may assume $f \not\equiv 0$ so that $u(z) > 0$. It follows that the functions $(u+iv)^p$ and $(u+iv)\log(u+iv)$ are holomorphic on the open unit disc, and hence their real parts are harmonic there.

Putting $u+iv = Re^{i\theta}$, $|\theta| < \pi/2$, and observing that $u(0) = \|f\|_1$ we obtain

$$(*) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}[R(re^{ix})]^p \cos p\theta(re^{ix})\,dx = [u(0)]^p = \|f\|_1^p$$

and (after some easy manipulations), if $\|f\|_1 = 1$,

$$(**) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}R(re^{ix})[\cos\theta(re^{ix})\log\cos\theta(re^{ix}) + \theta(re^{ix})\sin\theta(re^{ix})]\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}u(re^{ix})\log u(re^{ix})\,dx.$$

We apply now (2.2) with $\theta(re^{ix})$ instead of x, we multiply the resulting inequality by $[R(re^{ix})]^p$, and we integrate with respect to x. Letting $r\to 1$ and using Fatou's lemma we obtain (2.5) via (*) and the corresponding integrability conditions on f. By the same method we can obtain (2.6). In this case we make use of (2.3) and (**) instead of (2.2) and (*), and we multiply by $R(re^{ix})$ instead of $[R(re^{ix})]^p$. ■

The following two special cases of (2.5), corresponding to $\gamma = \pi/2p$ (if $p > 1$) and $\gamma = \pi/2$ (if $p < 1$), will be useful later.

$$(2.7) \qquad \|\tilde{f}\|_p < \tan(\pi/2p)\|f\|_p, \qquad 1 < p \leqslant 2,\ f \geqslant 0,\ f\in L^p,\ f \not\equiv 0.$$

$$(2.8) \qquad \frac{1}{\cos(p\pi/2)}(\|f\|_1^p - \|f\|_p^p) \leqslant \|\tilde{f}\|_p^p \leqslant \frac{1}{\cos(p\pi/2)}\|f\|_1^p,$$

$$0 < p < 1,\ f \geqslant 0,\ f\in L^1.$$

We list now some elementary properties of the constants appearing in (2.5) and (2.6). We omit the easy proofs.

(2.9) (a) *If $p > 1$, then: $A(p,\gamma)$ (considered as a function of γ) decreases from $+\infty$ to $\tan^p(\pi/2p)$ in $(0, \pi/2p)$ and increases from $\tan(\pi/2p)$ to $+\infty$ in $(\pi/2p, \pi/2)$, and $B(p,\gamma)$ (considered as a function of γ) decreases from $+\infty$ to $C(p)$ in $(0, \pi/2)$.*

(b) *If $p < 1$, then: $A(p,\gamma)$ (considered as a function of γ) increases from $-\infty$ to 0 in $(0,\pi/2)$, and $B(p,\gamma)$ (considered as a function of γ) increases from $-\infty$ to $C(p)$ in $(0,\pi/2)$.*

(c) *$(1/\gamma)\log(1/\cos\gamma)$ increases from 0 to $+\infty$ in $(0,\pi/2)$ and it is $O\left(\left|\log\left(\frac{1}{\gamma}-\frac{2}{\pi}\right)\right|\right)$ as $\gamma\to\frac{\pi}{2}$.*

$$(2.10) \qquad \begin{cases} \lim\limits_{p\to 1}(p-1)A(p,\gamma) = \lim\limits_{p\to 1}(p-1)B(p,\gamma) = \dfrac{1}{\gamma},\ \lim\limits_{p\to 1}(p-1)C(p) = \dfrac{2}{\pi}, \\ \lim\limits_{p\to 1}[A(p,\gamma)-B(p,\gamma)] = \dfrac{1}{\gamma}\log\dfrac{1}{\cos\gamma}, \\ A(p,\gamma)-B(p,\gamma) = O(2-p) \ \text{as}\ p\to 2. \end{cases}$$

$$(2.11) \qquad A(p,\gamma)-B(p,\gamma) = O(\gamma^p) \ \text{as}\ \gamma\to 0.$$

It follows from (2.9)(a) and (2.11) that (if $p > 1$) the only interesting values of γ are those in $(0, \pi/2p]$.

Formulae (2.10) enable us to say that (2.6) *is the limiting case of* (2.5) *as* $p\to 1$. Indeed, let $f\in L\log^+ L$ and $\|f\|_1 = 1$. If $p < 1$, (2.5) implies

$$C(p)(p-1)\frac{\|f\|_p^p-1}{p-1} \leqslant \|\tilde{f}\|_p^p \leqslant A(p,\gamma)(p-1)\frac{\|f\|_p^p-1}{p-1} + [A(p,\gamma)-B(p,\gamma)].$$

Letting $p\to 1$ we obtain (2.6) via (2.10) and the formula

$$\frac{\|f\|_p^p-1}{p-1} \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f\log f \quad \text{as } p\to 1,$$

which follows easily from Lebesgue's dominated convergence theorem.

Our next goal is to show that the constants appearing in (2.5) and (2.6) are best possible. Theorem 2.12 contains the desired results.

THEOREM 2.12. *Let $0 < p < 2$, $p\neq 1$, $0 < \gamma < \pi/2$, $f \geqslant 0$, $f\not\equiv 0$.*

(a) *Let $f\in L^p\cap L^1$. Then $\|\tilde{f}\|_p^p = A(p,\gamma)\|f\|_p^p - B(p,\gamma)\|f\|_1^p$ if and only if $|\tilde{f}| = (\tan\gamma)f$ almost everywhere.*

(b) *Let $f\in L\log^+ L$. Then, if $\|f\|_1 = 1$, $\|\tilde{f}\|_1 = \frac{1}{\gamma}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\log f\right) + \frac{1}{\gamma}\log\frac{1}{\cos\gamma}$ if and only if $|\tilde{f}| = (\tan\gamma)f$ almost everywhere.*

(c) *Let $\gamma < \pi/2p$. Then there exists a function $g\in L^p\cap L^1$ such that $g\geqslant 0$, $g\not\equiv 0$, and $|\tilde{g}| = (\tan\gamma)g$ almost everywhere.*

(d) *Let $p > 1$, $\gamma < \pi/2p$. Then $A(p,\gamma)$ is the smallest value of A such that $\|\tilde{f}\|_p^p \leqslant A\|f\|_p^p - B(p,\gamma)\|f\|_1^p$ for every $f\geqslant 0$ in L^p, and $B(p,\gamma)$ is the largest value of B such that $\|\tilde{f}\|_p^p \leqslant A(p,\gamma)\|f\|_p^p - B\|f\|_1^p$ for every $f\geqslant 0$ in L^p. Analogous results hold for the constants $\frac{1}{\gamma}$ and $\frac{1}{\gamma}\log\frac{1}{\cos\gamma}$ and for the case $p < 1$.*

(e) *$C(p)$ is the largest value of C such that $C\left|\|f\|_p^p - \|f\|_1^p\right| \leqslant \|\tilde{f}\|_p^p$ for every $f\geqslant 0$ in $L^p\cap L^1$, and $\frac{2}{\pi}$ is the largest value of C such that $C\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\log f\right) \leqslant \|\tilde{f}\|_1$ for every $f\leqslant 0$ in $L\log^+ L$ with $\|f\|_1 = 1$.*

Proof. (a) We use the same notation as in the proof of Theorem 2.4. Since $\lim R(re^{ix})$ (as $r\to 1$) is positive for almost all x, $\lim\theta(re^{ix})$ (as $r\to 1$) exists almost everywhere (see [4], Chapter VII, 7.25). If we denote the above limits by $R(x)$ and $\theta(x)$ respectively, we can easily verify that $f(x) = R(x)\cos\theta(x)$ and $\tilde f(x) = R(x)\sin\theta(x)$ and $|\theta(x)|\leqslant\frac{\pi}{2}$ for almost all x. Moreover, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[R(x)]^p\cos p\theta(x)\,dx = \|f\|_1^p.$$

If $|\tilde f| = (\tan\gamma)f$ almost everywhere, then $|\theta| = \gamma$ for almost all x and the right-hand side of (2.2) becomes equality if we write $\theta(x)$ instead of x. The same argument as in the proof of Theorem 2.4 shows that under these conditions the second inequality in (2.5) reduces to equality.

If $|\tilde f|\neq(\tan\gamma)f$ on a set E' of positive measure, then (by Lemma 2.1 (b)) there exists a set $E\subset E'$ of positive measure and a positive number ε such that for every $x\in E$ we have: $R(x)>\varepsilon$ and

$$|\sin\theta(x)|^p\leqslant A(p,\gamma)\cos^p\theta(x) - B(p,\gamma)\cos^p p\,\theta(x) - \varepsilon.$$

Arguing as in the proof of Theorem 2.4 we obtain

$$\|\tilde f\|_p^p\leqslant A(p,\gamma)\|f\|_p^p - B(p,\gamma)\|f\|_1^p - \left(\tfrac{1}{2\pi}\right)|E|\,\varepsilon^{p+1} < A(p,\gamma)\|f\|_p^p - B(p,\gamma)\|f\|_1^p,$$

which completes the proof of (a).

(b) The proof is the same as in (a).

(c) (Following a suggestion of Professor A. Calderòn). The function $\left(\frac{1+z}{1-z}\right)^{2\gamma/\pi}$, $\left|\arg\left(\frac{1+z}{1-z}\right)\right|<\frac{\pi}{2}$, is holomorphic on the open unit disc and (since $2\gamma/\pi<1/p$) it belongs to H^p. If $g(x)+i\tilde g(x)$ is its boundary value, then $\tilde g$ is the conjugate of g and $g\not\equiv 0$. Moreover, we have: $|\tilde g(x)| = (\tan\gamma)g(x)$ for all $x\neq 0$.

(d) This follows immediately from (a), (c) and (2.9).

(e) If $p<1$, (e) is a simple consequence of (2.8). Indeed, let $f\in L^1$, $f\geqslant 0$, $\|f\|_1 = 1$, and let us define f_n, $n = 1, 2, \ldots$, to be equal to $nf(nx)$ if $|x|\leqslant\frac{\pi}{n}$ and 0 in the remainder of $(-\pi,\pi]$. Since $\|f_n\|_1 = 1$ and $\|f_n\|_p^p = n^{p-1}$, (2.8) implies

$$\lim\|\tilde f_n\|_p^p = \frac{1}{\cos(p\pi/2)}\quad\text{as } n\to\infty, \tag{2.13}$$

which proves (e) for $p<1$.

If, in addition, $f\in L\log^+L$, then $\frac{1}{2\pi}\int_{-\pi}^{\pi}f_n\log f_n = \log n+\frac{1}{2\pi}\int_{-\pi}^{\pi}f\log f$, and (2.6), with $\frac{1}{\gamma} = \frac{2}{\pi}+\frac{1}{\log n}$, implies

$$o(1)+\frac{2}{\pi}\leqslant\frac{1}{\log n}\|\tilde f_n\|_1\leqslant\left(\frac{2}{\pi}+\frac{1}{\log n}\right)\left(o(1)+1\right)+\frac{1}{\log n}O(\log\log n)\quad\text{as } n\to\infty,$$

and hence

$$\|\tilde f_n\|_1\sim\frac{2}{\pi}\log n\quad\text{as } n\to\infty. \tag{2.14}$$

The second part of (e) follows.

The above method does not work if $p>1$. In this case we argue as follows (the argument works for $p<1$ as well):

Let $0<h<\frac{\pi}{2}(2-p)$, $a(\pi-h) = \frac{\pi}{2}$. The function

$$G(z) = [e^{2ih}(z-e^{-ih})/(z-e^{ih})]^a,\qquad |z|<1,$$

where the argument of the expression in square brackets is less than π in absolute value, belongs to H^p. The boundary value $g(x)+i\tilde g(x)$ of G is $(\cos ah)q(x)+i(\sin ah)q(x)$ if $|x|>h$, and $-iq(x)$ if $|x|\leqslant h$, where

$$q(x) = \left|\left(\sin\frac{x+h}{2}\right)\Big/\left(\sin\frac{x-h}{2}\right)\right|^a.$$

The norms $\|g\|_p$ and $\|\tilde g\|_p$ can be computed easily by substituting $y = \left(\sin\frac{x+h}{2}\right)\Big/\left(\sin\frac{x-h}{2}\right)$ and then applying the method of residua. It follows

$$\|g\|_p^p = \frac{\sin(p\pi/2)\cos^p ah}{\sin pa\pi}\quad\text{and}\quad\|\tilde g\|_p^p = \frac{\sin^p ah+\sin pah}{\sin pa\pi}.$$

Observing now that $\|g\|_1 = 1$ we obtain

$$\frac{\|\tilde g\|_p^p}{\|g_p^p\|-\|g\|_1^p} = \frac{\sin^p ah+\sin pah}{\sin(p\pi/2)\cos^p ah-\sin\left(pah+p\frac{\pi}{2}\right)}.$$

Applying now the de l'Hospital rules we see that the last expression tends to $C(p)$ as $ah\to 0$, and hence as $h\to 0$ (since a remains bounded as $h\to 0$). The proof of (e) is complete. ■

We give now an application of formulae (2.5) and (2.13).

Let $\tilde{f}_h$ denote the characteristic function of the interval $(-h, h)$, $0 < h < \pi$. We know (see [4], Chapter II, page 72, example 19) that

$$\tilde{f}_h(x) = \frac{1}{\pi}\log\left|\left(\sin\frac{x+h}{2}\right)\Big/\left(\sin\frac{x-h}{2}\right)\right|.$$

Splitting the interval $(-\pi, \pi)$ into the four subintervals $(-\pi, -h)$, $(-h, 0)$, $(0, h)$ and (h, π) and substituting $y = \log\left|\left(\sin\frac{x+h}{2}\right)\Big/\left(\sin\frac{x-h}{2}\right)\right|$ we can easily verify that

$$\|\tilde{f}_h\|_p^p = \frac{2\sin h}{\pi^{p+1}}\left[\int_0^\infty y^p\frac{e^y}{e^{2y}+2\cos h e^y+1}dy + \int_0^\infty y^p\frac{e^y}{e^{2y}-2\cos h e^y+1}dy\right]$$
$$= \frac{2\sin h}{\pi^{p+1}}[I_1(h)+I_2(h)].$$

We assume first that $p > 1$. The expressions under the sign of integration in the above equality are positive and they are majorized by the integrable (over the positive real axis) function $y^p e^y/(e^y-1)^2$. It follows

$$I_1(h) \to \int_0^\infty y^p\frac{e^y}{(e^y+1)^2}dy \quad\text{and}\quad I_2(h) \to \int_0^\infty y^p\frac{e^y}{(e^y-1)^2}dy \quad\text{as } h \to +0.$$

Using now the formulae ([3], Chapter XIII, pages 266 and 267)

$$\zeta(p)\Gamma(p) = \int_0^\infty \frac{y^{p-1}}{e^y-1}dy \quad\text{and}\quad (1-2^{1-p})\zeta(p)\Gamma(p) = \int_0^\infty\frac{y^{p-1}}{e^y+1}dy,\ p>1,$$

where ζ and Γ are the Zeta function of Riemann and the Gamma function respectively, and integrating by parts we obtain

$$I_1(h) \to p(1-2^{1-p})\zeta(p)\Gamma(p) \quad\text{and}\quad I_2(h) \to p\zeta(p)\Gamma(p) \text{ as } h \to 0.$$

Collecting results and applying (2.5) to the function f_h we deduce

$$(2.15)\qquad \frac{\pi^p}{4p(1-2^{-p})\Gamma(p)\sin\frac{(p-1)\pi}{2}} \leqslant \zeta(p) \leqslant \frac{\pi^p\tan^p(\pi/2p)}{4p(1-2^{-p})\Gamma(p)},\quad 1<p\leqslant 2.$$

If $p = 2$ (and assymptotically if $p \to 1$) (2.15) reduces to equality.

If $p < 1$, then we expand the functions appearing under the sign of integration in the definition of $I_1(h)$ and $I_2(h)$ into series of powers of e^{th} and we integrate termwise. A straightforward computation yields the formula (compare with formula 2.1, [4], Chapter V):

$$(2.16)\qquad \sum_{n=1}^\infty \frac{1}{n^{p+1}}\sin(2n-1)h \sim (2h)^p\Gamma(1-p)\sin(p\pi/2)$$
$$\text{as } h \to +0,\ 0<p<1.$$

We shall return to characteristic functions in Section 4.

Before we leave the case of non-negative functions we make a final remark concerning formulae (2.5) and (2.6). We shall restrict ourselves to the case $1 < p \leqslant 2$. It is very easy to show that the right-hand side of the second inequality in (2.5) attains its minimum when γ is the (unique) solution of the equation

$$\|f\|_p^p/\|f\|_1^p = \cos^p\gamma/\cos p\gamma,\quad 0 \leqslant \gamma < \pi/2p,$$

and the minimum is $\tan\gamma\|f\|_p^p$ (the case $p = 2$ is simpler and illuminates this fact). Similarly the minimum of the right-hand side of the second inequality in (2.6) (assuming $\|f\|_1 = 1$) is $\tan\gamma$, where γ is the (unique) solution of the equation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f\log f = \gamma\tan\gamma + \log\cos\gamma.$$

3. Functions of variable sign. In this section we consider real 2π-periodic functions. As usually, we denote by f_+ and f_- the positive and the negative part of such a function f ($f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$).

For the constant B_p in Kolmogorov's theorem we have very little to say. Let $p < 1$ and $f \in L^1$. Applying the second inequality in (2.8) to the functions f_+ and f_- and adding the resulting inequalities we obtain (see also [4], Chapter VII, Section 2)

$$\|\tilde{f}\|_p^p \leqslant \|\tilde{f}_+\|_p^p + \|\tilde{f}_-\|_p^p \leqslant \frac{1}{\cos(p\pi/2)}(\|f_+\|_1^p + \|f_-\|_1^p) \leqslant \frac{2^{1-p}}{\cos(p\pi/2)}\|f\|_1^p.$$

Theorem 2.12, (a) and (c), implies that for every $B < 1/\cos(p\pi/2)$ there exists an $f \in L^1$ such that $\|\tilde{f}\|_p^p > B\|f\|_1^p$. We have proved

THEOREM 3.1. *The least value B_p of the numbers B such that $\|\tilde{f}\|_p \leqslant B\|f\|_1$ for every $f \in L^1$ satisfies the inequality*

$$(3.2)\qquad \frac{1}{\cos(p\pi/2)} \leqslant B_p^p \leqslant \frac{2^{1-p}}{\cos(p\pi/2)},\quad 0<p<1,$$

and hence it is assymptotically equal to $\frac{2/\pi}{1-p}$ *as* $p \to 1$. *If we restrict ourselves to non-negative functions, then* $B_p = [\cos(p\pi/2)]^{-1/p}$.

We examine now the constants appearing in Zygmund's theorem. If $f \geqslant 0$ and $f \in L\log^+ L$, then (2.6) implies

$$(*)\qquad \|\tilde{f}\|_1 \leqslant \frac{1}{\gamma}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\log f\right) - \frac{1}{\gamma}\|f\|_1\log\|f\|_1 + \frac{1}{\gamma}\log\frac{1}{\cos\gamma}\|f\|_1$$
$$\leqslant \frac{1}{\gamma}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f\log^+ f\right) + \frac{1}{\gamma}\log\frac{1}{\cos\gamma}\|f\|_1 + \frac{1}{e\gamma},\quad 0<\gamma<\pi/2.$$

Applying (*) to the functions f_+ and f_- and adding the resulting inequalities we obtain

$$\|\tilde{f}\|_1 \leqslant \frac{1}{\gamma}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\log^+|f|\right)+\frac{1}{\gamma}\log\frac{1}{\cos\gamma}\,\|f\|_1+\frac{2}{e\gamma}, \tag{3.3}$$
$$0<\gamma<\pi/2,\quad f\in L\log^+L.$$

We observe now that for any $\varepsilon>0$ and any function $f\in L\log^+L$ there exists a number C (depending on ε only) such that

$$\int_{-\pi}^{\pi}|f| < \varepsilon\left(\int_{\pi}^{\pi}|f|\log^+|f|\right)+C.$$

Combining this remark with Theorem 2.12, (b) and (c), and (2.9)(c) we deduce

THEOREM 3.4. *A necessary and sufficient condition that there exists a number B such that*

$$\|\tilde{f}\|_1 \leqslant A\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}[|f|\log^+|f|\right)+B$$

for every $f\in L\log^+L$ is that A is greater than $\frac{2}{\pi}$.

A somewhat stronger result can be obtained by the same method. Namely: *Suppose that A, B, C are such that*

$$\|\tilde{f}\|_1 \leqslant A\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\log^+|f|\right)+B\|f\|_1+C$$

for every $f\in L\log^+L$. Then $A>\frac{2}{\pi}$ and for any fixed C and any positive $a<1$, B is greater than $aA\log\left(1/\cos\frac{1}{A}\right)$ provided that A is sufficiently close to $\frac{2}{\pi}$. We omit the easy proof.

We pass now to the constant A_p in M. Riesz's theorem. Our goal is to show that if $1<p\leqslant 2$, then the least value of A_p is $\tan(\pi/2p)$.

Let $f\in L^p$, $1<p\leqslant 2$, and let R, u, v be defined as in the proof of Theorem 2.4. We define first a function G on the open complex plane so that; $G(0)=0$ and if $z=x+iy\neq 0$ then $G(z)=|z|^p\cos p\theta(z)$, where $\theta(z)=\arctan(y/|x|)$, $|\theta(z)|\leqslant\pi/2$. We define now a function F on the open unit disc by putting $F(z)=G\big(u(z)+iv(z)\big)$.

The main idea for the proof of our assertion about A_p is to use the same method as in the case of non-negative functions. In order to overcome the difficulties arising from the fact that $(u+iv)^p$ can no longer be defined as a holomorphic function, we have introduced the function F. The proof will be based on the following

LEMMA 3.5. *F is subharmonic.*

Proof. Since $u+iv$ is holomorphic, it suffices to prove that G is subharmonic on the open complex plane (see [2], Chapter X, Section 8, Exercise 2). G is obviously continuous, and hence it is enough to show (see [2], Chapter X, (8.10)A) that for every complex number z there are arbitrarily small positive numbers r such that

$$G(z)\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}G(z+re^{it})\,dt. \tag{**}$$

Since G coincides with a harmonic function on the right open half plane and on the left open half plane ($\operatorname{Re}z^p$, $|\arg z|<\frac{\pi}{2}$, and $\operatorname{Re}(-z)^p$, $|\arg(-z)|<\frac{\pi}{2}$, respectively), (**) is satisfied for all z such that $\operatorname{Re}z\neq 0$.

If $z=0$, then for every positive number r we have (note that $1<p\leqslant 2$)

$$\frac{1}{2\pi}\int_{\pi}^{\pi}G(re^{it})\,dt = 2\left(\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}r^p\cos pt\,dt\right)=\frac{r^p}{p\pi}\sin(p\pi/2)\geqslant 0 = G(0)$$

which proves (**).

It remains to examine the case $z=iy$, where y is a real number different from zero. Let $H(z)=\operatorname{Re}z^p$, $z\neq 0$, $|\arg z|<\pi$. We observe that H coincides with G on the closed right half plane (the origin excluded). If $z=ae^{ix}$, $a>0$, $\frac{\pi}{2}<|x|<\pi$, then

$$G(z)-H(z)=a^p\cos p(x-\pi)-a^p\cos px = 2a^p\sin p\left(x-\frac{\pi}{2}\right)\sin(p\pi/2)$$

if $\frac{\pi}{2}<x<\pi$, and

$$G(z)-H(z)=a^p\cos p(x+\pi)-a^p\cos px = -2a^p\sin p\left(x+\frac{\pi}{2}\right)\sin(p\pi/2)$$

if $-\pi<x<-\frac{\pi}{2}$. In both cases $G(z)-H(z)$ is non-negative. Combining this result and the fact that H is harmonic in the complement of the non-positive real axis we obtain

$$G(iy)=H(iy)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(iy+re^{it})\,dt\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}G(iy+re^{it})\,dt$$

if $0<r<|y|$, which proves (**).

The proof of Lemma 3.5 is complete. ■

We return now to the constant A_p in M. Riesz's theorem. We put $\Phi = \theta(u+iv)$, where θ has the same measning as in the definition of F. Φ is defined on the open unit disc except for those points where $\operatorname{Re} z = 0$. Defining Φ arbitrarily at those points we can easily verify that

$$|u(z)| = R(z)\cos\Phi(z),\ |v(z)| = R(z)|\sin\Phi(z)| \text{ and (if } R(z)\neq 0)\ |\Phi(z)|\leqslant\frac{\pi}{2}.$$

Moreover, we have $F(z) = [R(z)]^p\cos p\Phi(z)$. Since F is subharmonic and

$$F(0) = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt\right|^p,$$

the same argument as in the proof of Theorem 2.4 implies

$$\|\tilde f\|_p^p \leqslant A(p,\gamma)\|f\|_p^p - B(p,\gamma)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt\right|^p,\quad 0<\gamma<\frac{\pi}{2},\quad 1<p\leqslant 2,\ f\in L^p. \tag{3.6}$$

Theorem 2.12 (c) shows that for any $\gamma<\pi/2p$ there exists an $f\in L_p$ such that $f\not\equiv 0$ and $|\tilde f| = (\tan\gamma)f$ almost everywhere. It follows that $\|\tilde f\|_p = \tan\gamma\|f\|_p$. Using this fact and applying (3.6) (with $\gamma = \pi/2p$) we deduce that the least value of A_p is $\tan(\pi/2p)$ if $1<p\leqslant 2$. If $p\geqslant 2$, then (by duality) the least value of A_p is $\tan\left(\left(\frac{\pi}{2}1-\frac{1}{p}\right)\right) = \cot(\pi/2p)$. We have proved

THEOREM 3.7. *The least value A_p of the numbers A such that $\|\tilde f\|_p \leqslant A\|f\|_p$, $p>1$, for every (real and 2π-periodic) $f\in L^p$, is $\tan(\pi/2p)$ if $p\leqslant 2$ and $\cot(\pi/2p)$ if $p\geqslant 2$.*

We remark finally that Lemma 3.5 remains valid under the weaker assumptions that $u+iv$ is a holomorphic function on the open unit disc (not necessarily in H^p, $1<p\leqslant 2$) and that $0<p\leqslant 2$.

4. Non-periodic functions. We begin this section by extending Theorem 3.7 to real (not necessarily periodic) functions on R^1. For such a function f we shall denote by Hf its Hilbert transform (if it exists):

$$Hf(x) = \text{P.V.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt,\ x\in R^1.$$

It is well known that if $f\in L^p(R^1)$ for some $p>1$, then Hf exists for almost all x. Moreover, there is a constant A_p, which depends on p only, such that $\|Hf\|_p\leqslant A_p\|f\|_p$. We also know that A_p can be taken the same as in the periodic case (see [4], Chapter XVI, Theorem 3.8. The result we are looking for is contained in the proof of this theorem, although it is not stated explicitly). It is very easy to show that if $1<p<2$ and $(p-1)\pi/2p<\gamma<\pi/2p$, then the function

$$F(z) = \frac{1}{z+1}\left(i\frac{z+1}{z-1}\right)^{2\gamma/\pi},\ \operatorname{Im} z>0,$$

belongs to H^p in the upper half plane (the argument of $i\frac{z+1}{z-1}$ is taken less than $\pi/2$ in absolute value). If $f+ig$ is the boundary value of F, then $g = Hf$ and $f\not\equiv 0$. Moreover, we have $|g(x)| = (\tan\gamma)|f(x)|$ for all x different from 1 and -1. (This example has been taken from [1]). Combining this result and Theorem 3.7 we obtain the following

THEOREM 4.1. *The least value A_p of the numbers A such that $\|Hf\|_p \leqslant A\|f\|_p$, $p>1$, for every (real) f in $L^p(R^1)$, is $\tan(\pi/2p)$ if $p\leqslant 2$ and $\cot(\pi/2p)$ if $p\geqslant 2$.*

We shall consider now the special case of characteristic functions of measurable sets E such that $|E|<\infty$ (the results which follow were communicated to the author by Professor A. Zygmund).

According to a result of E. Stein and G. Weiss, the distribution function of the Hilbert transform of the characteristic function f_E of such a set E is $2|E|/\sinh y$, $y>0$ (see [5], Chapter II, 2.4). It follows that the ratio $\|Hf_E\|_p/\|f_E\|_p$ is independent of E. A computation similar to that leading to formula (2.15) yields

$$\|Hf_E\|_p^p/\|f_E\|_p^p = 4p\pi^{-p}(1-2^{-p})\zeta(p)\Gamma(p),\quad 1<p\leqslant 2. \tag{4.2}$$

We observe that in the periodic case the ratio $\|\tilde f_E\|_p/\|f_E\|_p$ is not independent of E (e.g. if $|E| = 2\pi$, then the above ratio is zero), but it tends to the right-hand side of (4.2), as $|E|\to 0$. This is due to the fact that in the periodic case the distribution function of $\tilde f_E$ is not proportional to $|E|$, although it depends on $|E|$ only.

Following now a suggestion of Professor A. Zygmund, we shall consider functions of several variables.

Let $K(x)$, $x\in R^n$ (n is a positive integer), be an odd positively homogeneous real function (kernel) of degree $-n$, and let Σ denote the surface of the unit sphere in R^n. We know that if the integral of K over Σ is finite, and if we define $\tilde g_\varepsilon(x)$ by the formula

$$\tilde g_\varepsilon(x) = \int_{|y|\geqslant\varepsilon} g(x-y)K(y)\,dy,\quad x\in R^n,\ g\in L^p(R^n),\ p>1,\ \varepsilon>0,$$

then $\tilde g_\varepsilon$ converges as $\varepsilon\to 0$ (in the L^p norm) to a function $\tilde g$. Moreover, we have the following inequality

$$\|\tilde g\|_p\leqslant(\pi/2)A_p\Big(\int_\Sigma |K|\Big)\|g\|_p,$$

where A_p is the constant appearing in Theorem 4.1 (see [5], Chapter III, Theorem 11). It is clear that if K, considered as a mass distribution on Σ, is concentrated in the neighborhood of two antipodal points of Σ, then $\tilde{g}$ behaves like the Hilbert transform of the one-dimensional case. Thus, it should be conceivable that the constant

$$C = (\pi/2) A_p \Big(\int_{\Sigma} |K| \Big)$$

appearing in the previous inequality is in some sense best possible. The purpose of our last theorem is to make this statement precise.

THEOREM 4.3. *With the above established notation, if $p > 1$ and $a < 1$, then there exists an odd positively homogeneous kernel (of degree $-n$) such that for some function g in $L^p(R^n)$ we have: $g \not\equiv 0$ and $\|\tilde{g}\|_p \geqslant aC\|g\|_p$.*

Proof. Let e_1 be the unit vector along the x_1 axis. Let also $h(x_1)$, $x_1 \in R^1$, and $q(v)$, $v \in R^{n-1}$, be two continuously differentiable functions with compact support. We assume that h and q do not vanish identically and that $\|Hh\|_p > bA_p\|h\|_p$ for some b such that $1 > b > a$ (this is possible because of Theorem 4.1). We define now a function g on R^n by putting

$$g(x) = g(x_1, x_2, \ldots, x_n) = h(x_1)q(v), \quad x_1 \in R^1,\ v = (x_2, \ldots, x_n) \in R^{n-1}.$$

Let K_m, $m = 1, 2, \ldots$, be positively homogeneous kernels (of degree $-n$ which are odd and such that K_m vanishes on Σ except for the points of $\Sigma_m = \Sigma \cap \{x\colon |x-e_1| \leqslant 1/m\}$ and $-\Sigma_m = \{x\colon -x \in \Sigma_m\}$, where K_m is $1/(\text{area } \Sigma_m)$ and $-1/(\text{area } \Sigma_m)$ respectively. It follows that

$$\int_{\Sigma} |K| = 2,$$

and hence the corresponding constant C is πA_p for all m. Let finally

$$\tilde{g}_m(x) = \int_{R^n} g(x-y) K_m(y)\,dy,$$

and write $a_m = \text{area } \Sigma_m$. We observe that if $x = (x_1, v)$, $x_1 \in R^1$, $v \in R^{n-1}$, then

$$\begin{aligned}
&\tilde{g}_m(x) - \pi Hh(x_1)q(v) \\
&= \int_{R^n} g(x-y)K_m(y)\,dy - \Big(\int_0^\infty \frac{h(x_1-t)-h(x_1+t)}{t}\,dt \Big) q(v) \\
&= \int_{\Sigma} K(y') \Big(\int_0^\infty \frac{g(x-ty')}{t}\,dt \Big) dy' - \int_0^\infty \frac{g(x-te_1)-g(x+te_1)}{t}\,dt \\
&= \int_{\Sigma_m} \frac{1}{a_m} \Big\{ \int_0^\infty \Big[\frac{g(x-ty')-g(x-te_1)}{t} - \frac{g(x+ty')-g(x+te_1)}{t} \Big] dt \Big\} dy'.
\end{aligned}$$

The integrand inside the square brackets is zero if

$$t \geqslant |x| + \sup\{|y| : g(y) \neq 0\}$$

and it is majorized by $2(\sup \nabla g)|y'-e_1|$. Since

$$2(\sup \nabla g)|y'-e_1| = O(1/m) \quad \text{as } m \to \infty,$$

$\tilde{g}_m(x)$ converges to $\pi Hh(x_1)q(v)$ as $m \to \infty$. Using now Fatou's lemma we obtain

$$\liminf_{m\to\infty} \|\tilde{g}_m\|_p \geqslant \|\pi q(Hh)\|_p = \pi\|Hh\|_p\|q\|_p \geqslant b\pi A_p\|h\|_p\|q\|_p = bC\|g\|_p > aC\|g\|_p.$$

The proof of Theorem 4.3 is complete. ■

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