Lévy's probability measures on Euclidean spaces

by

E. URBANIEK (Wrocław)

Dedicated to Professor Antoni Zygmund
in honour of the fiftieth anniversary of his scientific activity

Abstract. The limit laws arising from an affine modification of sequences of partial sums of independent random variables whose values belong to the Euclidean space are characterized in terms of operator-decomposability of probability measures. Our next aim is to give a representation of the characteristic function of these limit laws. The method of proof consists in finding the extreme points of a certain convex set of measures. Then the Choquet Theorem yields the representation formula.

Introduction. A Lévy's probability measure on the Euclidean space $\mathbb{R}^N$ is a limit law arising, roughly speaking, from affine modification of the partial sums of a sequence of independent and identically distributed random variables, i.e. the operator stable probability measures on $\mathbb{R}^N$ were considered by H. Sharpe in [12].

Throughout this paper we denote by $\mathcal{P}(\mathbb{R}^N)$ or, shortly, by $\mathcal{P}$ the set of all probability measures on $\mathbb{R}^N$. With the topology of weak convergence and multiplication defined by the convolution $\mathcal{P}$ becomes a topological semigroup. We denote the convolution of two measures $\lambda$ and $\mu$ by $\lambda \ast \mu$. Moreover, by $\delta_a(\mathbb{R}^N)$ we denote the probability measure concentrated at the point $a$. The characteristic function $\hat{\lambda}$ of a measure $\lambda \in \mathcal{P}$ is defined by the formula

$$\hat{\lambda}(x) = \int_{\mathbb{R}^N} \exp i(\langle x, y \rangle) \lambda(dy),$$

where $\langle x, y \rangle$ denotes the inner product in $\mathbb{R}^N$.

Given $\lambda \in \mathcal{P}$, we define $\lambda^-$ by the formula $\hat{\lambda^-}(E) = \hat{\lambda}(-E)$, where $-E = \{-x : x \in E\}$. The mapping $\lambda \mapsto \lambda^-$ is an involutive automorphism of $\mathcal{P}$. It is easy to see that $\lambda^- = \bar{\lambda}$, the last bar denoting the complex
conjugate. For any $\lambda \in \mathcal{D}$, the measure $\lambda^* = \lambda^* \lambda^*$ is called the symmetrization of $\lambda$.

We call a measure from $\mathcal{D}(R^n)$ full if its support is not contained in any $(n-1)$-dimensional hyperplane of $R^n$. We denote by $\mathcal{D}(R^n)$ or, shortly by $\mathcal{D}$, the set of all full probability measures on $R^n$. We mention that the set $\mathcal{D}$ is an open subsemigroup of $\mathcal{P}$.

Let $\text{End } R^n$ denote the semigroup of all linear operators on $R^n$ with the composition as a semigroup operation. Further, let $\text{Aut } R^n$ denote the group of all non-singular linear operators on $R^n$. For any $\lambda \in \text{End } R^n$ and $\lambda \in \mathcal{D}(R^n)$ let $\lambda \lambda$ denote the measure defined by the formula $\lambda \lambda(E) = \lambda(\lambda^{-1}(E))$ for all Borel subsets $E$ of $R^n$. It is easy to check that the equations for all $A, B : \text{End } R^n$

$A(Bx) = (ABx), \quad \lambda(\lambda \lambda) = \lambda \lambda \lambda \lambda, \quad A \lambda \lambda = \lambda \lambda \lambda$,

where $A^*$ denotes the adjoint operator. Moreover, the mapping $(A, \lambda) \mapsto A \lambda$ from $\text{End } R^n \times \mathcal{D}(R^n)$ onto $\mathcal{D}(R^n)$ is jointly continuous, where $\text{End } R^n$ is provided with a norm topology. Consequently, we have the following statement:

(i) If a sequence $(\lambda_n)$ is precompact in $\text{End } R^n$, then for every $\lambda \in \mathcal{D}(R^n)$ the sequence $(\lambda_n \lambda)$ is precompact in $\mathcal{D}(R^n)$.

For full measures the converse implication is also true. Namely, we shall prove the following statement:

(ii) Let $\lambda \in \mathcal{D}(R^n)$ and $\lambda \in \text{End } R^n$ $(n = 1, 2, \ldots)$. If the sequence $(\lambda \lambda_n)$ is precompact in $\mathcal{D}(R^n)$, then the sequence $(\lambda \lambda_n)$ is precompact in $\text{End } R^n$.

Proof. We shall assume for the purpose of obtaining a contradiction that $(\lambda \lambda_n)$ is precompact and $(\lambda \lambda_n)$ is bounded. Let us choose vectors $x_n$ in $R^n$ such that $\|x_n\| = 1$ and $\|x_n\| = \|x_n\| (s = 1, 2, \ldots)$. Passing to a subsequence, if necessary, it may be assumed that $\|x_n\| \to \infty$ and the sequence of vectors $\lambda \lambda_n^{-1} x_n$ tends to a vector $u \in R^n$ with $\|u\| = 1$. Since the sequence $(\lambda \lambda_n)$ is precompact and $\|x_n\| \to 0$, we infer that

$$\lim_{n \to \infty} A \lambda_\infty(\lambda \lambda_n^{-1} x_n) = 1$$

for every $\sigma \in E$. Consequently, by the transformation rule of $\lambda \lambda$, we have

$$\lim_{n \to \infty} \lambda_\infty(\lambda \lambda_n^{-1} A \lambda_n x_n) = 1,$$

which yields the equation $\lambda(\sigma) = 1$ for all $\sigma \in E$. In other words, we proved that the characteristic function of $\lambda$ is equal to 1 on a one-dimensional subspace of $R^n$. Hence it follows that $\lambda$ is not a full measure (see [12], p. 52, Proposition 1). The contradiction implies that $\|x_n\|$ is bounded and, consequently, the sequence $(\lambda \lambda_n)$ is precompact in $\text{End } R^n$.

1. Operator-decomposability of measures. Let $\lambda \in \mathcal{D}(R^n)$ and $\lambda \in \text{End } R^n$. Suppose that there exists a measure $\lambda_\infty \in \mathcal{D}(R^n)$ for which the equation

$$\lambda = \lambda_\infty \lambda_\infty$$

holds. Then we say that the measure $\lambda$ is $\lambda_\infty$-decomposable. We denote by $\mathcal{E}(\lambda)$ the set of all operators $A$ such that the measure $\lambda$ is $\lambda_\infty$-decomposable. Further, by $\mathcal{A}(\lambda)$ we denote the subset of $\mathcal{E}(\lambda)$ consisting of those operators $A$ for which in (1.1) we may take $\lambda_\infty = \delta_0$ for some vector $\sigma \in R^n$.

It is obvious that the identity operator $I$ belongs to $\mathcal{A}(\lambda)$ for all $\lambda \in \mathcal{D}$. Moreover, since $0 \in \mathcal{D}$, we infer that $0 \in \mathcal{E}(\lambda)$ for all $\lambda \in \mathcal{D}$.

In this section we shall establish some simple properties of the sets $\mathcal{E}(\lambda)$ and $\mathcal{A}(\lambda)$.

**Proposition 1.1.** For every $\lambda \in \mathcal{D}(R^n)$ the set $\mathcal{E}(\lambda)$ is a compact subsemigroup of $\text{End } R^n$.

Proof. Given $A, B \in \mathcal{E}(\lambda)$, we put $C = AB$ and $\lambda_\infty = A \lambda \lambda_\infty$. It is easy to check that $\lambda = C \lambda \lambda_\infty$. Thus $\mathcal{E}(\lambda)$ is a subsemigroup of $\text{End } R^n$. Suppose now that $\lambda \in \mathcal{E}(\lambda)$ $(n = 1, 2, \ldots)$. It is clear that the symmetrization $\lambda_\infty$ of $\lambda$ is full and $\lambda_\infty = A \lambda_\infty \lambda_\infty$. By Theorems 2.2. and 5.1. in [10] (pp. 59 and 71) we infer that both sequences $(\lambda \lambda_n)$ and $(\lambda \lambda_n)$ are precompact in $\mathcal{D}$. Moreover, by the property (ii), the sequence $(\lambda_n)$ is precompact in $\text{End } R^n$. Let $A$ be its limit point. Without loss of generality we may assume that the sequence $(\lambda_n)$ converges to $A$. Then

$$\lim_{n \to \infty} \lambda \lambda_n \lambda = \lambda \lambda_\infty$$

It remains to prove that the sequence of measures $(\lambda_n \lambda)$ is precompact. Since the sequence of the symmetrizations $(\lambda \lambda_n)$ is precompact, we infer, by Theorem 2.2. in [10] (p. 59), that there exists a sequence $(\lambda_n)$ of vectors in $R^n$ for which the sequence of measures $(\lambda \lambda_n \lambda \lambda_n)$ is precompact in $\mathcal{D}$.

Thus, by (1.2), the sequence $\mu_n = A \lambda \lambda_n \lambda \lambda_n \lambda \lambda_n \lambda \lambda_n$ $(n = 1, 2, \ldots)$ is precompact in $\mathcal{D}$. But $\mu_n = \delta_{\lambda \lambda_n \lambda \lambda_n} \lambda_\infty$ $(n = 1, 2, \ldots)$. Now it is easy to prove that the sequence $(\lambda_n)$ is precompact in $R^n$ (see e.g. [12], The Compactness Lemma, p. 55). Hence it follows that the sequence of measures $(\lambda \lambda_n)$ is precompact. Denoting by $\lambda_\infty$ its limit point we have, by (1.2), the formula $\lambda = A \lambda_\infty \lambda_\infty$ which shows that $A \in \mathcal{E}(\lambda)$. Thus the set $\mathcal{E}(\lambda)$ is compact which completes the proof.

**Proposition 1.2.** For every $\lambda \in \mathcal{D}(R^n)$ the set $\mathcal{A}(\lambda)$ is a compact subsemigroup of $\text{Aut } R^n$.

Proof. Suppose that $A, B \in \mathcal{A}(\lambda)$ and $\lambda_\infty = \delta_0, \lambda_\infty = \delta_0$. Setting $C = AB$ and $c = A \lambda_\infty \lambda_\infty$, we get the equation $\lambda = C \lambda_\infty$. Consequently, $\mathcal{A}(\lambda)$ is a semigroup. Further, for every $A \in \mathcal{A}(\lambda)$ the measure $\lambda \lambda$ is also full. Since the support of $\lambda \lambda$ is contained in the image $A(R^n)$, we infer
that the operator \( A \) is invertible. Setting \( d = -A^{-1}a \) we have the formula \( \lambda = A^{-1}A \hat{\lambda}_d \) which shows that \( A^{-1}A(\lambda) \) and, consequently, that \( A(\lambda) \) is a subgroup of \( \text{Aut} R^N \). Suppose now that \( A_{n+1}A(\lambda) \) and for some vectors \( \alpha_n \in R^N \) the equations \( \lambda = A_n \hat{\lambda} \hat{\lambda}_{n+1} \) (\( n = 1, 2, \ldots \)) hold. By the Compactness Lemma in [12] (p. 55) we infer that both sequences \( \{\alpha_n\} \) and \( \{\alpha_n \} \) are precompact in \( \text{Aut} R^N \) and \( R^N \) respectively. Moreover, if \( A \) and \( a \) are their limit points, then \( \lambda = A \hat{\lambda} \hat{\lambda}_a \). Thus \( A \in A(\lambda) \) which completes the proof.

**Proposition 1.3.** If \( A \) and \( A^{-1} \) belong to \( E(\lambda) \), then \( A \in A(\lambda) \).

Proof. From the formula \( \lambda = A^{-1}A \) \( \hat{\lambda}_a \) we get the following one \( A \hat{\lambda}_a = A \hat{\lambda}_a \). Hence and from the formula \( \lambda = A \hat{\lambda} \) \( \hat{\lambda}_a \) we get the inequality for characteristic functions

\[
|\hat{\lambda}(y)| = |A \hat{\lambda}(y)| |\hat{\lambda}_a(y)| \leq |\hat{\lambda}(y)| |\hat{\lambda}_a(y)| \quad (y \in R^N),
\]

which yields the equation \( |\hat{\lambda}_a(y)| = 1 \) in a neighborhood of the origin.

By elementary properties of the characteristic function the last relation implies the formula \( |\hat{\lambda}_a(y)| = 1 \) for all \( y \in R^N \). Thus \( \lambda_a = \delta_a \) for a vector \( \alpha \in R^N \) which shows that \( A \in A(\lambda) \).

In what follows for any operator \( A \in \text{End} R^N \) \( \det A \) will denote the determinant of the matrix representation of \( A \) with respect to an orthonormal basis in \( R^N \).

**Proposition 1.4.** Let \( \lambda \in F(R^N) \). If \( A \in E(\lambda) \) and \( |\det A| = 1 \), then \( A \in A(\lambda) \).

Proof. Consider the monothetic compact subgroup \( S \) of \( E(\lambda) \) generated by the operator \( A \). By a Theorem of Numakura (see [8], [9] p. 109) the limit points of the sequence \( \{\lambda_n\} \) form a group \( G \) which is the minimal ideal of \( S \) and \( S \) contains exactly one idempotent, namely the unit \( J \) of \( G \). Of course, \( \det J = 1 \) and, consequently, \( J \) is the identity operator \( I \). Hence it follows that \( S = G \) and, consequently \( S \) is a group. Now our assertion is a consequence of Proposition 1.3.

**Proposition 1.5.** For every idempotent \( J \) from \( E(\lambda) \) the equation

\[
\lambda = JA \hat{\lambda}_J = \hat{\lambda}_J J \lambda
\]

holds. Consequently, \( I - J \in E(\lambda) \).

**Proposition 1.6.** Let \( J \) be an idempotent and

\[
\hat{\lambda}_J = \hat{\lambda}_J J \lambda
\]

Hence we get the formula \( J \lambda = J \hat{\lambda}_J J \lambda \). Consequently, \( \hat{\lambda}_J = \hat{\lambda}_J J \lambda \) which implies the equation \( J \lambda(y) = 1 \) in a neighborhood of the origin.

It is well-known that the last condition implies the formula \( J \lambda(y) = 1 \) for all \( y \in R^N \). Thus \( J \lambda = \delta_1 \). Hence, in particular, it follows that the measure \( \lambda(y) \) is concentrated on the subspace \( (I - J)(R^N) \). In other words,

\[
(I - J) \lambda(y) = \delta_1.
\]

Since \( 0 = \delta_1 \), equation (1.3) yields the formula

\[
(I - J) \lambda = (I - J) \lambda - (I - J) \delta_1 = (I - J) \lambda.
\]

Thus, by (1.4), \( (I - J) \lambda = \lambda J \) which, together with (1.3), implies the equation \( \lambda = J \lambda * (I - J) \hat{\lambda}_J \).

**Proposition 1.7.** Let \( J \) be an arbitrary idempotent from \( E(\lambda) \). Then for every pair \( A, B \) of operators from \( E(\lambda) \) the operator \( J A J + (I - J) B (I - J) \) belongs to \( E(\lambda) \) too.

**Proposition 1.8.** Let \( A \), \( B \in E(\lambda) \), i.e.,

\[
(\lambda = \delta_1 \hat{\lambda}_J \lambda)
\]

and

\[
(\lambda = \delta_1 \hat{\lambda}_J \lambda)
\]

If \( J \) is an idempotent from \( E(\lambda) \), then by Proposition 1.5,

\[
\lambda = J \lambda *(I - J) \lambda
\]

Thus

\[
(\lambda = J \lambda *
\]

and

\[
(\lambda = J \lambda *)
\]

The equations (1.5) and (1.6) imply the equation

\[
(\lambda = J \lambda *)
\]

Further, from the equations (1.6) and (1.9) we obtain

\[
(I - J) \lambda = (I - J) \lambda *(I - J) \lambda
\]

Taking into account (1.7), (1.10) and (1.11) we get the formula

\[
(\lambda = J \lambda *)
\]

where \( \lambda_J = \lambda J \lambda J \lambda J \lambda J \). Setting \( C = J A J + (I - J) B (I - J) \), we get, by virtue of (1.7),

\[
C \lambda = C \lambda J \lambda C \lambda = J A J \lambda J \lambda C \lambda J \lambda C \lambda.
\]

Hence and from (1.12) we get the formula \( \lambda = C \) \( \lambda \) which yields \( C \lambda \). The proposition is thus proved.

---

3 — Studia Mathematica XLIV.
Proposition 1.7. If \( E_n \in \mathcal{E}(\lambda) \) \((n = 1, 2, \ldots)\),
\[
\lim_{n \to \infty} R_n = 0 \quad (n = 1, 2, \ldots),
\]
\[
\lim_{n \to \infty} E_n = I
\]
and the set \( \{E_n^k; k = 0, 1, \ldots, n = 1, 2, \ldots\} \) is precompact, then \( \lambda(y) \neq 0 \) for every \( y \in \mathbb{R}^n \).

Proof. Suppose the contrary and assume that \( \lambda(a) = 0 \) and \( \lambda(y) \neq 0 \) whenever \( |y| < |a| \). First we note that the equations \( \lambda(y) = \lambda(B_n y) \Delta_n(y) \) \((n = 1, 2, \ldots)\) and the assumption (1.14) imply the relation \( \lim_{n \to \infty} \Delta_n(y) = 1 \) whenever \( |y| < |a| \). But the last relation is equivalent to the following one
\[
\lim_{n \to \infty} \Delta_n = \delta_n.
\]

Let \( E_n \) be the closure of the set \( \{E_n^k; k = 0, 1, \ldots, n = 1, 2, \ldots\} \). By the assumption the set \( E_n \) is compact. Thus, by (1.15), \( \lim_{n \to \infty} \Delta_n(y) = 1 \) uniformly on \( E_n \). Consequently, without loss of generality we may assume that
\[
\Delta_n(y) \neq 0 \quad (n = 1, 2, \ldots; y \in E_n).
\]

Now we shall prove that \( \lambda(a) = 0 \) for all \( a \in E_n \). Since \( \lambda(a) = 0 \) to prove this it suffices to prove that \( \lambda(B_n y) = 0 \) \((n = 1, 2, \ldots)\) whenever \( \lambda(y) = 0 \) and \( y \in E_n \). But this implication is a consequence of the equation
\[
\lambda(y) = \lambda(B_n y) \Delta_n(y)
\]
and the inequality (1.16). In particular, we have the formula \( \Delta(0) = 0 \) because, in view of (1.13), \( a \in E_n \). But this contradicts the obvious formula \( \lambda(0) = 1 \). The Proposition is thus proved.

2. Statement of the problem. A triangular array of probability measures \( \mu_{ij} \) \((i = 1, 2, \ldots; j = 1, 2, \ldots)\) on \( \mathbb{R}^n \) is said to be uniformly infinitesimal if for every neighborhood \( U \) of the origin the relation
\[
\lim_{n \to \infty} \max_{1 \leq i < j < n} \mu_{ij}(R^n \setminus U) = 0
\]
holds.

In terms of random variables, the problem we study is enunciated as follows: suppose that \( \{X_n\} \) is a sequence of independent \( \mathbb{R}^n \)-valued random variables and assume that \( \{A_n\} \) and \( \{a_n\} \) are sequences from \( \text{Aut} \mathbb{R}^n \) and \( \mathbb{R}^n \) respectively such that the probability distributions of
\[
A_n X_n \quad (h = 1, 2, \ldots; n = 1, 2, \ldots)
\]
form a uniformly infinitesimal triangular array and the distribution of
\[
A_n \sum_{k=1}^{n} X_k + a_n
\]
converges to a measure \( \mu \); what can be said about the limit measure \( \mu \)? Converting this to a problem involving only measures we ask which measures \( \mu \) can arise as limits of sequences \( A_n \mu_1 \ast \mu_2 \ast \cdots \ast \mu_n \ast \delta_{a_1} \), where \( \{a_n\} \) is an arbitrary sequence of probability measures in \( \mathbb{R}^n \), such that \( A_n \mu \) \((h = 1, 2, \ldots; n = 1, 2, \ldots)\) form a uniformly infinitesimal triangular array.

The limit measures \( \mu \) will be called Lévy's measures. The set of all Lévy's measures on \( \mathbb{R}^n \) will be denoted by \( \mathcal{L} \).

We refer the reader to M. Loève [6] (p. 319) for an account of the set \( \mathcal{L} \). The problem of characterizing of this set was proposed by A. Ya. Khintchine in 1936 and solved by P. Lévy in [5] (p. 195). He proved that a measure belongs to \( \mathcal{L} \) if and only if, it is self-decomposable. Self-decomposability of a measure \( \mu \) means here that \( \mathcal{E}(\mu) \) contains the open interval \((0,1)\) (see [5], p. 319 and [6] p. 332). It is possible in this case to describe the set \( \mathcal{L} \) in terms of characteristic functions. Namely, the set \( \mathcal{L} \) coincides with the set of probability measures with the characteristic function \( \varphi \) of the form
\[
\varphi(y) = \exp \left\{ iy + \int_{-\infty}^{\infty} \left( e^{itx} - 1 - itx \right) \frac{1 + x^2}{x^2} \, dM(x) \right\}
\]
where \( a \in \mathcal{E} \) and \( M \) is a bounded monotone non-decreasing function such that on \((-\infty, 0)\) and \((0, \infty)\) its left and right derivatives, denoted invariably by \( M'(x) \), exist and \( 1 + x^2 M'(x) \) do not increase.

Another characterization of \( \mathcal{L} \) was given in [13]. Namely, I proved that a function \( \varphi \) is the characteristic function of a measure from \( \mathcal{L} \) if and only if
\[
\varphi(y) = \exp \left\{ iy + \int_{-\infty}^{\infty} \left( \int_{y}^{\infty} \frac{e^{-u} - 1}{u} \, du - i y \arctan x \right) \frac{x \, dx}{\log(1 + x^2)} \right\}
\]
where \( a \in \mathcal{E} \) and \( \nu \) is a finite Borel measure on \( R \) and the integrand is defined as its limiting value \( -\frac{1}{2} y^2 \) when \( x = 0 \).

All that has been done so far in the multi-dimensional case is to describe limits of distributions of sequences
\[
A_n (X_1 + X_2 + \cdots + X_n) + a_n
\]
where $\mathcal{A}_n$ is a multiple of the identity operator. By the same techniques as in the one-dimensional case, one finds a representation of characteristic functions (see [14]).

Our aim is to characterize all full Lévy’s measures on $\mathbb{R}^n$. Before proceeding to state and prove the main results of this paper we shall establish auxiliary propositions.

3. Norming sequences. We say that a norming sequence $(A_n)$ of operators from $\text{Aut}^n\mathbb{R}^n$ corresponds to a measure $\mu$ if there exist sequences $(\mu_n)$ and $(\beta_n)$ of elements of $\mathcal{P}(\mathbb{R}^n)$ and $\mathbb{R}^n$ respectively, such that $A_n(\mu_1*\mu_2*\ldots*\mu_n)*\beta_n$ converges to $\mu$ and $A_n\beta_n$ $(k = 1, 2, \ldots, n; \ n = 1, 2, \ldots)$ form a uniformly infinitesimal triangular array.

**Proposition 3.1.** For every norming sequence $(A_n)$ corresponding to a full measure the relation $\lim_{n\to\infty} A_n = 0$ holds.

**Proof.** Suppose that $(A_n)$ corresponds to a full measure $\mu$. Taking into account the integrability of the measures in question, we may assume that $a_n = 0 (n = 1, 2, \ldots)$, i.e. that the sequence $A_n(\mu_1*\mu_2*\ldots*\mu_n)$ converges to $\mu$. Contrary to our statement let us suppose that there exists a subsequence of indices $n_1 < n_2 < \ldots$ for which

$$\lim_{k\to\infty} ||A_{n_k}|| > 0.$$  

(3.1)

Let us choose vectors $v_k \in \mathbb{R}^n$ with $||v|| = 1$ and $||A^*_n v|| = ||A^*_n \beta_n||$. Passing if necessary to a subsequence, we may assume that the sequence $v_n = ||A^*_n v||^{-1}A^*_n v$ converges to a vector $v \in \mathbb{R}^n$. Of course, $||v|| = 1$. From (3.1) it follows that the sequence $||A^*_n v||^{-1}v_n$ is bounded. Since, by the assumption, $\lim_{n\to\infty} A_n \beta_n = \beta$ $(j = 1, 2, \ldots)$, we have the relation

$$\lim_{k\to\infty} A_{n_k} \beta_{n_k} (v ||A^*_n v||^{-1}v_n) = 1 \quad (j = 1, 2, \ldots)$$

for every $c \in \mathbb{R}$. Consequently, by the transformation rule of $A_n \beta_n$, we get the formula

$$\lim_{k\to\infty} \mu_n (c v) = \mu (c v) = 1 \quad (j = 1, 2, \ldots)$$

for all $c \in \mathbb{R}$. Introducing the notation $v_n = \mu_1*\mu_2*\ldots*\mu_n$, we have

$$\mu_n (c v) = 1 \quad (n = 1, 2, \ldots; c \in \mathbb{R}).$$  

(3.2)

We note that the vectors $v_n = ||A^*_n v||^{-1}v_n$ are different from 0 because $||v|| = 1$. Let $c$ be a limit point of the sequence $||A^*_n v||^{-1}v_n$, say $\lim_{k\to\infty} ||A^*_n v||^{-1}v_n = c$. Since $A_n v_n$ converges to $\mu$, we have for all $c \in \mathbb{R}$

$$\lim_{k\to\infty} A_{n_k} v_n (c ||A^*_n v||^{-1}v_n) = \mu (c v).$$

On the other hand, by (3.3),

$$A_n v_n (c ||A^*_n v||^{-1}v_n) = \mu (c ||A^*_n v||^{-1}v_n) = 1 \quad (n = 1, 2, \ldots)$$

and, consequently, $\mu (c v) = 1$ for all $c \in \mathbb{R}$. Hence, by Proposition 1 in [13] (p. 69), it follows that the measure $\mu$ is full. But this contradicts the assumption. The Proposition is thus proved.

**Proposition 3.2.** To every full Lévy’s measure there corresponds a norming sequence $(A_n)$ with the property

$$\lim_{n\to\infty} A_{n+1}^{-1} = I.$$

**Proof.** Let $\mu$ be a full Lévy’s measure. Suppose that a sequence $(\beta_n)$ corresponds to $\mu$, i.e. $B_n(\mu_1*\mu_2*\ldots*\mu_n)*\beta_n$ converges to $\mu$ for some sequences $(\mu_n)$ and $(\beta_n)$. Setting $v_n = \mu_1*\mu_2*\ldots*\mu_n$ we have for some vectors $v_n$

$$B_{n+1} v_{n+1} * \delta_n = B_{n+1} B_n v_n * \beta_n \in B_{n+1} B_n \beta_n * \beta_n.$$

Since the measures $B_n \beta_n$ $(k = 1, 2, \ldots; n = 1, 2, \ldots)$ form a uniformly infinitesimal triangular array, we infer that the sequence $(B_{n+1} \beta_n)$ converges to $\delta_n$. Consequently, from (3.3) and the Compactness Lemma in [12] (p. 63) it follows that the sequence $(B_{n+1} B_n)$ is precompact in $\text{Aut}^n \mathbb{R}^n$. Moreover, for every its limit point $J$ one can find a vector $v \in \mathbb{R}^n$ such that $\mu = J v \beta_n$. Consequently, $J \in \text{Aut} (\mu)$.

Let $T$ be the set of all limit points of the sequence $(B_{n+1} B_n)$. The set $\{J\mu\}$, according to Proposition 1.2., is compact. The set $T$ being a closed its subset is compact too. Consequently, for every integer $n$ we can find an operator $J_n$ in $T$ such that

$$v_n = [J_n^{-1} B_{n+1}^{-1} B_n^{-1}] = \min \{\|J_n^{-1} B_{n+1}^{-1} B_n^{-1} - J_n\| : J_n \in T\}.$$

Obviously, $\lim_{n\to\infty} v_n = 0$. Moreover, the operators $J_n$ belong to $\{J\mu\}$. Since, by the Proposition 1.2., $\{J\mu\}$ is a group the operators $H_n$ defined by the formulae $H_n = I - J_n B_n$ $(n = 1, 2, \ldots)$ belong to $\{J\mu\}$ too. Put $A_n = H_n \beta_n$ $(n = 1, 2, \ldots)$. It is clear that $A_n \in \text{Aut}^n \mathbb{R}^n$. Moreover,

$$A_{n+1}^{-1} - I = H_n \beta_n - J_n H_n \beta_n = H_n \beta_n - J_n \beta_n \in \text{Aut}^n \mathbb{R}^n.$$

(3.4)

Since the set $\{J\mu\}$ is compact, all operators from $\{J\mu\}$ have the norm bounded in common; say $|A| \leq c$ for all $A \in \{J\mu\}$. Consequently, by (3.4),

$$\lim_{n\to\infty} ||A_{n+1}^{-1} - I|| = ||H_n \beta_n - J_n \beta_n|| \in \delta \beta_c \quad (n = 1, 2, \ldots)$$

which implies the relation

$$\lim_{n\to\infty} A_{n+1}^{-1} = I.$$
It remains to prove that the sequence \((A_n)\) corresponds to the measure \(\mu\). We note that sequence \((H_n)\) contained in the compact set \(A(\mu)\) is precompact. Consequently, the sequence \((H_n, B_{n} B_{n+1} \cdots B_{n+k})\), i.e. the sequence \((A_n, n_{k} = \cdots = n_{k+1})\), is precompact in \(\mathcal{S}(R^N)\). Moreover, its limit points are of the form \(H \mu \triangleq \delta_h\), where \(H\) is a limit point of the sequence \((H_n)\). Since \(H \in \mathcal{A} (\mu)\), we have the equation

\[H \mu \triangleq \delta_h = \mu \triangleq \delta_h\]

where \(h \in R^N\). Hence it follows that we can choose a sequence \((\delta_{a_n})\) of vectors such that the sequence \((A_n, n_{k} = \cdots = n_{k+1})\) converges to \(\mu\). Thus the sequence \((A_n)\) corresponds to \(\mu\) which completes the proof.

**Proposition 3.3.** Let \(m_n \leq m_k (k = 1, 2, \ldots)\) and \(n_k \to \infty\). For every norming sequence \((a_n)\) corresponding to a full measure \(\mu\) the sequence \((A_n, A_n^{-1})\) is precompact in \(End R^N\). Moreover, all its limit points belong to \(E(\mu)\).

**Proof.** Suppose that

\[
\lim_{n \to \infty} A_n a_n \triangleq \delta_{a_n} = \mu,
\]

where \(a_n = \mu_{1} \mu_{2} \cdots \mu_{k}\) and \((\mu_{k})\), \((a_{k})\) are suitably chosen sequences from \(\mathcal{S}(R^N)\) and \(R^N\) respectively. For simplicity of notation we put

\[
C_k = A_{m_k} A_{m_k}^{-1} (k = 1, 2, \ldots).
\]

Then we have the equation

\[
A_{m_k} a_{m_k} \triangleq \delta_{a_{m_k}} = C_k (A_{m_k} a_{m_k} \triangleq \delta_{a_{m_k}}) \triangleq a_{m_k},
\]

where \(a_{m_k}\) is a probability measure. The symmetrisation of (3.6) yields the formula

\[
A_{m_k} a_{m_k} = C_k A_{m_k} a_{m_k} \triangleq a_{m_k}.
\]

Hence, by virtue of Theorem 2.2 in [10] (p. 59), we get the precompactness of the sequence \((C_k A_{m_k} a_{m_k})\). Passing if necessary to a subsequence we may assume that the last sequence is convergent to a probability measure, say \(\lambda\). Thus

\[
\lim_{k \to \infty} C_k A_{m_k} a_{m_k} (g) = \lambda (g)
\]

uniformly on every compact subset of \(R^N\).

First we shall prove that the sequence \((C_k)\) is precompact in \(End R^N\). To prove this it suffices to prove that the sequence of norms \((\|C_k\|)\) is bounded. Contrary to this let us suppose that the sequence of norms is unbounded. Of course we may, assume without loss of generality, that \(\|C_k\| \to \infty\). Let us choose vectors \(a_{m_k} \in R^N\) such that \(\|a_{m_k}\| = 1\) and \(\|C_k a_{m_k}\| = \|C_k a_{m_k}\|(k = 1, 2, \ldots)\). Passing to a subsequence, if necessary, it may be assumed that the sequence of vectors \(u_{m_k} = \|C_k a_{m_k}\|^{-1} C_k a_{m_k}\) tends to a vector \(w \in \mathcal{R}^N\) with \(\|w\| = 1\). Thus, by (3.5),

\[
\lim_{k \to \infty} \lambda (C_k a_{m_k} (w_{m_k})) = \lambda (w)
\]

for all \(w \in \mathcal{R}\). Since \(\|C_k a_{m_k}\|^{-1} a_{m_k} \to 0\), we have, by (3.7),

\[
\lim_{k \to \infty} \lambda (C_k a_{m_k} (c \|C_k a_{m_k}\|^{-1} a_{m_k})) = \lambda (0) = 1
\]

for all \(c \in \mathcal{R}\). But the last formula can be written in the form

\[
\lim_{k \to \infty} \lambda (C_k a_{m_k} (w_{m_k})) = 1 (c \in \mathcal{R})
\]

Comparing it with (3.8) we get the equation \(\lambda (w) = 1\) for all \(w \in \mathcal{R}\). Hence it follows that \(\mu\) is not a full measure (see [12] p. 32, Proposition 1). The contradiction shows that the sequence \((C_k)\) is precompact in \(End R^N\).

Let \(A\) be a limit point of \((C_k)\). Without loss of generality we may assume that the sequence \((C_k)\) tends to \(A\). From (3.5) and (3.6), by virtue of Theorem 2.2 in [10], we get the precompactness of the sequence \((a_{m_k} \triangleq \delta_{a_{m_k}})\) for suitably chosen vectors \(b_{m_k} \in R^N\). Without loss of generality we may assume that the last sequence is convergent. By (3.5) and (3.6) it is easy to show that the sequence \((b_{m_k})\) is convergent to 0. For instance one can apply the Compactness Lemma in [12] p. 35. Thus the sequence \((\delta_{a_{m_k}})\) is convergent to a probability measure which will be denoted by \(\mu_\lambda\). Finally, from (3.5) and (3.6) we get the equation \(\mu = A \mu \triangleq \mu_\lambda\) which shows that \(A \in E(\mu)\).

4. Decomposability properties of Lévy’s measures. Let \(J\) be a non-zero idempotent in \(End R^N\), i.e. a projector from \(R^N\) onto \(J (R^N)\). For every operator \(A\) in \(End R^N\) by \(det_J A\) we shall denote the determinant of the matrix representation of the operator \(JA\) in \(J (R^N)\) relatively to an orthonormal basis of \(J (R^N)\). It is easy to prove the following formulæ

\[
det_J A = \det_J JA = \det_J A J = \det_J A J A J,
\]

\[
det_J A J B = \det_J A \det_J B.
\]

Moreover, if the projectors \(J_1\) and \(J_2\) satisfy the conditions \(J_1 J_2 = J_2 J_1 = 0\), then for every \(A, B \in End R^N\) we have the equation

\[
det_{J_1 + J_2} (A J_1 J_1 + J_1 B J_1) = \det_{J_1} A \det_{J_2} B.
\]

**Lemma 4.1.** Let \(\mu\) be a full Lévy’s measure and let \(J\) be a non-zero idempotent from \(E(\mu)\). Then for every number \(c\) satisfying the condition \(0 < c < 1\) there exists an operator \(B\), in \(E(\mu)\) such that \(\det_J B = c\).
Proof. Let \( \{A_n\} \) be a norming sequence corresponding to \( \mu \). By Proposition 3.3 we may assume that
\[
\lim_{n \to \infty} A_{n+1}^{-1} A_n^{-1} = I.
\]
(4.4) Put \( b_{mn} = \det_j A_m A_n^{-1} \quad (n \leq m) \). Obviously,
\[
b_{mn} = 1 \quad (n = 1, 2, \ldots).
\]
(4.5) Moreover, by Proposition 3.1,
\[
\lim_{m \to \infty} b_{mn} = 0 \quad (n = 1, 2, \ldots).
\]
(4.6) By Proposition 3.3, the set \( \{A_m A_n^{-1} \mid m \geq n, n = 1, 2, \ldots\} \) is precompact in \( End R^N \). Consequently, all its elements have the norm bounded in common, say by a number \( d \). Thus
\[
\|A_m A_n^{-1} - A_m A_n^{-1} - I\| \leq \|A_m A_n^{-1} - I\| \|A_m A_n^{-1}\| \leq d \|A_m A_n^{-1} - I\|.
\]
Consequently, by (4.4),
\[
\lim_{n \to \infty} \sup_{m \geq n} \|A_m A_n^{-1} - A_m A_n^{-1} - I\| = 0.
\]
Hence we get the relation
\[
\lim_{n \to \infty} \sup_{m \geq n} |b_{mn} - b_{mn}| = 0.
\]
(4.7) Given a number \( c \) satisfying the condition \( 0 < c < 1 \), we can find, by virtue of (4.5) and (4.6), an index \( m_0 \geq n \) such that \( b_{mn_0} > c \) and \( b_{m_0+1, n} < c \) \( (n = 1, 2, \ldots) \). From (4.7) it follows that
\[
\lim_{n \to \infty} b_{m_0n} = c.
\]
(4.8) By Proposition 3.3 the sequence \( \{A_m A_n^{-1}\} \) is precompact in \( End R^N \). Let \( B_n \) be its limiting point. By the same Proposition we infer that \( B_n \in E(\mu) \).
Finally, by (4.8), \( \det_j B_n = c \) which completes the proof.

**Lemma 4.2.** Let \( \mu \) be a full Lévy’s measure and let \( J \) be a non-zero idempotent from \( E(\mu) \). There exists then a sequence \( \{S_n\} \) of operators from \( E(\mu) \) which converges to \( J \) and satisfies the conditions \( J S_n = S_n J = S_n \) \( (n = 1, 2, \ldots) \) and
\[
\lim_{n \to \infty} S_n^k = 0 \quad (n = 1, 2, \ldots).
\]

**Proof.** We shall prove the Lemma by induction with respect to the dimension of the subspace \( J(R^N) \).

First consider the case \( \dim J(R^N) = 1 \). Let us choose, by virtue of Lemma 4.1, operations \( C_n \) from \( E(\mu) \) for which \( \det_j C_n = 1 - \frac{1}{n} \) \( (n = 1, 2, \ldots) \). Put \( S_n = JC_n J \). Since the subspace \( J(R^N) \) is one-dimensional, the operator \( S_n \) is a multiple of the operator \( J \). Moreover, by (4.3),
\[
\det_j S_n = 1 - \frac{1}{n}.
\]
Thus \( S_n = \left( \frac{1}{n} \right) J \). Now it is obvious that the operators \( S_n \) fulfil the conditions of the Lemma.

Suppose now that \( \dim J(R^N) = d > 1 \) and for all projectors \( K \) belonging to \( E(\mu) \) for which \( K(R^N) \) is of dimension less than \( d \) the Lemma is true.

First suppose that there exists a non-zero idempotent \( L \) in \( E(\mu) \) different from \( J \) and satisfying the condition
\[
L = JL = LJ.
\]
In other words \( L \) maps \( R^N \) into a proper subspace of \( J(R^N) \). By (4.9) the operator \( J - L \) is an idempotent. Moreover, by Proposition 1.3, \( J - L \in E(\mu) \). Consequently, by the equation \( J(J - L) = J - L \), the idempotent \( J - L \) belongs to \( E(\mu) \). We note that both subspaces \( L(R^N) \) and \( (J - L)(R^N) \) have the dimension less than \( d \). Consequently, by the induction assumption we can find two sequences \( \{U_n\} \) and \( \{V_n\} \) in \( E(\mu) \) which converge to \( L \) and \( J - L \) respectively and for every \( n \) satisfy the conditions \( L U_n = U_n L = U_n \) and \( J - L V_n = V_n (J - L) = V_n \) and \( \lim_{n \to \infty} V_n = 0 \). Setting \( S_n = U_n + V_n \), we infer that the sequence \( \{S_n\} \) converges to \( J \). Further, from the equation \( S_n = (L R_n + (J - L) V_n) \) and Proposition 1.6 we obtain the relation \( S_n \in E(\mu) \). Moreover, by (4.9),
\[
S_n = S_n J = S_n.
\]
Since \( S_n = U_n + V_n \), finally we have the equation
\[
\lim_{n \to \infty} S_n^k = 0.
\]
Thus the sequence \( \{S_n\} \) fulfills the conditions of the Lemma.

It remains to consider the case when \( E(\mu) \) does not contain non-zero idempotents \( L \) different from \( J \) and satisfying (4.9). By Lemma 4.1 we can find operations \( D_n \) from \( E(\mu) \) such that
\[
0 < \det_j D_n < 1
\]
(4.10) and
\[
\lim_{n \to \infty} \det_j D_n = 1.
\]
(4.11) Moreover, by (4.1), we may assume that
\[
J D_n = D_n J = D_n
\]
(4.12) and, by the compactness of \( E(\mu) \) (see Proposition 1.1) that the sequence \( \{D_n\} \) converges to an operator \( D \) in \( E(\mu) \). Obviously,
\[
J D = DJ = D
\]
(4.13) and
\[
\det_j D = 1.
\]
(4.14)
Put $A = D + I - J$. By Proposition 1.6, $A \in \mathbb{E}(\mu)$ and, by (4.3) and (4.14) $\det A = \det A \det A = 1$. Hence and from Proposition 1.4 it follows that $A \in \mathbb{E}(\mu)$. By Proposition 1.2 $A(\mu)$ is a compact group. Consequently, there exists a sequence $t_1 < t_2 < \ldots$ of indices such that $A^n$ converges to the identity operator $I$ (see [9], p. 109). Since $J A^n = D^n$, the sequence $\{D^n\}$ converges to $J$. Consequently, we can find a sequence $k_1 < k_2 < \ldots$ of indices such that $D^n \to J$. Setting $S_n = D^n (n = 1, 2, \ldots)$, we get a sequence from $\mathbb{E}(\mu)$ convergent to $J$. Moreover, by (4.12), $J S_n = S_n J = S_n$ and, by (4.10), $\det S_n < 1$. From the compactness of $\mathbb{E}(\mu)$ it follows that for every $n$ the sequence $\{S_n\}$ is precompact in $\mathbb{E}(\mu)$. Consequently, the limit points of this sequence form a group (see Numakura Theorem, [9], p. 109). The unit $L$ of this group is an idempotent belonging to $\mathbb{E}(\mu)$ and satisfying the equation $L = J L = L J$. Taking into account formula (4.2), we infer that $\det L = 0$. Consequently, $L \neq J$. We have assumed that $\mathbb{E}(\mu)$ does not contain non-zero idempotents different from $J$ and satisfying (4.9). Thus $L = 0$ and, consequently, the group of all limit points of $\{S_n\}$ is the one-element group $\{0\}$. In other words, $\lim_{n \to \infty} S_n = 0$ for all $n$ and the sequence $\{S_n\}$ fulfills the conditions of the Lemma which completes the proof.

5. A characterization of full Lévy’s measures. The aim of this section is a characterization of full Lévy’s measures in terms of operator-decomposability.

Proposition 5.1. Let $\mu$ be a full Lévy’s measure. Then the set $\mathbb{E}(\mu)$ contains a one-parameter semigroup $\exp Q_t (t \geq 0)$ with the property $\lim_{t \to \infty} \exp Q_t = 0$.

Proof. By Propositions 1.5 and 1.6 the identity operator $I$ can be written in the form $I = J + J_2 + \ldots + J_q$, where $J_t$ are non-zero idempotents from $\mathbb{E}(\mu)$ for $r \neq s$ and for every $s$ there is a non-zero idempotent K from $\mathbb{E}(\mu)$ different from $J_t$ and satisfying the condition $K J_t = J_t K = K$. By consecutive application of Proposition 1.6 we conclude that $\sum_{r=1}^{q} J_{r} A_{r} J_{r} \in \mathbb{E}(\mu)$ whenever $A_1, A_2, \ldots, A_q \in \mathbb{E}(\mu)$.

By Lemma 4.2 for every $r$ ($1 \leq r \leq q$) we can find a sequence $\{S_{n}, r\}$ of operators from $\mathbb{E}(\mu)$ satisfying the conditions
\begin{equation}
J_r S_n, r = S_n, r J_r = S_n, r, \quad (n = 1, 2, \ldots),
\end{equation}
\begin{equation}
\lim_{n \to \infty} S_n, r = J_r
\end{equation}
and
\begin{equation}
\lim_{n \to \infty} S_n, r = 0, \quad (n = 1, 2, \ldots).
\end{equation}

Moreover, by (5.2) and (5.3), we may assume that
\begin{equation}
0 < \det S_n, r < 1, \quad (n = 1, 2, \ldots).
\end{equation}
Put
\begin{equation}
eq n, r = (\log \det S_n, r)^{-1},
\end{equation}
where square brackets denote the integer part. Let $W$ be the set of all non-negative rational numbers. By the Proposition 1.1, i.e. by the compactness of $\mathbb{E}(\mu)$, the sequences $\left\{S_n, r^m_\log \det S_n, r\right\}$ ($n \in \mathbb{N}$, $r = 1, 2, \ldots, q$) are precompact in $\mathbb{E}(\mu)$. Passing, if necessary, to subsequences we may assume, without loss of generality, that all these sequences are convergent. Put
\begin{equation}
\lim_{n \to \infty} S_n, r^m_\log \det S_n, r = B_\infty, \quad (n \in \mathbb{N}, r = 1, 2, \ldots, q).
\end{equation}
Since $\sum_{n=1}^{q} S_n, r^m_\log \det S_n, r = \sum_{n=1}^{q} J_n S_n, r^m_\log \det S_n, r J_n$, we infer that $B_\infty \in \mathbb{E}(\mu)$ for $w \in W$.

Moreover, by (5.5),
\begin{equation}
\det B_\infty = \lim_{n \to \infty} \det S_n, r^m_\log \det \det S_n, r = e^{-m} \quad (r = 1, 2, \ldots, q).
\end{equation}

Hence, by (4.3), we get the formula
\begin{equation}
\det B_\infty = e^{-\infty} \quad (w = \infty).
\end{equation}

Consequently, $B_\infty \in \mathbb{E}(\mu)$. Moreover, it is easy to verify the equation
\begin{equation}
B_{a \cdot w} = B_{a} B_{w} \quad (w \in W).
\end{equation}

Consequently, the set $H = \{B_a : w \in W\}$ is a subsemigroup of the group $\mathbb{E}(\mu)$. Let us introduce the notation $H^{-1} = \{B^{-1}_a : w \in W\}$. To prove that the union $H \cup H^{-1}$ is a group it suffices to prove that for every pair $w, w \in W$ belongs to $H$. By symmetry we may assume that $w \geq u$. Then, by (5.8), $B_w B^{-1}_u = B_{w-u} B_u B^{-1}_u = B_{w-u} H$. Let $S$ be the closure of $H$ in $\mathbb{E}(\mu)$. It is clear that $S \in \mathbb{E}(\mu)$ and $G = S \cup S^{-1}$ is a closed subgroup of the group $\mathbb{E}(\mu)$. Moreover, by (5.6),
\begin{equation}
\det J_r A = \det J_r A = \ldots = \det J_r A \quad (\mathbb{A} \mathbb{S}),
\end{equation}
and, by (5.7),
\begin{equation}
0 < \det A \leq 1 \quad (\mathbb{A} \mathbb{S}).
\end{equation}

Since, by (5.8), $B_a = I$, the set $S_a = S \cap \mathbb{A}(\mu)$ is non-void. Moreover, being a closed subsemigroup of the compact group $\mathbb{A}(\mu)$ it is a compact group (see [9], p. 20). From (5.10) we obtain the equation
\begin{equation}
\det A = 1 \quad \text{for } A \in S_a.
The mapping \( h(A) = \log \det A \) is a homomorphism of the topological group \( G \) onto the additive group \( R \). We shall prove that \( S_0 \) is the kernel of this homomorphism. By (5.11), \( S_0 \) is contained in the kernel of \( h \). Suppose that \( A \in G \) and \( h(A) = 0 \). Of course, \( A \) or \( A^{-1} \) belongs to \( S_0 \). Without loss of generality we may assume that \( A \in S_0 \). Consequently, \( A \in E(\mu) \) and, by Proposition 1.4, \( A \in A(\mu) \) which implies the relation \( A \in S_0 \). Thus \( S_0 \) is the kernel of \( h \). Hence it follows that the factor group \( G/S_0 \) is isomorphic to \( R \). Since the group \( G \) is commutative and compactly generated, we infer, by Pontrjagin Theorem ([7], p. 187; [15], § 29), that \( G \) is isomorphic to the direct sum of \( R \) and \( S_0 \). Let \( g: G \to R \times S_0 \) be such isomorphism. Since \( R \times S_0 = g(G) = g(S) \cup g(S)^{-1} \) and \( g(S) \) is a closed subgroup, we infer that either \( g(S) = R^+ \times S_0 \) or \( g(S) = R^- \times S_0 \), where \( R^+ \) and \( R^- \) denote the right and left half-lines respectively. For \( t \geq 0 \) we put \( T_t = g^{-1}((t, \cdot)) \) in the first case and \( T_t = g^{-1}((-t, \cdot)) \) in the remaining one, where \( t \) is the unit of \( S_0 \). It is clear that \( T_t(t \geq 0) \) is a continuous one-parameter semigroup of operators from \( S \) satisfying the condition \( \lim_{t \to 0} T_t = I \). By Theorem 8.4.2 in [3] it can be represented in an exponential form \( T_t = \exp tQ \) (\( t > 0 \)). Moreover, \( T_tS_0 \) for \( t > 0 \). Consequently, by (5.10) and (5.11),

\[
0 < \det T_t < 1 \quad \text{for } t > 0.
\]

From the definition of the operators \( S_0, S_0 \) and the semigroup \( S \) it follows that the idempotents \( J_1, J_2, \ldots, J_g \) commute with the elements of \( S \). We note that the semigroup \( T_t(t \geq 0) \) is precompact in \( E(\mu) \). Consequently, to prove the relation \( \lim_{t \to 0} T_t = 0 \) it suffices to prove that 0 is a limit point of this semigroup. It is well known that the set of limit points of the semigroup \( T_t(t \geq 0) \) contains an idempotent \( K \) (see [9], p. 109). By (5.12), we have the equation \( \det K = 0 \). Consequently, by (5.9) and (4.3),

\[
\det_j K = 0 \quad (r = 1, 2, \ldots, g).
\]

Since \( J_r \) commutes with \( K \), the operator \( J_rK \) is an idempotent in \( E(\mu) \). Taking into account (4.1) and (5.13), we have the inequality \( J_rK \neq J_r \). On the other hand \( J_r(J_rK) = (J_rK)J_r = J_rK \), which by the definition of the idempotents \( J_1, J_2, \ldots, J_g \) yields the equation \( J_rK = 0 \). Thus \( K = J_1K + J_2K + \ldots + J_gK = 0 \) and, consequently, 0 is a limit point of the semigroup \( T_t(t \geq 0) \). The Proposition is thus proved.

\[\text{Proposition 5.2. Suppose that a one-parameter semigroup } \exp tQ \text{ fulfills the condition } \lim_{t \to 0} \exp tQ = 0. \text{ Then each } \exp tQ \text{-decomposable probability measure } \mu \text{ is a Lévy's measure. Moreover, for every } t \geq 0 \text{, we have } \mu = \exp tQ \text{.}\]

\[\mu = \exp tQ \text{, where } \mu \text{ is an infinitely divisible measure.}\]

**Proof.** Setting \( B_n = \exp -\frac{1}{n} Q \) (\( n = 1, 2, \ldots \)) we have the formula

\[(5.14) \quad \mu = B_n \mu \ast B_n, \]

where \( \mu_n \in \mathcal{P}(R^d) \). It is easy to verify the relation

\[(5.15) \quad \lim_{n \to \infty} \mu_n = \delta_0.\]

Moreover, the operators \( B_n \) satisfy conditions of the Proposition 1.7. Thus

\[(5.16) \quad \hat{\mu}(y) = 0 \quad \text{for all } y \in R^d.\]

Put

\[(5.17) \quad A_n = \prod_{k=1}^n B_k = \exp \sum_{k=1}^n \frac{1}{k} Q \quad (n = 1, 2, \ldots)\]

and

\[(5.18) \quad \mu_1 = A_1 \mu, \quad \mu_2 = A_2 \mu_1. \quad \mu_k = A_k \mu_{k-1}. \quad (k = 2, 3, \ldots).\]

From (5.17) it follows that the set \( \{A_nA_k^{-1}; k = 1, 2, \ldots; n = 1, 2, \ldots \} \) is precompact in \( E \). Moreover, for every \( k \) \( \lim_{n \to \infty} A_nA_k^{-1} = 0 \). Consequently, by (5.18), \( \{A_n, \mu_n\} \) \( (k = 1, 2, \ldots; n = 1, 2, \ldots) \) are uniformly infinitesimal too.

From (5.14) and (5.15), by virtue of (5.16), we get the formulæ

\[(5.19) \quad \mu_1(y) = \hat{\mu}(A_1^{-1}y), \quad \mu_2(y) = \hat{\mu}(A_1^{-1}y) = \hat{\mu}(A_2^{-1}y) = \cdots = \hat{\mu}(A_k^{-1}y). \quad (k = 2, 3, \ldots).\]

Thus

\[(5.20) \quad A_n \mu_1 \ast \mu_2 \ast \cdots \ast \mu_n(y) = \prod_{k=1}^n \hat{\mu}_k(y) = \hat{\mu}(y) \]

and, consequently,

\[(5.21) \quad A_n \mu_1 \ast \mu_2 \ast \cdots \ast \mu_n = \mu \quad (n = 1, 2, \ldots),\]

which shows that \( \mu \) is a Lévy's measure.

Given \( t > 0 \), we can choose a sequence of integers \( n_k > n \) such that

\[(5.22) \quad \lim_{n \to \infty} \frac{1}{k} = t. \]

Then, by (5.17),

\[\lim_{n \to \infty} A_n A_k^{-1} = \exp tQ.\]

Further, by (5.19),

\[(5.23) \quad \mu = A_n A_k^{-1} \mu \ast A_k \mu_1 \ast \mu_2 \ast \cdots \ast \mu_n.\]
The characteristic functions of the measures in question are, according to (5.16), different from 0 everywhere on $\mathbb{R}^N$. Thus the last equation yields the existence of the limit:

$$
\lambda_i = \lim_{n \to \infty} A_{\delta_n}(\mu_{n_1} \ast \cdots \ast \mu_{n_k});
$$

and the equation $\mu = \exp Q \ast \lambda_i$. Since $(\delta_n \mu_j) (j = n+1, n+2, \ldots, n_k; n = 1, 2, \ldots)$ are uniformly infinitesimal, the limit distribution $\lambda_i$ is infinitely divisible (see [10], p. 52) which completes the proof.

The class of infinitesimal operators $Q$ which can occur in Propositions 5.1 and 5.2 is closed under similarity transformations and is simply describable through spectral properties. Namely, $\lim_{t \to \infty} \exp Q_t = 0$ if and only if all eigenvalues of $Q$ have negative real part. As a consequence of Propositions 5.1 and 5.2 we get a characterization of full Lévy's measures.

**Theorem 5.1.** A full probability measure on $\mathbb{R}^N$ is a Lévy's measure if and only if it is $\exp Q_t$-decomposable for $t > 0$ where $Q$ is an operator whose all eigenvalues have negative real part.

6. An extreme point method. Our next aim is to give a representation of the characteristic functions of $\exp Q_t$-decomposable for $t > 0$ measures in $\mathbb{R}^N$. By Proposition 5.2 all such measures are Lévy's measures and, consequently, are infinitely divisible. The method of proof consists in finding the extreme points of a certain convex set formed by Khintchine measures corresponding to $\exp Q_t$-decomposable measures. Once the extreme points are found one can apply a Theorem by Choquet on representation of the points of a compact convex set as barycentres of the extreme points.

First we introduce some auxiliary spaces. Let $Q$ be an operator on $\mathbb{R}^N$ whose eigenvalues have negative real part. Let $S^m$ be the $m$-dimensional unit sphere and $R$ the compactified real line: $R = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Put $\mathbb{R}^N = S^{N-1} \times R$. Obviously, the space $\mathbb{R}^N$ is compact.

We define a congruence relation on $\mathbb{R}^N$ as follows: $(x, t) \sim (y, u)$ where $x, y \in \mathbb{R}^{N-1}$ and $t, u \in R$ if and only if there exists a real number $s$ such that $\exp Q_s = y$ and $u = t + s$. Suppose that $(x_n, t_n) \sim (y_n, u_n)$ for all $n = 1, 2, \ldots$. Since all eigenvalues of $Q$ have negative real part, the last equations and the compactness of $S^{N-1}$ imply that the sequence $(x_n)$ is bounded. If $s$ is its limit point, then $\exp Q_s = y$ and $u = t + s$. Thus $(x, t) \sim (y, u)$ and, consequently, the quotient space $\mathbb{R}^N/\sim$ denoted by $M^N$ is compact (see [1], p. 97).

The element of $M^N$, i.e., the equivalence class containing $(x, t)$ from $\mathbb{R}^N$ will be denoted by $[x, t]$. We define a one-parameter group $T_s (s \in R)$ of transformations of $M^N$ by assuming

$$
T_s (x, t) = (x, s + t);
$$

Further, for every element $[x, t] \in M^N$ we put

$$
[x, t] = [\exp Q_t x] \text{ if } t \in R, \quad [x, \infty] = 0 \text{ and } [x, -\infty] = \infty.
$$

Since $\lim_{t \to \infty} \exp Q_t = 0$ and for every $z \in \mathbb{R}^N \setminus \{0\}$ $\lim_{t \to \infty} \exp Q_t = \infty$, each element $z \in \mathbb{R}^N \setminus \{0\}$ can be represented in the form $z = \exp Q_t x$, where $x \in S^{N-1}$ and $t \in R$. In general this representation is not unique. But $z = \exp Q_t y$, where $y \in S^{N-1}$ and $t \in R$ if and only if $(x, t) \sim (y, u)$. Thus the mapping

$$
\pi(\exp Q_t x) = [x, t] \quad (x \in S^{N-1}, t \in R)
$$

is an embedding of $\mathbb{R}^N \setminus \{0\}$ into $M^N$. Obviously,

$$
[y] = [\pi(y)]
$$

and

$$
\pi(\exp Q_t y) = T_s \pi(y)
$$

for all $x \in \mathbb{R}^N \setminus \{0\}$ and $t \in R$.

We say a subset $B$ of $M^N$ is bounded from below if $\inf \{|x| : a \in B\} > 0$. Let $\lambda$ be a finite Borel measure on $M^N$. For any Borel subset $E$ of $M^N$ from below we put

$$
I_\lambda(E) = \int_E (1 + |x|^{-2}) \lambda(dx),
$$

where the integrand is assumed to be 1 if $|x| = \infty$.

Let $\lambda$ be the set of all finite Borel measures $\lambda$ on $M^N$ satisfying the condition

$$
I_\lambda(E) - I_\lambda_1(E) \geq 0
$$

for all $t > 0$ and all Borel subsets $E$ bounded from below. It is clear that the set $\lambda$ is convex. Let $\lambda$ be the subset of $\lambda$ consisting of probability measures. The set $\lambda$ is convex and compact.

Suppose that a Borel subset $F$ of $M^N$ is $T_s$-invariant for all $t \in R$ and $s \in \mathbb{R}.$ Then the restriction $\lambda|F$ belongs to $\lambda$ too because of the equation

$$
I_\lambda(F) - I_\lambda_1(E) = I_\lambda(E \cap F) - I_\lambda_1(E \cap F).
$$

Hence it follows that the extreme points of the set $\lambda$ are measures concentrated on orbits of elements of $M^N$. In other words, we have the following proposition:
Proposition 6.1. The extreme points of $\mathcal{X}$ are measures concentrated on one of the following sets: $[x_i, -\infty[[, [x_i, \infty[[, (x_i, t): t \epsilon R]$ where $x \epsilon S^{n-1}$.

We proceed now to the investigation of extreme points of $\mathcal{X}$ concentrated on the set $F_x = \{(x_i, t): t < u\}$. Let $\lambda$ be a probability measure concentrated on $F_x$. Put

(6.8) $J_t(u) = \mathcal{I}_1([x_i, t): t < u) = \lambda(R)$.

It is easy to verify that $\lambda \epsilon \mathcal{X}$ if and only if the inequality (6.7) holds for all $t \geq 0$ and all subsets $\mathcal{R}$ of the form $\{(x_i, t): a \leq t < b\}$, where $a < b$ and $a, b \epsilon R$. Taking into account the formulae

I_1([x_i, t): a \leq t < b) = J_t(b) - J_t(a),

$T_x I_1([x_i, t): a \leq t < b) = I_1([x_i, t): a \leq s \leq t < b - s)$

we infer that $\lambda \epsilon \mathcal{X}$ if and only if for every triplet $a, b, t \epsilon R$ satisfying the conditions $a < b$ and $t \geq 0$ the inequality

(6.9) $J_t(b) - J_t(a) - J_t(b - t) + J_t(a) \geq 0$

is fulfilled.

Now we shall give more convenient description in terms of the function $J_t$ of measures $\lambda$ from $\mathcal{X}$. Let $f$ be a continuous bounded function on $F_x$. By (6.6) and (6.8) we have the formula

(6.10) $\int_{F_x} f(t)\lambda(dt) = \frac{\int_{F_x} f([v, u])}{\int_{F_x} f([v, u])} \frac{[[v, u]]}{1 + [[v, u]]^t} dJ_t(u)$.

Substituting $b = a + t$ into (6.9) we get the inequality

$J_t(a) \geq \frac{1}{2}(J_t(a + t) + J_t(a))$

for all $a \epsilon R$ and $t \geq 0$. Thus the function $J_t$ is convex. Moreover, by (6.8), it is also monotone non-decreasing with $J_t(-\infty) = 0$. Consequently,

$J_t(t) = \int_{-\infty}^t q_t(u) du \quad (t \epsilon R)$,

where the function $q_t$ is non-negative and monotone non-decreasing. Of course, we may assume that $q_t$ is continuous from the left. In this case the function $q_t$ is uniquely determined by $\lambda$. Further, by (6.10), we have the equation

$\int_{-\infty}^\infty \frac{[[v, u]]}{1 + [[v, u]]^t} q_t(u) du = 1$.

Suppose now that we have a non-negative monotone non-decreasing function $q$ on $R$ satisfying the condition

(6.11) $\int_{-\infty}^\infty \frac{[[v, u]]}{1 + [[v, u]]^t} q(u) du = 1$.

We define a measure $\lambda$ on $F_x$ by means of the formula

(6.12) $\int_{F_x} f([v, u]) \frac{[[v, u]]}{1 + [[v, u]]^t} q(u) du$,

for any bounded continuous function $f$ on $F_x$. It is obvious, by (6.11), that $\lambda$ is a probability measure on $F_x$. Moreover, $J_t(t) = \int_{-\infty}^t q(u) du$. Since $q$ is monotone non-decreasing, the function $J_t$ fulfills the condition (6.9).

Consequently, $\lambda \epsilon \mathcal{X}$. Thus we proved the following proposition.

Proposition 6.2. Equation (6.12) defines a one-to-one correspondence between all measures $\lambda$ from $\mathcal{X}$ concentrated on $F_x$ and all non-negative monotone non-decreasing continuous from the left functions $q$ on $R$ satisfying the condition (6.11).

In the sequel we shall use the following Lemma.

Lemma 6.1. For every $a \epsilon R$ and $x \epsilon E^R$ the integral

$P_x(a) = \int_a^\infty \frac{\exp(x^t)\exp(a)}{1 + \exp(x^t)\exp(a)} dt$

is finite. Moreover, for every $a \epsilon E$ there exist positive constants $b_1$ and $b_2$ such that for all $x \epsilon E^R$ the inequality

$b_1 \log(1 + ||x||^p) \leq P_x(a) \leq b_2 \log(1 + ||x||^p)$

is true.

Proof. We assumed that all eigenvalues, say $a_1, a_2, ..., a_p$, of $Q$ have negative real part. Consequently, for $0 > b \epsilon R$ and $c < \Re a_j$ ($j = 1, 2, ..., p$) we have the relations

$\lim_{t \to \infty} e^{-cb} \exp(t) = 0 = \lim_{t \to -\infty} e^{-cb} \exp(t)$.

Hence we get the inequalities

$\sup_{t > 0} |e^{-cb} \exp(t)| = c_1 < \infty$

and

$\sup_{t < 0} |e^{-cb} \exp(t)| = c_2 < \infty$.

3 — Studia Mathematica XLIV, 1.
Thus for \( z \in \mathbb{R}^n \) and \( t \geq c \)
\[ |\exp\{tq(z)\}| \leq c e^{dt} |z|^{2} \]
and
\[ e^{t} |z| = |e^{t} \exp(-tQ) \exp\{tq(z)\}| \leq c_{1} |\exp\{tq(z)\}|. \]

Since the function \( \frac{t e^{t} z}{1 + t e^{t}} \) is monotone non-decreasing on the right half-line \( \mathbb{R}^+ \) we have the inequalities
\[ \int_{c}^{\infty} \frac{c_{1} e^{t} z^{2}}{1 + c_{2} e^{t} z^{2}} dt \leq P_{a}(z) \leq \int_{c}^{\infty} \frac{c_{1} e^{t} z^{2}}{1 + c_{2} e^{t} z^{2}} dt. \]

Hence and from the formula for \( m < 0 \)
\[ \int_{c}^{\infty} \frac{e^{mt} t^{n}}{1 + s^{2} e^{nt}} dt = -\frac{1}{2m} \log(1 + s^{2} e^{nt}) \]

by a simple computation we get the assertion of the Lemma.

As a consequence of the definition (6.2) and Lemma 6.1 we get the following Corollary:

**Corollary.** For every \( a \in \mathbb{E} \) and \( a \in \mathbb{S}^{N-1} \) the integral
\[ \int_{c}^{\infty} \frac{[x, u]^{2}}{1 + [x, u]^{2}} du \]
is finite.

We define a family \( m_{(r, a)} (a \in \mathbb{E}) \) of probability measures on \( F_{a} \) as follows. Put
\[ g_{a}(t) = \begin{cases} 0 & \text{if } t \leq a, \\ c_{a} & \text{if } t > a, \end{cases} \]
where
\[ c_{a} = \int_{c}^{\infty} \frac{[x, u]^{2}}{1 + [x, u]^{2}} du. \]

By Corollary to Lemma 6.1, the constant \( c_{a} \) is finite. It is very easy to verify that the function \( g_{a} \) fulfills the conditions of the Proposition 6.2. Consequently, by formula (6.13), the measure \( m_{(r, a)} \) belonging to \( \mathbb{E} \) and concentrated on \( F_{a} \).

**Proposition 6.3.** The set of measures \( m_{(r, a)} (a \in \mathbb{E}) \) is identical with the set of extreme points of \( \mathbb{E} \) concentrated on \( F_{a} \).

**Proof.** First we shall prove that each measure \( m_{(r, a)} \) is an extreme point of \( \mathbb{E} \). Suppose that
\[ m_{(r, a)} = c_{1} \mu_{1} + (1 - c_{1}) \mu_{2}, \]
where \( 0 < c < 1 \) and \( \mu_{1}, \mu_{2} \in \mathbb{E} \). It is clear that both measures \( \mu_{1} \) and \( \mu_{2} \) are concentrated on the set \( F_{a} \). Let \( g_{1} \) and \( g_{2} \) be the functions corresponding, by Proposition 6.2, to \( \mu_{1} \) and \( \mu_{2} \) respectively. Then
\[ g_{a} = c_{1} g_{1} + (1 - c_{1}) g_{2}. \]

Since both functions \( g_{1} \) and \( g_{2} \) are non-negative monotone non-decreasing and continuous from the left, we infer that the last equation is possible if they are constant on the half-lines \( (-\infty, a] \) and \( [a, \infty) \). Furthermore, by condition (6.11) \( g_{1} = g_{2} = g_{a} \) which proves that the measures \( m_{(r, a)} \) are extreme points of \( \mathbb{E} \).

Now we shall prove that each extreme point \( \lambda \) of \( \mathbb{E} \) concentrated on \( F_{a} \) is one of the measures \( m_{(r, a)} (a \in \mathbb{E}) \). Let \( g_{a} \) be the function corresponding to \( \lambda \) according to Proposition 6.2. Suppose that there exists a real number \( v_{a} \) such that the function \( g_{1} \) is constant on both halflines \( (-\infty, v_{a}) \) and \( (v_{a}, \infty) \). Setting
\[ c = g_{1}(v_{a}) = \int_{-\infty}^{v_{a}} \frac{[x, u]^{2}}{1 + [x, u]^{2}} du + \int_{v_{a}}^{\infty} \frac{[x, u]^{2}}{1 + [x, u]^{2}} ds g_{1}(u) du \]
we have, by (6.11), the inequalities \( 0 < c < 1 \). Further, the functions
\[ k_{1}(t) = \begin{cases} e^{-c} g_{1}(t) & \text{if } t \leq v_{a}, \\ e^{-c_{a}} q_{1}(t) & \text{if } t > v_{a}, \end{cases} \]
and
\[ k_{2}(t) = \begin{cases} 0 & \text{if } t \leq v_{a}, \\ (1 - e^{-c_{a}}) g_{2}(t) - g_{2}(v_{a}) & \text{if } t > v_{a}, \end{cases} \]
satisfy the conditions of the Proposition 6.2 and, consequently, determine the probability measures, say \( \lambda_{1} \) and \( \lambda_{2} \) belonging to \( \mathbb{E} \) and concentrated on \( F_{a} \). Since \( h_{1} \neq h_{2} \) and \( q_{1} = c_{a} h_{1} + (1 - c_{a}) h_{2} \), we infer that \( \lambda_{1} \neq \lambda_{2} \) and
\[ \lambda = c_{1} \lambda_{1} + (1 - c_{1}) \lambda_{2}, \]
which contradicts the assumption that \( \lambda \) is an extreme point. Thus for every real number \( v \) the function \( g_{1} \) is constant on at least one half-line \( (-\infty, v) \) and \( (v, \infty) \). But, according to (6.11), it is not constant on the whole line \( \mathbb{E} \). Let \( a \) be a point of increase of \( g_{1} \). Then the function \( g_{2} \) is constant on both half-lines \( (-\infty, a] \) and \( [a, \infty) \). Taking into account condition (6.11), we infer that \( g_{2} \) is equal to 0 on the half-line \( (-\infty, a] \) and is equal to \( g_{a} \) on the remaining half-line. Thus \( g_{a} = g_{a} \) and, consequently, \( \lambda = m_{(r, a)} \) which completes the proof.

From the definition (6.13), in view of (6.2) and (6.12), we get the relation \( m_{(r, a)} = m_{(r, a)} \) if and only if \( [x, a] = [y, b] \). This fact permits us to introduce the notation
\[ m_{(r, a)} = m_{(r, a)} \] if \( x \in \mathbb{S}^{N-1} \) and \( a \in \mathbb{E} \).
By (6.1), (6.12) and (6.13), for any function $f$ continuous on $M^N$ we have the formula

$$
\int_{M^N} f(x) \mu_{\lambda(a)}(dx) = e^{\lambda(a)} \int_0^\infty \frac{|T\{T[x, a]\}|^2}{1 + |T\{T[x, a]\]|^2} dt,
$$

where

$$
e^{\lambda(a)} \int_0^\infty \frac{|T\{T[x, a]\}|^2}{1 + |T\{T[x, a]\]|^2} dt.
$$

Further, we put

$$
m_{\lambda(a)} = \delta_{\lambda(a)} \quad \text{if either} \quad a = -\infty \quad \text{or} \quad a = \infty.
$$

The mapping $x \rightarrow m_x$ from $M^N$ onto the set of extreme points of $\mathcal{X}$ is one-to-one. From (6.15) and (6.16) it follows that this mapping is continuous at every point $[x, a]$ with $a \in R$. Further, it is obvious that $m_{\lambda(a)}$ tends to $m_{\lambda(a)}$ whenever $x_n \rightarrow x$ in $S^{N-1}$ and either $a = -\infty$ or $a = \infty$. Suppose that $x_n \rightarrow x$ in $S^{N-1}$, $a_n \in R$ and $a_n \rightarrow -\infty$, i.e. $[x_n, a_n] \rightarrow [x, -\infty]$. Then, by (6.1) and (6.2),

$$
\lim_{n \rightarrow \infty} |T\{T[x, a_n]\}| = |T\{T[x, a]\}|
$$

uniformly in $t$ in every finite interval. Hence and from (6.16) it follows that

$$
\lim_{a \rightarrow \infty} e^{\lambda(a)} = 0.
$$

Given $\varepsilon > 0$ and a continuous function $f$ on $M^N$, we can choose a number $t_0$ and an integer $n_0$ such that for all $t < t_0$ and $n \geq n_0$ the inequality

$$
|f([x_n, t]) - f([x, -\infty])| < \varepsilon
$$

holds. Consequently,

$$
\int_{\lambda(a)} \frac{|T\{T[x_n, a_n]\}|^2}{1 + |T\{T[x_n, a_n]\]|^2} dt = \int_0^\infty \frac{|T\{T[x_n, w]\}|^2}{1 + |T\{T[x_n, w]\]|^2} dw
$$

and $|\lambda_n| = 1$, we infer, by virtue of Lemma 6.1, that the integrals (6.20) are bounded in common. Thus, by (6.18),

$$
\lim_{n \rightarrow \infty} \int_{\lambda(a)} \frac{|T\{T[x_n, a_n]\}|^2}{1 + |T\{T[x_n, a_n]\]|^2} dt = 0.
$$

Further, by (6.19),

$$
\lim_{n \rightarrow \infty} \int_{\lambda(a)} \frac{|T\{T[x_n, a_n]\]|^2}{1 + |T\{T[x_n, a_n]\]|^2} dt = \varepsilon
$$

whenever $a_n \rightarrow a$. The arbitrariness of $a$ and (6.21) show that $m_{\lambda(a)} \rightarrow m_{\lambda(a)}$. Thus the mapping $x \rightarrow m_x$ is also continuous at the points $x$ of the form $[x, -\infty]$. Suppose now that $x_n \rightarrow x$ in $S^{N-1}$, $b_n \in R$ and $b_n \rightarrow \infty$, i.e. $[x_n, b_n] \rightarrow [x, \infty]$. Then, by (6.1),

$$
\lim_{n \rightarrow \infty} |T\{T[x_n, b_n]\}| = |T\{T[x, \infty]\}|
$$

uniformly in $t$ (i.e. $0$) which, by (6.15), implies the relation $m_{\lambda(a)} \rightarrow m_{\lambda(a)}$. Thus the mapping $x \rightarrow m_x$ is continuous at the points $x$ of the form $[x, \infty]$. This completes the proof of continuity of the mapping $x \rightarrow m_x$. Hence, by well known Theorem (see [4], p.11), we conclude that this mapping is a homeomorphism between $M^N$ and the set of extreme points of $\mathcal{X}$. Thus we have the following Proposition:

**Proposition 6.4.** The set of measures $m_x (x \in M^N)$ defined by formula (6.14) and (6.17) coincides with the set of extreme points of $\mathcal{X}$. Moreover, the mapping $x \rightarrow m_x$ is a homeomorphism between $M^N$ and the set of extreme points of $\mathcal{X}$.

Once the extreme points of $\mathcal{X}$ are found we can apply a Theorem by Choquet ([2], see also [11], Chapter 3). Since each element of $\mathcal{X}$ is of the form $\sigma_\theta$ where $\theta \geq 0$ and $\sigma \in \mathcal{X}$, we then get the following proposition:

**Proposition 6.5.** A measure $\mu$ belongs to $\mathcal{X}$ if and only if there exists a finite Borel measure $\lambda$ on $M^N$ such that for each continuous function $f$ on $M^N$ the equation

$$
\int_{M^N} f(u) \mu(du) = \int_{M^N} f(u) \lambda(du) \lambda(dx)
$$

holds.

7. A representation of characteristic functions. Suppose that all eigenvalues of $Q$ have negative real part. By Proposition 5.2 each exp$Q$-decomposable for $t > 0$ probability measure $\mu$ is infinitely divisible. Consequently, the characteristic function $\hat{\mu}$ has a Lévy–Khinchine representation

$$
\hat{\mu}(y) = \exp \left\{ i(a, y) - \frac{1}{2}(Dy, y) + \int_{S^{N-1} \setminus \{0\}} \left( e^{i(y, w)} - 1 - i(y, w) \frac{1}{2} + \frac{|w|^2}{2} \right) \frac{1}{|w|^2} \lambda(\{w\}) \right\},
$$
where \(a\) is a vector from \(\mathbb{R}^N\), \(D\) is a symmetric non-negative operator in \(\mathbb{R}^N\) and \(\gamma\) is a finite Borel measure on \(\mathbb{R}^N\). The triplet \(a\), \(D\) and \(\gamma\) is uniquely determined by \(\mu\). The operator \(D\) and the measure \(\gamma\) will be called the Lévy–Khintchine operator and measure of \(\mu\) respectively. In what follows \(\pi\) will denote the embedding of \(\mathbb{R}^N\) into \(M^N\) defined by the formula (6.3).

**Proposition 7.1.** A symmetric non-negative operator \(D\) and a finite Borel measure \(\gamma\) on \(\mathbb{R}^N\) are Lévy–Khintchine operators and measures respectively of an \(\exp Q\)-decomposable for \(t \geq 0\) probability measure if and only if the operator \(QD + DQ^\ast\) is non-positive and the induced measure \(\pi\gamma\) belongs to \(\mathcal{M}\).

**Proof.** Put for simplicity of notation \(T_t = \exp Q(t\mathbb{R}^N)\). Suppose that the characteristic function of a measure \(\mu\) is given by (7.1). By Proposition 5.2, \(\mu\) is \(T_t\)-decomposable for \(t \geq 0\) if and only if for any \(t \geq 0\)

\[\mu = T_t \mu \ast \mu,\]

where \(\mu\) is an infinitely divisible measure. This condition can be formulated in terms of the characteristic functions as follows: \(\mu\) is \(T_t\)-decomposable for \(t \geq 0\) if and only if for any \(t \geq 0\) \(\mu^{1/2} T_t \mu^{1/2}\) is the characteristic function of an infinitely divisible measure. From (7.1) by a simple computation we get the formula

\[
\mu(y) = \exp \left( i(a, y) - \frac{1}{2} (D(y), y) + \int \frac{1}{\rho} \frac{1}{|y|} \frac{1}{|y|} |y| \gamma(dy) \right),
\]

(7.2)

where \(a \in \mathbb{R}^N\),

\[
D = D - T_t D T_t^\ast,
\]

and

\[
\gamma(E) = \gamma(E) - \frac{1}{E} \frac{1}{|E|} \frac{1}{|E|} \gamma(T_t \sigma dv).
\]

(7.3)

Hence if follows that \(\mu\) is \(T_t\)-decomposable for \(t \geq 0\) if and only if for any \(t \geq 0\) \(D_t\) is non-negative, and \(\gamma_t\) is a non-negative measure.

First we shall prove that the operator \(D_t\) is non-negative for \(t \geq 0\) if and only if the operator \(QD + DQ^\ast\) is non-positive. Suppose that \(D_t\) is non-negative for all \(t \geq 0\). By (7.2), we have the expansion in a neighborhood of 0 \(D_t = -t(QD - DQ^\ast) + o(t)\). Hence it follows that \(QD - DQ^\ast\) is non-positive.

Assume now that \(QD + DQ^\ast\) is non-positive. Given \(x \in \mathbb{R}^N\), we put \(w_x(t) = (D_t x, x)\). By a simple computation we get the formula

\[
\frac{d}{dt} w_x(t) = -((QD + DQ^\ast) T_t x, T_t x).
\]

which implies the inequality \(\frac{d}{dt} w_x(t) \geq 0\). Taking into account the initial condition \(w_x(0) = 0\), we get the inequality \(w_x(t) \geq 0\) for all \(t \geq 0\) and all \(x \in \mathbb{R}^N\). Thus the operators \(D_t\) are for \(t \geq 0\) non-negative.

Taking into account (6.2), (6.3), (6.4), (6.5) and (7.3) for each Borel subset \(E\) of \(M^N\) bounded from below we have the formula

\[
\int_{\mathbb{R}^N} \int \frac{1}{|y|} \frac{1}{|y|} \gamma(dy) = \int_{\mathbb{R}^N} \int \frac{1}{|y|} \frac{1}{|y|} \gamma(dy).
\]

Consequently, \(\gamma\) is non-negative for \(t \geq 0\) if and only if \(\pi \gamma \in \mathcal{M}\). The Proposition is thus proved.

**Theorem 7.1.** Suppose that all eigenvalues of \(Q\) have negative real part. A function \(\psi\) on \(\mathbb{R}^N\) is the characteristic function of an \(\exp Q\)-decomposable for \(t \geq 0\) probability measure if and only if

\[
\psi(y) = \exp \left( i(a, y) - \frac{1}{2} (D(y), y) + \int \frac{1}{\rho} \frac{1}{|y|} \frac{1}{|y|} |y| \gamma(dy) \right),
\]

where \(a\) is a vector from \(\mathbb{R}^N\), \(D\) is a symmetric non-negative operator in \(\mathbb{R}^N\) for which the operator \(QD + DQ^\ast\) is non-positive and \(\gamma\) is a finite Borel measure on \(\mathbb{R}^N\). Moreover, the function \(\psi\) determines the triplet \(a\), \(D\) and \(\gamma\) uniquely.

**Proof.** The necessity. Suppose that \(\mu\) is an \(\exp Q\)-decomposable for \(t \geq 0\) probability measure. By Proposition 5.2 \(\mu\) is infinitely divisible and its characteristic function can be written in the form (7.1) with parameters \(a\), \(D\) and \(\gamma\). Moreover, by Proposition 7.1, the operator \(QD + DQ^\ast\) is non-positive and the induced measure \(\pi \gamma\) on \(\mathbb{R}^N\) belongs to \(\mathcal{M}\). By Proposition 6.5 there exists a finite Borel measure \(\pi \mu\) on \(\mathbb{R}^N\) such that for every continuous function \(f\) on \(\mathbb{R}^N\), the equation

\[
\int f(u) \pi \gamma (du) = \int_{\mathbb{R}^N} f(u) \mu(u) (du)
\]

holds. Here \(m_0 (\pi \mathbb{M}^N)\) denote the extreme points of \(\mathcal{M}\) defined by the formulas (6.14) and (6.17). It is clear that the measure \(\pi \gamma\) is concentrated on the set \(\pi \mathbb{M}^N\). Consequently, by (7.5), the measure \(\mu\) is also concentrated on \(\pi \mathbb{M}^N\). Since for \(x \in \pi \mathbb{M}^N\) the measures \(m_0\) are concentrated on \(\pi \mathbb{M}^N\) (see (6.15)), the formula (7.5) can be rewritten in the form

\[
\int f(u) \mu (du) = \int_{\pi \mathbb{M}^N} f(u) m_0 (du) (du)
\]
for any function $f$ continuous and bounded on $U_N$. Let us introduce the notation $\lambda = \pi^{-1}/2$ and

$$\nu(E) = \int_E c(x) \log(1 + |x|^2) \lambda(dx)$$

where $E$ are Borel subsets of $R^N \setminus \{0\}$ and

$$c(x)^{-1} = \int_0^\infty \frac{\exp|Qx|^2}{1 + |x|^2} \, dt.$$  

By Lemma 6.1 $\nu$ is a finite measure on $R^N \setminus \{0\}$. By a simple computation, in view of (6.2), (6.3), (6.4), (6.5) and (7.6), for every continuous and bounded function $g$ on $R^N \setminus \{0\}$ we get the formula

$$g(x) \gamma(dx) = \int_{R^N \setminus \{0\}} g(\exp Qx) \frac{|\exp|Qx|^2}{1 + |x|^2} \, dt \frac{\nu(dx)}{1 + |x|^2}. \tag{7.7}$$

Setting

$$g(x) = \begin{cases} e^{i(x,a)} - 1 - \frac{i(x,y)}{1 + |y|^2} & y \in R^N \end{cases}$$

into the last formula and taking into account (7.1) we get the representation (7.4). The necessity of the conditions is thus proved.

The sufficiency. Suppose that the function $\varphi$ is given by formula (7.4). First we note that $\varphi$ is a limit of products of a Gaussian characteristic function $\exp \{i(a, y) - \frac{1}{2} (D_y, y)\}$ and Poissonian characteristic functions of the form

$$\exp \left[ e^{i(x,b)} - 1 - \frac{i(x,b)}{1 + |b|^2} \right],$$

where $x \geq 0$ and $b \in R^N \setminus \{0\}$. Thus $\varphi$ is the characteristic function of an infinitely divisible measure, say $\mu$ (see [10], Theorems 4.1 and 4.10). For every $x \geq 0$ from (7.4) by a simple computation we get the formula

$$\mu'(y)/\exp Q \mu(y) = \exp \{i(a, y) - \frac{1}{2} (D_y, y)\} +$$

$$+ \int_{R^N \setminus \{0\}} \left[ e^{i(x,\exp Qx)} - 1 - \frac{i(x,\exp Qx)}{1 + |\exp Qx|^2} \right] \frac{\nu(dx)}{1 + |x|^2} \gamma(dx),$$

where $a \in R^N$ and $D_y = (\exp Q + D)^{-1}(\exp Q)^*$. From the assumption that the operator $Q D + D^* Q$ is non-positive and from Proposition 7.1 it follows that the Gaussian probability measure with the characteristic function $\exp \{-\frac{1}{2} (D_y, y)\}$ is $\exp Q$-decomposable for $t \geq 0$. Hence we infer that the operator $D_t$ is non-negative. Consequently, the function (7.8) is a limit of products of Gaussian and Poissonian characteristic functions. Thus (7.8) is the characteristic function of a probability measure, say $\lambda_t$. Obviously, $\mu = \exp Q \mu \ast \lambda_t$ for $t \geq 0$ which shows that the function $\varphi$ is the characteristic function of an $\exp Q$-decomposable for $t \geq 0$ probability measure. The sufficiency of the conditions is thus proved.

It remains to prove the uniqueness of the triplet $a, D$ and $\nu$ in the representation (7.4). First we note that the formula (7.7) establishes a one-to-one correspondence between the Lévy-Khintchine measure $\gamma$ and the measure $\nu$. In fact, it is evident that $\nu$ determines $\gamma$ uniquely. To prove the converse let us take an arbitrary continuously differentiable function $f$ in $R^N$ with a compact support and vanishing in a neighborhood of the origin. Then the function

$$g(x) = -\frac{1 + |x|^2}{|x|^2} \frac{d}{dt} f(\exp Q x) \log(1 + |\exp Q x|^2)|_{t=0}$$

is continuous and bounded on $R^N$. Setting it into (7.7) we get the formula

$$\int_{R^N \setminus \{0\}} g(x) \gamma(dx) = \int_{R^N \setminus \{0\}} f(x) \nu(dx)$$

which shows that $\nu$ determines $\gamma$ uniquely.

Suppose now that the function $g$ has two representations (7.4) with the triplets $(a_1, D_1, \gamma_1)$ and $(a_2, D_2, \gamma_2)$ respectively. Then denoting by $\gamma_1$ and $\gamma_2$ the measures corresponding in (7.7) to $\gamma_1$ and $\gamma_2$ respectively we have, by (7.4) and (7.7), the equations

$$\varphi(x) = \exp \{i(a_1, y) - \frac{1}{2} (D_1 y, y)\} +$$

$$+ \int_{R^N \setminus \{0\}} \left[ e^{i(x,\exp Qx)} - 1 - \frac{i(x,\exp Qx)}{1 + |\exp Qx|^2} \right] \frac{\nu(dx)}{1 + |x|^2} \gamma_1(dx)$$

$$= \exp \{i(a_1, y) - \frac{1}{2} (D_2 y, y)\} +$$

$$+ \int_{R^N \setminus \{0\}} \left[ e^{i(x,\exp Qx)} - 1 - \frac{i(x,\exp Qx)}{1 + |\exp Qx|^2} \right] \frac{\nu(dx)}{1 + |x|^2} \gamma_2(dx)$$

Hence, by the uniqueness of the Lévy-Khintchine representation (see [10] Theorem 4.10), $a_1 = a_2$, $D_1 = D_2$, and $\gamma_1 = \gamma_2$. Since the measures $\gamma_1$ and $\gamma_2$ determine the measures $\nu_1$ and $\nu_2$, respectively, we have the equation $\nu_1 = \nu_2$ which completes the proof.
On the Riesz–Fischer theorem for vector-valued functions
by
W. MATUZSEWSKA and W. ORLIK (Poznań)

Dedicated to Professor A. Zygmund on the occasion of 65th anniversary of his scientific research

Abstract. Let \( \varphi : (0, \infty) \to K \) be a nondecreasing continuous function satisfying conditions \( \varphi(u)/u \to 0 \) if \( u \to 0 \), \( \varphi(u)/u \to \infty \) if \( u \to \infty \), \( X \) denote a Banach space, \( \mathcal{E} \) its dual space. Let, further, \( \mathcal{E}^* \) denote a vector space consisting of sequences \( x^* = \{x_n\} \), \( x_n \in \mathcal{E} \).

Assuming that \( \varphi \) is a convex function on \( \mathcal{E}^* \) one can define a modular \( q_\varphi(x) = \sup \sum \varphi(\|x_i\|) \), where supremum is taken over the ball \( \mathcal{B}_q = \{x : \|x\| < 1\} \).

Investigations are the properties of the space \( P^*(X) \), elements of which are the sequences \( x^* \in \mathcal{E}^* \) such that \( q_\varphi(\|x^*\|) < \infty \) for some \( \lambda > 0 \). Section 2 of the paper deals with the spaces of vector functions \( x(\cdot) : (a, b) \to \mathcal{E} \), of finite Riesz \( \varphi \)-variation (as defined in 2.1) and with the spaces \( P^*(X) \).

In Section 3 certain remarks are made about orthogonal series of the form \( \{x_n\} \in \mathcal{E}^* \) and \( \{y_n\} \in \mathcal{E}^* \) where \( x_n \in \mathcal{E} \), and \( \{y_n\} \) is an orthogonal system in \( \mathcal{E} \).

If \( x(\cdot) : (a, b) \to \mathcal{E} \) is a vector function absolutely continuous in \( (a, b) \), then its Fourier coefficients are represented by \( c_n = f q_\varphi(t) dt \) where the integral in this formula is a (Dunford) integral \( \langle a, b \rangle \) with respect to the vector measure \( x(\cdot) \) associated with \( x(\cdot) \).

Using the spaces \( P^*(X) \), \( P^*(\mathcal{E}) \), where \( f(\cdot) = \omega(\cdot) \), authors obtain the analogue of Riesz–Fischer theorem for series of the form \( \{x(\cdot)\} \).

1. In this note \( \mathcal{E} \) always stands for a real Banach space provided with a norm \( \|\cdot\| \), \( \mathcal{E} \) for its conjugate space, \( \mathcal{E} = (\mathcal{E}, \mathcal{E}^*) \). \( \mathcal{E} \) denotes the class of all zero-one sequences \( \{q_n\} \), \( \mathcal{E} \) denotes the algebra of subsets of an interval \( T = (a, b) \) whose elements are finite unions of intervals \( (a, b) \), \( 0 \leq a < c < d < b, (a, b), a < c < d < b \), \( \langle a, b \rangle \), \( a < d < b \) and the empty set, \( \mathcal{E} \) is the \( s \)-algebra of Lebesgue measurable subsets of \( T \) and \( \mu \) is the Lebesgue measure on \( \mathcal{E} \). Measurability of sets and functions are always understood with respect to \( \mu \).

\[ x(\cdot), y(\cdot), \ldots \text{ or } x_n, y_n, \ldots \text{ always denote vector-valued functions from } T \text{ into } \mathcal{E}, f(\cdot), g(\cdot), \ldots \text{ or } f_n, g_n, \ldots \text{ real-valued function on } T. \] A series \( \sum \eta \), \( \eta \) of elements belonging to a Banach space is said to be perfectly con-