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Lévy's probability measures on Euclidean spaces

by

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Dedicated to Professor Antoni Zygmund in honour of the fiftieth anniversary of his scientific activity

Abstract. The limit laws arising from an affine modification of sequences of partial sums of independent random variables whose values belong to the Euclidean space are characterized in terms of operator-decomposability of probability measures. Our next aim is to give a representation of the characteristic function of these limit laws. The method of proof consists in finding the extreme points of a certain convex set of measures. Then the Choquet Theorem yields the representation formula.

Introduction. A Lévy's probability measure on the Euclidean space R^N is a limit law arising, roughly speaking, from affine modification of the partial sums of a sequence of independent R^N-valued random variables. This paper is concerned with a description of Lévy's probability measures. The limit laws in the case of a sequence of independent and identically distributed random variables, i.e. the operator stable probability measures on R^N were considered by H. Sharpe in [12].

Throughout this paper we denote by $\mathscr{P}(R^N)$ or, shortly, by $\mathscr{P}$ the set of all probability measures on R^N. With the topology of weak convergence and multiplication defined by the convolution $\mathscr{P}$ becomes a topological semigroup. We denote the convolution of two measures λ and μ by $\lambda * \mu$. Moreover, by δ_a $(a \in R^N)$ we denote the probability measure concentrated at the point a. The characteristic function $\hat{\lambda}$ of a measure $\lambda \in \mathscr{P}$ is defined by the formula

$$\hat{\lambda}(x) = \int_{R^N} \exp i(x, y)\lambda(dy),$$

where (x, y) denotes the inner product in R^N.

Given $\lambda \in \mathscr{P}$, we define λ^- by the formula $\lambda^-(E) = \lambda(-E)$, where $-E = \{-x: x \in E\}$. The mapping $\lambda \to \lambda^-$ is an involutive automorphism of $\mathscr{P}$. It is easy to see that $\hat{\lambda}^- = \bar{\hat{\lambda}}$, the last bar denoting the complex conjugate. For any $\lambda \in \mathscr{P}$, the measure $^\circ\lambda = \lambda * \lambda^-$ is called the *symmetrization* of λ.

We call a measure from $\mathscr{P}(R^N)$ *full* if its support is not contained in any $(N-1)$-dimensional hyperplane of R^N. We denote by $\mathscr{F}(R^N)$ or, shortly, by $\mathscr{F}$ the set of all full probability measures on R^N. We mention that the set $\mathscr{F}$ is an open subsemigroup of $\mathscr{P}$.

Let $\mathbf{End}\, R^N$ denote the semigroup of all linear operators in R^N with the composition as a semigroup operation. Further, let $\mathbf{Aut}\, R^N$ denote the group of all non-singular linear operators in R^N. For any $A \in \mathbf{End}\, R^N$ and $\lambda \in \mathscr{P}(R^N)$ let $A\lambda$ denote the measure defined by the formula $A\lambda(E) = \lambda(A^{-1}(E))$ for all Borel subsets E of R^N. It is easy to check the equations for all $A, B \in \mathbf{End}\, R^N$

$$A(B\lambda) = (AB)\lambda, \quad A(\lambda * \mu) = A\lambda * A\mu, \quad \widehat{A\lambda}(y) = \hat{\lambda}(A^* y),$$

where A^* denotes the adjoint operator. Moreover, the mapping $\langle A, \lambda\rangle \to A\lambda$ from $\mathbf{End}\, R^N \times \mathscr{P}(R^N)$ onto $\mathscr{P}(R^N)$ is jointly continuous, where $\mathbf{End}\, R^N$ is provided with a norm topology. Consequently, we have the following statement:

(i) *If a sequence $\{A_n\}$ is precompact in $\mathbf{End}\, R^N$, then for every $\lambda \in \mathscr{P}(R^N)$ the sequence $\{A_n\lambda\}$ is precompact in $\mathscr{P}(R^N)$.*

For full measures the converse implication is also true. Namely, we shall prove the following statement:

(ii) *Let $\lambda \in \mathscr{F}(R^N)$ and $A_n \in \mathbf{End}\, R^N$ $(n = 1, 2, \ldots)$. If the sequence $\{A_n\lambda\}$ is precompact in $\mathscr{P}(R^N)$, then the sequence $\{A_n\}$ is precompact in $\mathbf{End}\, R^N$.*

Proof. We shall assume for the purpose of obtaining a contradiction that $\{A_n\lambda\}$ is precompact and $\{\|A_n\|\}$ is unbounded. Let us choose vectors x_n in R^N such that $\|x_n\| = 1$ and $\|A_n^*\| = \|A_n^* x_n\|$ $(n = 1, 2, \ldots)$. Passing to a subsequence, if necessary, it may be assumed that $\|A_n\| \to \infty$ and the sequence of vectors $\|A_n^*\|^{-1} A_n^* x_n$ tends to a vector $u \in R^N$ with $\|u\| = 1$. Since the sequence $\{A_n\lambda\}$ is precompact and $\|A_n^*\|^{-1} x_n \to 0$, we infer that

$$\lim_{n\to\infty} \widehat{A_n\lambda}(c\|A_n^*\|^{-1} x_n) = 1$$

for every $c \in R$. Consequently, by the transformation rule of $\widehat{A_n\lambda}$,

$$\lim_{n\to\infty} \hat{\lambda}(c\|A_n^*\|^{-1} A_n^* x_n) = 1,$$

which yields the equation $\hat{\lambda}(cu) = 1$ for all $c \in R$. In other words, we proved that the characteristic function of λ is equal to 1 on a one-dimensional subspace of R^N. Hence it follows that λ is not a full measure (see [12], p. 52, Proposition 1). The contradiction implies that $\{\|A_n\|\}$ is bounded and, consequently, the sequence $\{A_n\}$ is precompact in $\mathbf{End}\, R^N$.

1. Operator-decomposability of measures. Let $\lambda \in \mathscr{P}(R^N)$ and $A \in \mathbf{End}\, R^N$. Suppose that there exists a measure $\lambda_A \in \mathscr{P}(R^N)$ for which the equation

$$\lambda = A\lambda * \lambda_A \tag{1.1}$$

holds. Then we say that the measure λ is *A-decomposable*. We denote by $\mathbf{E}(\lambda)$ the set of all operators A such that the measure λ is A-decomposable. Further, by $\mathbf{A}(\lambda)$ we denote the subset of $\mathbf{E}(\lambda)$ consisting of those operators A for which in (1.1) we may take $\lambda_A = \delta_a$ for some vector $a \in R^N$. It is obvious that the identity operator I belongs to $\mathbf{A}(\lambda)$ for all $\lambda \in \mathscr{P}$. Moreover, since $0\lambda = \delta_0$, we infer that $0 \in \mathbf{E}(\lambda)$ for all $\lambda \in \mathscr{P}$.

In this section we shall establish some simple properties of the sets $\mathbf{E}(\lambda)$ and $\mathbf{A}(\lambda)$.

PROPOSITION 1.1. *For every $\lambda \in \mathscr{F}(R^N)$ the set $\mathbf{E}(\lambda)$ is a compact subsemigroup of $\mathbf{End}\, R^N$.*

Proof. Given $A, B \in \mathbf{E}(\lambda)$, we put $C = AB$ and $\lambda_C = A\lambda_B * \lambda_A$. It is easy to check that $\lambda = C\lambda * \lambda_C$. Thus $\mathbf{E}(\lambda)$ is a subsemigroup of $\mathbf{End}\, R^N$. Suppose now that $A_n \in \mathbf{E}(\lambda)$ $(n = 1, 2, \ldots)$. It is clear that the symmetrization $^\circ\lambda$ of λ is also full and $^\circ\lambda = A_n{}^\circ\lambda * {}^\circ\lambda_{A_n}$ $(n = 1, 2, \ldots)$. By Theorems 2.2. and 5.1. in [10] (pp. 59 and 71) we infer that both sequences $\{A_n{}^\circ\lambda\}$ and $\{^\circ\lambda_{A_n}\}$ are precompact in $\mathscr{P}$. Moreover, by the property (ii), the sequence $\{A_n\}$ is precompact in $\mathbf{End}\, R^N$. Let A be its limit point. Without loss of generality we may assume that the sequence $\{A_n\}$ converges to A. Then

$$\lim_{n\to\infty} A_n\lambda = A\lambda. \tag{1.2}$$

It remains to prove that the sequence of measures $\{\lambda_{A_n}\}$ is precompact. Since the sequence of the symmetrizations $\{^\circ\lambda_{A_n}\}$ is precompact, we infer, by Theorem 2.2. in [10] (p. 59), that there exists a sequence $\{a_n\}$ of vectors in R^N for which the sequence of measures $\{\lambda_{A_n} * \delta_{a_n}\}$ is precompact in $\mathscr{P}$. Thus, by (1.2), the sequence $\mu_n = A_n\lambda * \lambda_{A_n} * \delta_{a_n}$ $(n = 1, 2, \ldots)$ is precompact in $\mathscr{P}$. But $\mu_n * \delta_{-a_n} = \lambda$ $(n = 1, 2, \ldots)$. Now it is easy to prove that the sequence $\{a_n\}$ is precompact in R^N (see e.g. [12], The Compactness Lemma, p. 55). Hence it follows that the sequence of measures $\{\lambda_{A_n}\}$ is precompact. Denoting by λ_A its limit point we have, by (1.2), the formula $\lambda = A\lambda * \lambda_A$ which shows that $A \in \mathbf{E}(\lambda)$. Thus the set $\mathbf{E}(\lambda)$ is compact which completes the proof.

PROPOSITION 1.2. *For every $\lambda \in \mathscr{F}(R^N)$ the set $\mathbf{A}(\lambda)$ is a compact subgroup of $\mathbf{Aut}\, R^N$.*

Proof. Suppose that $A, B \in \mathbf{A}(\lambda)$ and $\lambda_A = \delta_a$, $\lambda_B = \delta_b$. Setting $C = AB$ and $c = Ab + a$, we get the equation $\lambda = C\lambda * \delta_c$. Consequently, $\mathbf{A}(\lambda)$ is a semigroup. Further, for every $A \in \mathbf{A}(\lambda)$ the measure $A\lambda$ is also full. Since the support of $A\lambda$ is contained in the image $A(R^N)$, we infer that the operator A is invertible. Setting $a = -A^{-1}a$ we have the formula $\lambda = A^{-1}\lambda * \delta_a$ which shows that $A^{-1} \epsilon \mathbf{A}(\lambda)$ and, consequently, that $\mathbf{A}(\lambda)$ is a subgroup of **Aut** R^N. Suppose now that $A_n \epsilon \mathbf{A}(\lambda)$ and for some vectors $a_n \epsilon R^N$ the equations $\lambda = A_n\lambda * \delta_{a_n}$ $(n = 1, 2, \ldots)$ hold. By the Compactness Lemma in [12] (p. 55) we infer that both sequences $\{A_n\}$ and $\{a_n\}$ are precompact in **Aut** R^N and R^N respectively. Moreover, if A and a are their limit points, then $\lambda = A\lambda * \delta_a$. Thus $A \epsilon \mathbf{A}(\lambda)$ which completes the proof.

PROPOSITION 1.3. *If A and A^{-1} belong to $\mathbf{E}(\lambda)$, then $A \epsilon \mathbf{A}(\lambda)$.*

Proof. From the equation $\lambda = A^{-1}\lambda * \lambda_{A-1}$ we get the following one $A\lambda = \lambda * A\lambda_{A^{-1}}$. Hence and from the formula $\lambda = A\lambda * \lambda_A$ we get the inequality for characteristic functions

$$|\hat{\lambda}(y)| = |\widehat{A\lambda}(y)|\,|\hat{\lambda}_A(y)| \leqslant |\hat{\lambda}(y)|\,|\hat{\lambda}_A(y)| \quad (y \epsilon R^N),$$

which yields the equation $|\hat{\lambda}_A(y)| = 1$ in a neighborhood of the origin. By elementary properties of the characteristic function the last relation implies the formula $|\hat{\lambda}_A(y)| = 1$ for all $y \epsilon R^N$. Thus $\lambda_A = \delta_a$ for a vector $a \epsilon R^N$ which shows that $A \epsilon \mathbf{A}(\lambda)$.

In what follows for any operator $A \epsilon \mathbf{End} R^N$ $\det A$ will denote the determinant of the matrix representation of A with respect to an orthonormal basis in R^N.

PROPOSITION 1. 4. *Let $\lambda \epsilon \mathscr{F}(R^N)$. If $A \epsilon \mathbf{E}(\lambda)$ and $|\det A| = 1$, then $A \epsilon \mathbf{A}(\lambda)$.*

Proof. Consider the monothetic compact subsemigroup $\mathbf{S}$ of $\mathbf{E}(\lambda)$ generated by the operator A. By a Theorem of Numakura (see [8], [9] p. 109) the limit points of the sequence $\{A^n\}$ form a group $\mathbf{G}$ which is the minimal ideal of $\mathbf{S}$ and $\mathbf{S}$ contains exactly one idempotent, namely the unit J_0 of $\mathbf{G}$. Of course, $\det J_0 = 1$ and, consequently, J_0 is the identity operator I. Hence it follows that $\mathbf{S} = \mathbf{G}$ and, consequently $\mathbf{S}$ is a group. Now our assertion is a consequence of Proposition 1.3.

PROPOSITION 1.5. *For every idempotent J from $\mathbf{E}(\lambda)$ the equation $\lambda = J\lambda * (I-J)\lambda$ holds. Consequently, $I-J \epsilon \mathbf{E}(\lambda)$.*

Proof. Let J be an idempotent and

$$(1.3) \qquad \lambda = J\lambda * \lambda_J.$$

Hence we get the formula $J\lambda = J\lambda * J\lambda_J$. Consequently, $\widehat{J\lambda} = \widehat{J\lambda} \cdot \widehat{J\lambda_J}$ which implies the equation $\widehat{J\lambda_J}(y) = 1$ in a neighborhood of the origin. It is well-known that the last condition implies the formula $\widehat{J\lambda_J}(y) = 1$

for all $y \epsilon R^N$. Thus $J\lambda_J = \delta_0$. Hence, in particular, it follows that the measure λ_J is concentrated on the subspace $(I-J)(R^N)$. In other words,

$$(1.4) \qquad (I-J)\lambda_J = \lambda_J.$$

Since $0\lambda = \delta_0$, equation (1.3) yields the formula

$$(I-J)\lambda = (I-J)J\lambda * (I-J)\lambda_J = 0\lambda * (I-J)\lambda_J = (I-J)\lambda_J.$$

Thus, by (1.4), $(I-J)\lambda = \lambda_J$ which, together with (1.3), implies the equation $\lambda = J\lambda * (I-J)\lambda$.

PROPOSITION 1.6. *Let J be an arbitrary idempotent from $\mathbf{E}(\lambda)$. Then for every pair A, B of operators from $\mathbf{E}(\lambda)$ the operator $JAJ + (I-J)B(I-J)$ belongs to $\mathbf{E}(\lambda)$ too.*

Proof. Let $A, B \epsilon \mathbf{E}(\lambda)$, i.e.

$$(1.5) \qquad \lambda = A\lambda * \lambda_A$$

and

$$(1.6) \qquad \lambda = B\lambda * \lambda_B.$$

If J is an idempotent from $\mathbf{E}(\lambda)$, then by Proposition 1.5,

$$(1.7) \qquad \lambda = J\lambda * (I-J)\lambda.$$

Thus

$$(1.8) \qquad JA\lambda = JAJ\lambda * JA(I-J)\lambda$$

and

$$(1.9) \qquad (I-J)B\lambda = (I-J)BJ\lambda * (I-J)B(I-J)\lambda.$$

The equations (1.5) and (1.8) imply the equation

$$(1.10) \qquad J\lambda = JA\lambda * J\lambda_A = JAJ\lambda * JA(I-J)\lambda * J\lambda_A.$$

Further, from the equations (1.6) and (1.9) we obtain

$$(1.11) \qquad (I-J)\lambda = (I-J)B\lambda * (I-J)\lambda_B = (I-J)B(I-J)\lambda * (I-J)BJ\lambda * (I-J)\lambda_B.$$

Taking into account (1.7), (1.10) and (1.11) we get the formula

$$(1.12) \qquad \lambda = JAJ\lambda * (I-J)B(I-J)\lambda * \lambda_C,$$

where $\lambda_C = JA(I-J)\lambda * J\lambda_A * (I-J)BJ\lambda * (I-J)\lambda_B$. Setting $C = JAJ + (I-J)B(I-J)$, we get, by virtue of (1.7),

$$C\lambda = CJ\lambda * C(I-J)\lambda = JAJ\lambda * (I-J)B(I-J)\lambda.$$

Hence and from (1.12) we get the formula $\lambda = C\lambda * \lambda_C$ which yields $C \epsilon \mathbf{E}(\lambda)$. The proposition is thus proved.

PROPOSITION 1.7. *If* $B_n \in \mathbf{E}(\lambda)$ $(n = 1, 2, \ldots)$,

(1.13) $$\lim_{k\to\infty} B_n^k = 0 \quad (n = 1, 2, \ldots),$$

(1.14) $$\lim_{n\to\infty} B_n = I$$

and the set $\{B_n^{*k}: k = 0, 1, \ldots, n = 1, 2, \ldots\}$ *is precompact, then* $\hat{\lambda}(y) \neq 0$ *for every* $y \in R^N$.

Proof. Suppose the contrary and assume that $\hat{\lambda}(a) = 0$ and $\hat{\lambda}(y) \neq 0$ whenever $\|y\| < \|a\|$. First we note that the equations $\hat{\lambda}(y) = \hat{\lambda}(B_n y)\hat{\lambda}_{B_n}(y)$ $(n = 1, 2, \ldots)$ and the assumption (1.14) imply the relation $\lim_{n\to\infty} \hat{\lambda}_{B_n}(y) = 1$ whenever $\|y\| < \|a\|$. But the last relation is equivalent to the following one

(1.15) $$\lim_{n\to\infty} \lambda_{B_n} = \delta_0.$$

Let E_0 be the closure of the set $\{B_n^{*k} a: k = 0, 1, \ldots, n = 1, 2, \ldots\}$. By the assumption the set E_0 is compact. Thus, by (1.15), $\lim_{n\to\infty} \hat{\lambda}_{B_n}(y) = 1$ uniformly on E_0. Consequently, without loss of generality we may assume that

(1.16) $$\hat{\lambda}_{B_n}(y) \neq 0 \quad (n = 1, 2, \ldots; y \in E_0).$$

Now we shall prove that $\hat{\lambda}(x) = 0$ for all $x \in E_0$. Since $\hat{\lambda}(a) = 0$ to prove this it suffices to prove that $\hat{\lambda}(B_n y) = 0$ $(n = 1, 2, \ldots)$ whenever $\hat{\lambda}(y) = 0$ and $y \in E_0$. But this implication is a consequence of the equation

$$\hat{\lambda}(y) = \hat{\lambda}(B_n y)\hat{\lambda}_{B_n}(y)$$

and the inequality (1.16). In particular, we have the formula $\hat{\lambda}(0) = 0$ because, in view of (1.13), $0 \in E_0$. But this contradicts the obvious formula $\hat{\lambda}(0) = 1$. The Proposition is thus proved.

2. Statement of the problem. A triangular array of probability measures μ_{ij} $(i = 1, 2, \ldots, k_j;\ j = 1, 2, \ldots)$ on R^N is said to be *uniformly infinitesimal* if for every neighborhood U of the origin the relation

$$\lim_{n\to\infty} \max_{1\leqslant i\leqslant k_n} \mu_{in}(R^N \setminus U) = 0$$

holds.

In terms of random variables, the problem we study is enunciated as follows: suppose that $\{X_n\}$ is a sequence of independent R^N-valued random variables and assume that $\{A_n\}$ and $\{a_n\}$ are sequences from $\mathbf{Aut}\, R^N$ and R^N respectively such that the probability distributions of $A_n X_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ form a uniformly infinitesimal triangular array and the distribution of

$$A_n \sum_{k=1}^{n} X_k + a_n$$

converges to a measure μ; what can be said about the limit measure μ? Converting this to a problem involving only measures we ask which measures μ can arise as limits of sequences $A_n(\mu_1 * \mu_2 * \ldots * \mu_n) * \delta_{a_n}$ where $\{\mu_n\}$ is an arbitrary sequence of probability measures in R^N, such that $A_n \mu_k$ $(k = 1, 2, \ldots, n;\ n = 1, \ldots)$ form a uniformly infinitesimal triangular array. The limit measures μ will be called *Lévy's measures*. The set of all Lévy's measures on R^N will be denoted by $\mathscr{L}_N$.

We refer the reader to M. Loéve [6] (p. 319) for an account of the set $\mathscr{L}_1$. The problem of characterizing of this set was proposed by A. Ya. Khintchine in 1936 and solved by P. Lévy in [5] (p. 195). He proved that a measure belongs to $\mathscr{L}_1$ if and only if, it is self-decomposable. Self-decomposability of a measure μ means here that $\mathbf{E}(\mu)$ contains the open interval (0,1) (see [5], p. 319 and [6] p. 323). It is possible in this case to describe the set $\mathscr{L}_1$ in terms of characteristic functions. Namely, the set $\mathscr{L}_1$ coincides with the set of probability measures with the characteristic function φ of the form

$$\varphi(y) = \exp\left\{iay + \int_{-\infty}^{\infty} \left(e^{ixy} - 1 - \frac{ixy}{1+x^2}\right) \frac{1+x^2}{x^2}\, dM(x)\right\},$$

where $a \in R$ and M is a bounded monotone non-decreasing function such that on $(-\infty, 0)$ and $(0, \infty)$ its left and right derivatives, denoted invariably by $M'(x)$, exist and $\frac{1+x^2}{x} M'(x)$ do not increase.

Another characterization of $\mathscr{L}_1$ was given in [13]. Namely, I proved that a function φ is the characteristic function of a measure from $\mathscr{L}_1$ if and only if

$$\varphi(y) = \exp\left\{iay + \int_{-\infty}^{\infty} \left(\int_0^{xy} \frac{e^{iu}-1}{u}\, du - iy \arctan x\right) \frac{\nu(dx)}{\log(1+x^2)}\right\},$$

where $a \in R$, ν is a finite Borel measure on R and the integrand is defined as its limiting value $-\frac{1}{4}y^2$ when $x = 0$.

All that has been done so far in the multi-dimensional case is to describe limits of distributions of sequences

$$A_n(X_1 + X_2 + \ldots + X_n) + a_n,$$

where A_n is a multiple of the identity operator. By the same techniques as in the one-dimensional case, one finds a representation of characteristic functions (see [14]).

Our aim is to characterize all full Lévy's measures on R^N. Before proceeding to state and prove the main results of this paper we shall establish auxiliary propositions.

3. Norming sequences. We say that a norming sequence $\{A_n\}$ of operators from $\mathbf{Aut}\,R^N$ corresponds to a measure μ if there exist sequences $\{\mu_n\}$ and $\{a_n\}$ of elements of $\mathscr{P}(R^N)$ and R^N respectively, such that $A_n(\mu_1*\mu_2*\ldots*\mu_n)*\delta_{a_n}$ converges to μ and $A_n\mu_k$ $(k = 1, 2, \ldots, n;$ $n = 1, 2, \ldots)$ form a uniformly ininitesimal triangular array.

PROPOSITION 3.1. *For every norming sequence $\{A_n\}$ corresponding to a full measure the relation $\lim_{n\to\infty} A_n = 0$ holds.*

Proof. Suppose that $\{A_n\}$ corresponds to a full measure μ. Taking if necessary the symmetrization of the measures in question, we may assume that $a_n = 0$ $(n = 1, 2, \ldots)$, i.e. that the sequence $A_n(\mu_1*\mu_2*\ldots*\mu_n)$ converges to μ. Contrary to our statement let us suppose that there exists a subsequence of indices $n_1 < n_2 < \ldots$ for which

$$\lim_{k\to\infty} \|A_{n_k}\| > 0. \tag{3.1}$$

Let us choose vectors $z_n \in R^N$ with $\|z_n\| = 1$ and $\|A_n^*\| = \|A_n^* z_n\|$. Passing if necessary to a subsequence, we may assume that the sequence $u_k = \|A_{n_k}^*\|^{-1} A_{n_k}^* z_{n_k}$ converges to a vector $u \in R^N$. Of course, $\|u\| = 1$. From (3.1) it follows that the sequence $\|A_{n_k}^*\|^{-1} z_{n_k}$ is bounded. Since, by the assumption, $\lim_{n\to\infty} A_n\mu_j = \delta_0$ $(j = 1, 2, \ldots)$, we have the relation

$$\lim_{k\to\infty} \widehat{A_{n_k}\mu_j}(c\|A_{n_k}^*\|^{-1} z_{n_k}) = 1 \qquad (j = 1, 2, \ldots)$$

for every $c \in R$. Consequently, by the transformation rule of $\widehat{A_n\mu_j}$, we get the formula

$$\lim_{k\to\infty} \hat{\mu}_j(cu_k) = \hat{\mu}_j(cu) = 1 \qquad (j = 1, 2, \ldots)$$

for all $c \in R$. Introducing the notation $\nu_n = \mu_1*\mu_2*\ldots*\mu_n$ we have

$$\hat{\nu}_n(cu) = 1 \qquad (n = 1, 2, \ldots;\ c \in R). \tag{3.2}$$

We note that the vectors $y_n = (A_n^*)^{-1} u$ $(n = 1, 2, \ldots)$ are different from 0 because $\|u\| = 1$. Let v be a limit point of the sequence $\{\|y_n\|^{-1} y_n\}$, say $\lim_{k\to\infty} \|y_{m_k}\|^{-1} y_{m_k} = v$. Since $A_n\nu_n$ converges to μ, we have for all $c \in R$

$$\lim_{k\to\infty} A_{m_k}\nu_{m_k}(c\|y_{m_k}\|^{-1} y_{m_k}) = \hat{\mu}(cv).$$

On the other hand, by (3.2),

$$\widehat{A_n\nu_n}(c\|y_n\|^{-1} y_n) = \hat{\nu}_n(c\|y_n\|^{-1} u) = 1 \qquad (n = 1, 2, \ldots)$$

and, consequently, $\hat{\mu}(cv) = 1$ for all $c \in R$. Hence, by Proposition 1 in [12] (p. 52), it follows that the measure μ is not full. But this contradicts the assumption. The Proposition is thus proved.

PROPOSITION 3.2. *To every full Lévy's measure there corresponds a norming sequence $\{A_n\}$ with the property*

$$\lim_{n\to\infty} A_{n+1}A_n^{-1} = I.$$

Proof. Let μ be a full Lévy's measure. Suppose that a sequence $\{B_n\}$ corresponds to μ, i.e. $B_n(\mu_1*\mu_2*\ldots*\mu_n)*\delta_{b_n}$ converges to μ for some sequences $\{\mu_n\}$ and $\{b_n\}$. Setting $\nu_n = \mu_1*\mu_2*\ldots*\mu_n$ we have for some vectors c_n

$$B_{n+1}\nu_{n+1}*\delta_{b_{n+1}} = B_{n+1}B_n^{-1}(B_n\nu_n*\delta_{b_n})*B_{n+1}\mu_{n+1}*\delta_{c_n}. \tag{3.3}$$

Since the measures $B_n\mu_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ form a uniformly infinitesimal triangular array, we infer that the sequence $\{B_{n+1}\mu_{n+1}\}$ converges to δ_0. Consequently, from (3.3) and the Compactness Lemma in [12] (p. 55) it follows that the sequence $\{B_{n+1}B_n^{-1}\}$ is precompact in $\mathbf{Aut}\,R^N$. Moreover, for every its limit point J one can find a vector $c_J \in R^N$ such that $\mu = J\mu*\delta_{c_J}$. Consequently, $J \in A(\mu)$.

Let $\mathbf{T}$ be the set of all limit points of the sequence $\{B_{n+1}B_n^{-1}\}$. The set $\mathbf{A}(\mu)$, according to Proposition 1.2., is compact. The set $\mathbf{T}$ being a closed its subset is compact too. Consequently, for every interger n we can find an operator J_n in $\mathbf{T}$ such that

$$\varepsilon_n = \|J_n - B_{n+1}B_n^{-1}\| = \min\{\|J - B_{n+1}B_n^{-1}\|:\ J \in \mathbf{T}\}.$$

Obviously, $\lim_{n\to\infty} \varepsilon_n = 0$. Moreover, the operators J_n belong to $\mathbf{A}(\mu)$. Since, by the Proposition 1.2., $\mathbf{A}(\mu)$ is a group the operators H_n defined by the formulae $H_1 = I$, $H_n = J_1^{-1}J_2^{-1}\ldots J_{n-1}^{-1}$ $(n = 2, 3, \ldots)$ belong to $\mathbf{A}(\mu)$ too. Put $A_n = H_nB_n$ $(n = 1, 2, \ldots)$. It is clear that $A_n \in \mathbf{Aut}\,R^N$. Moreover,

$$A_{n+1}A_n^{-1} - I = H_{n+1}(B_{n+1}B_n^{-1} - J_n)H_n \qquad (n = 1, 2, \ldots). \tag{3.4}$$

Since the set $\mathbf{A}(\mu)$ is compact, all operators from $\mathbf{A}(\mu)$ have the norm bounded in common; say $\|A\| \leqslant c$ for all $A \in \mathbf{A}(\mu)$. Consequently, by (3.4),

$$\|A_{n+1}A_n^{-1} - I\| \leqslant \|H_{n+1}\|\,\|B_{n+1}B_n^{-1} - J_n\|\,\|H_n\| \leqslant c^2\varepsilon_n \qquad (n = 1, 2, \ldots)$$

which implies the relation

$$\lim_{n\to\infty} A_{n+1}A_n^{-1} = I.$$

It remains to prove that the sequence $\{A_n\}$ corresponds to the measure μ. We note that sequence $\{H_n\}$ contained in the compact set $\mathbf{A}(\mu)$ is precompact. Consequently, the sequence $\{H_n B_n \nu_n * \delta_{H_n b_n}\}$, i.e. the sequence $\{A_n \nu_n * \delta_{H_n b_n}\}$ is precompact in $\mathscr{P}(R^N)$. Moreover, its limit points are of the form $H\mu * \delta_a$ where H is a limit point of the sequence $\{H_n\}$. Since $H \in \mathbf{A}(\mu)$, we have the equation

$$H\mu * \delta_a = \mu * \delta_b$$

where $b \in R^N$. Hence it follows that we can choose a sequence $\{a_n\}$ of vectors such that the sequence $\{A_n \nu_n * \delta_{a_n}\}$ converges to μ. Thus the sequence $\{A_n\}$ corresponds to μ which completes the proof.

PROPOSITION 3.3. *Let $n_k \leqslant m_k$ $(k = 1, 2, \ldots)$ and $n_k \to \infty$. For every norming sequence $\{A_n\}$ corresponding to a full measure μ the sequence $\{A_{m_k} A_{n_k}^{-1}\}$ is precompact in* $\mathbf{End}\, R^N$. *Moreover, all its limit points belong to* $\mathbf{E}(\mu)$.

Proof. Suppose that

$$\lim_{n\to\infty} A_n \nu_n * \delta_{a_n} = \mu, \tag{3.5}$$

where $\nu_n = \mu_1 * \mu_2 * \ldots * \mu_n$ and $\{\mu_n\}$, $\{a_n\}$ are suitably chosen sequences from $\mathscr{P}(R^N)$ and R^N respectively. For simplicity of notation we put

$$C_k = A_{m_k} A_{n_k}^{-1} \quad (k = 1, 2, \ldots).$$

Then we have the equation

$$A_{m_k} \nu_{m_k} * \delta_{a_{m_k}} = C_k (A_{n_k} \nu_{n_k} * \delta_{a_{n_k}}) * \omega_k, \tag{3.6}$$

where ω_k is a probability measure. The symmetrization of (3.6) yields the formula

$$A_{m_k}^{\circ} \nu_{m_k} = C_k A_{n_k}^{\circ} \nu_{n_k} * {}^{\circ}\omega_k.$$

Hence, by virtue of Theorem 2.2 in [10] (p. 59), we get the precompactness of the sequence $\{C_k A_{n_k} {}^{\circ}\nu_{n_k}\}$. Passing if necessary to a subsequence we may assume that the last sequence is convergent to a probability measure, say λ. Thus

$$\lim_{k\to\infty} \widehat{C_k A_{n_k} {}^{\circ}\nu_{n_k}}(y) = \hat{\lambda}(y) \tag{3.7}$$

uniformly on every compact subset of R^N.

First we shall prove that the sequence $\{C_k\}$ is precompact in $\mathbf{End}\, R^N$. To prove this it suffices to prove that the sequence of norms $\{\|C_k\|\}$ is bounded. Contrary to this let us suppose that the sequence of norms is unbounded. Of course, we may assume, without loss of generality, that $\|C_k\| \to \infty$. Let us choose vectors x_k in R^N such that $\|x_k\| = 1$ and $\|C_k^*\| = \|C_k^* x_k\|$ $(k = 1, 2, \ldots)$. Passing to a subsequence, if necessary, it may be assumed that the sequence of vectors $u_k = \|C_k^*\|^{-1} C_k^* x_k$ tends to a vector $u \in R^N$ with $\|u\| = 1$. Thus, by (3.5),

$$\lim_{k\to\infty} \widehat{A_{n_k} {}^{\circ}\nu_{n_k}}(cu_k) = {}^{\circ}\hat{\mu}(cu) \tag{3.8}$$

for all $c \in R$. Since $\|C_k^*\|^{-1} x_k \to 0$, we have, by (3.7),

$$\lim_{k\to\infty} \widehat{C_k A_{n_k} {}^{\circ}\nu_{n_k}}(c\|C_k^*\|^{-1} x_k) = \hat{\lambda}(0) = 1$$

for all $c \in R$. But the last formula can be written in the form

$$\lim_{k\to\infty} \widehat{A_{n_k} {}^{\circ}\nu_{n_k}}(cu_k) = 1 \quad (c \in R).$$

Comparying it with (3.8) we get the equation ${}^{\circ}\hat{\mu}(cu) = 1$ for all $c \in R$. Hence it follows that μ is not a full measure (see [12] p. 52, Proposition 1). The contradiction shows that the sequence $\{C_k\}$ is precompact in $\mathbf{End}\, R^N$.

Let A be a limit point of $\{C_k\}$. Without loss of generality we may assume that the sequence itself tends to A. From (3.5) and (3.6), by virtue of Theorem 2.2 in [10], we get the precompactness of the sequence $\{\omega_k * \delta_{b_k}\}$ for suitably chosen vectors $b_k \in R^N$. Without loss of generality we may assume that the last sequence is convergent. By (3.5) and (3.6) it is easy to show that the sequence $\{b_k\}$ is convergent to 0. For instance one can apply the Compactness Lemma in ([12], p. 55). Thus the sequence $\{\omega_k\}$ is convergent to a probability measure which will be denoted by μ_A. Finally, from (3.5) and (3.6) we get the equation $\mu = A\mu * \mu_A$ which shows that $A \in \mathbf{E}(\mu)$.

4. Decomposability properties of Lévy's measures. Let J be a non-zero idempotent in $\mathbf{End}\, R^N$, i.e. a projector from R^N onto $J(R^N)$. For every operator A in $\mathbf{End}\, R^N$ by $\det_J A$ we shall denote the determinant of the matrix representation of the operator JA in $J(R^N)$ relatively to an orthonormal basis of $J(R^N)$. It is easy to prove the following formulae

$$\det_J A = \det_J JA = \det_J AJ = \det_J JAJ, \tag{4.1}$$

$$\det_J (AJB) = \det_J A \det_J B. \tag{4.2}$$

Moreover, if the projectors J_1 and J_2 satisfy the conditions $J_1 J_2 = 0$, then for all $A, B \in \mathbf{End}\, R^N$ we have the equation

$$\det_{J_1+J_2}(J_1 A J_1 + J_2 B J_2) = \det_{J_1} A \det_{J_2} B. \tag{4.3}$$

LEMMA 4.1. *Let μ be a full Lévy's measure and let J be a non-zero idempotent from $\mathbf{E}(\mu)$. Then for every number c satisfying the condition $0 < c < 1$ there exists an operator B_c in $\mathbf{E}(\mu)$ such that $\det_J B_c = c$.*

Proof. Let $\{A_n\}$ be a norming sequence corresponding to μ. By Proposition 3.2 we may assume that

$$\lim_{n\to\infty} A_{n+1}A_n^{-1} = I. \tag{4.4}$$

Put $b_{mn} = \det_J A_m A_n^{-1}$ $(n \leqslant m)$. Obviously,

$$b_{nn} = 1 \quad (n = 1, 2, \ldots). \tag{4.5}$$

Moreover, by Proposition 3.1,

$$\lim_{m\to\infty} b_{mn} = 0 \quad (n = 1, 2, \ldots). \tag{4.6}$$

By Proposition 3.3. the set $\{A_m A_n^{-1};\ m \geqslant n,\ n = 1, 2, \ldots\}$ is precompact in $\mathbf{End}\, R^N$. Consequently, all its elements have the norm bounded in common, say by a number d. Thus

$$\|A_{m+1}A_n^{-1} - A_m A_n^{-1}\| \leqslant \|A_{m+1}A_m^{-1} - I\|\,\|A_m A_n^{-1}\| \leqslant d\,\|A_{m+1}A_m^{-1} - I\|.$$

Consequently, by (4.4),

$$\lim_{n\to\infty} \sup_{m\geqslant n} \|A_{m+1}A_n^{-1} - A_m A_n^{-1}\| = 0.$$

Hence we get the relation

$$\lim_{n\to\infty} \sup_{m\geqslant n} |b_{m+1,n} - b_{mn}| = 0. \tag{4.7}$$

Given a number c satisfying the condition $0 < c < 1$, we can find, by virtue of (4.5) and (4.6), an index $m_n \geqslant n$ such that $b_{m_n,n} \geqslant c$ and $b_{m_n+1,n} < c$ $(n = 1, 2, \ldots)$. From (4.7) it follows that

$$\lim_{n\to\infty} b_{m_n,n} = c. \tag{4.8}$$

By Proposition 3.3 the sequence $\{A_{m_n}A_n^{-1}\}$ is precompact in $\mathbf{End}\, R^N$. Let B_c be its limit point. By the same Proposition we infer that $B_c \in \mathbf{E}(\mu)$. Finally, by (4.8), $\det_J B_c = c$ which completes the proof.

Lemma 4.2. *Let μ be a full Lévy's measure and let J be a non-zero idempotent from $\mathbf{E}(\mu)$. There exists then a sequence $\{S_n\}$ of operators from $\mathbf{E}(\mu)$ which converges to J and satisfies the conditions $JS_n = S_nJ = S_n$ $(n = 1, 2, \ldots)$ and*

$$\lim_{k\to\infty} S_n^k = 0 \quad (n = 1, 2, \ldots).$$

Proof. We shall prove the Lemma by induction with respect to the dimension of the subspace $J(R^N)$.

First consider the case $\dim J(R^N) = 1$. Let us choose, by virtue of Lemma 4.1, operations C_n from $\mathbf{E}(\mu)$ for which $\det_J C_n = 1 - \frac{1}{n}$ $(n = 1, 2, \ldots)$. Put $S_n = JC_nJ$. Since the subspace $J(R^N)$ is one-dimensional, the operator S_n is a multiple of the operator J. Moreover, by (4.1), $\det_J S_n = 1 - \frac{1}{n}$. Thus $S_n = \left(1 - \frac{1}{n}\right) J$. Now it is obvious that the operators S_n fulfil the conditions of the Lemma.

Suppose now that $\dim J(R^N) = d > 1$ and for all projectors K belonging to $\mathbf{E}(\mu)$ for which $K(R^N)$ is of dimension less than d the Lemma is true.

First suppose that there exists a non-zero idempotent L in $\mathbf{E}(\mu)$ different from J and satisfying the condition

$$L = JL = LJ. \tag{4.9}$$

In other words L maps R^N into a proper subspace of $J(R^N)$. By (4.9) the operator $J - L$ is an idempotent. Moreover, by Proposition 1.5, $I - L \in \mathbf{E}(\mu)$. Consequently, by the equation $J(I-L) = J - L$, the idempotent $J - L$ belongs to $\mathbf{E}(\mu)$. We note that both subspaces $L(R^N)$ and $(J-L)(R^N)$ have the dimension less than d. Consequently, by the induction assumption we can find two sequences $\{U_n\}$ and $\{V_n\}$ in $\mathbf{E}(\mu)$ which converge to L and $J - L$ respectively and for every n satisfy the conditions $LU_n = U_nL = U_n$, $(J-L)V_n = V_n(J-L) = V_n$ and $\lim_{k\to\infty} U_n^k = \lim_{k\to\infty} V_n^k = 0$. Setting $S_n = U_n + V_n$, we infer that the sequence $\{S_n\}$ converges to J. Further, from the equation $S_n = LU_nL + (I-L)V_n(I-L)$ and Proposition 1.6 we obtain the relation $S_n \in \mathbf{E}(\mu)$. Moreover, by (4.9), $JS_n = S_nJ = S_n$. Since $S_n^k = U_n^k + V_n^k$, we finally have the equation $\lim_{k\to\infty} S_n^k = 0$. Thus the sequence $\{S_n\}$ fulfils the conditions of the Lemma.

It remains to consider the case when $\mathbf{E}(\mu)$ does not contain non-zero idempotents L different from J and satisfying (4.9). By Lemma 4.1 we can find operations D_n from $\mathbf{E}(\mu)$ such that

$$0 < \det_J D_n < 1 \tag{4.10}$$

and

$$\lim_{n\to\infty} \det_J D_n = 1. \tag{4.11}$$

Moreover, by (4.1), we may assume that

$$JD_n = D_nJ = D_n, \tag{4.12}$$

and, by the compactness of $\mathbf{E}(\mu)$ (see Proposition 1.1) that the sequence $\{D_n\}$ converges to an operator D in $\mathbf{E}(\mu)$. Obviously,

$$JD = DJ = D \tag{4.13}$$

and

$$\det_J D = 1. \tag{4.14}$$

Put $A = D+I-J$. By Proposition 1.6, $A \in \mathbf{E}(\mu)$ and, by (4.3) and (4.14) $\det A = \det_J D \det_{I-J} I = 1$. Hence and from Proposition 1.4 it follows that $A \in \mathbf{A}(\mu)$. By Proposition 1.2 $\mathbf{A}(\mu)$ is a compact group. Consequently, there exists a sequence $r_1 < r_2 < \ldots$ of indices such that $\{A^{r_n}\}$ converges to the identity operator I (see [9], p. 109). Since $JA^n = D^n$, the sequence $\{D^{r_n}\}$ converges to J. Consequently, we can find a sequence $k_1 < k_2 < \ldots$ of indices such that $D^{r_n}_{k_n} \to J$. Setting $S_n = D^{r_n}_{k_n}$ $(n = 1, 2, \ldots)$, we get a sequence from $E(\mu)$ convergent to J. Moreover, by (4.12), $JS_n = S_n J = S_n$ and, by (4.10), $\det_J S_n < 1$. From the compactness of $\mathbf{E}(\mu)$ it follows that for every n the sequence $\{S_n^k\}$ is precompact in $\mathbf{E}(\mu)$. Consequently, the limit points of this sequence form a group (see Numakura Theorem, [9], p. 109). The unit L of this group is an idempotent belonging to $\mathbf{E}(\mu)$ and satisfying the equation $L = JL = LJ$. Taking into account formula (4.2), we infer that $\det_J L = 0$. Consequently, $L \neq J$. We have assumed that $\mathbf{E}(\mu)$ does not contain non-zero idempotents different from J and satisfying (4.9). Thus $L = 0$ and, consequently, the group of all limit points of $\{S_n^k\}$ is the one-element group $\{0\}$. In other words, $\lim_{k\to\infty} S_n^k = 0$ for all n and the sequence $\{S_n\}$ fulfils the conditions of the Lemma which completes the proof.

5. A characterization of full Lévy's measures. The aim of this section is a characterization of full Lévy's measures in terms of operator-decomposability.

PROPOSITION 5.1. *Let μ be a full Lévy's measure. Then the set $\mathbf{E}(\mu)$ contains a one-parameter semigroup* $\exp tQ$ $(t \geqslant 0)$ *with the property* $\lim_{\to\infty} \exp tQ = 0$.

Proof. By Propositions 1.5 and 1.6 the identity operator I can be written in the form $I = J_1 + J_2 + \ldots + J_q$, where J_s are non-zero idempotents from $\mathbf{E}(\mu)$, $J_r J_s = J_s J_r = 0$ for $r \neq s$ and for every s there is no non-zero idempotent K in $\mathbf{E}(\mu)$ different from J_s and satisfying the condition $KJ_s = J_s K = K$. By consecutive application of Proposition 1.6 we conclude that $\sum_{s=1}^{q} J_s A_s J_s \in \mathbf{E}(\mu)$ whenever $A_1, A_2, \ldots, A_q \in \mathbf{E}(\mu)$. By Lemma 4.2 for every r $(1 \leqslant r \leqslant q)$ we can find a sequence $\{S_{n,r}\}$ of operators from $\mathbf{E}(\mu)$ satisfying the conditions

(5.1) $$J_r S_{n,r} = S_{n,r} J_r = S_{n,r} \quad (n = 1, 2, \ldots),$$

(5.2) $$\lim_{n\to\infty} S_{n,r} = J_r$$

and

(5.3) $$\lim_{k\to\infty} S_{n,r}^k = 0 \quad (n = 1, 2, \ldots).$$

Moreover, by (5.2) and (5.3), we may assume that

(5.4) $$0 < \det_{J_r} S_{n,r} < 1 \quad (n = 1, 2, \ldots).$$

Put

(5.5) $$c(n, r) = [-(\log \det_{J_r} S_{n,r})^{-1}],$$

where square brackets denote the integral part. Let W be the set of all non-negative rational numbers. By the Proposition 1.1, i.e. by the compactness of $\mathbf{E}(\mu)$, the sequences $\{S_{n,r}^{[c(n,r)w]}\}$ $(w \in W,\ r = 1, 2, \ldots, q)$ are precompact in $\mathbf{E}(\mu)$. Passing, if necessary, to subsequences we may assume, without loss of generality, that all these sequences are convergent. Put

$$\lim_{n\to\infty} \sum_{r=1}^{q} S_{n,r}^{[c(n,r)w]} = B_w \quad (w \in W).$$

Since $\sum_{r=1}^{q} S_{n,r}^{[c(n,r)w]} = \sum_{r=1}^{q} J_r S_{n,r}^{[c(n,r)w]} J_r$, we infer that $B_w \in \mathbf{E}(\mu)$ for $w \in W$. Moreover, by (5.5),

(5.6) $$\det_{J_r} B_w = \lim_{n\to\infty} (\det_{J_r} S_{n,r})^{[c(n,r)w]} = e^{-w} \quad (r = 1, 2, \ldots, q).$$

Hence, by (4.3), we get the formula

(5.7) $$\det B_w = e^{-qw} \quad (w \in W).$$

Consequently, $B_w \in \mathbf{Aut}\, R^N$. Moreover, it is easy to verify the equation

(5.8) $$B_{u+w} = B_u B_w \quad (u, w \in W).$$

Consequently, the set $\mathbf{H} = \{B_w : w \in W\}$ is a subsemigroup of the group $\mathbf{Aut}\, R^N$. Let us introduce the notation $\mathbf{H}^{-1} = \{B_w^{-1} : w \in W\}$. To prove that the union $\mathbf{H} \cup \mathbf{H}^{-1}$ is a group it suffices to prove that for every pair $u, w \in W$ $B_u B_w^{-1}$ or $B_w B_u^{-1}$ belongs to $\mathbf{H}$. By symmetry we may assume that $w \geqslant u$. Then, by (5.8), $B_w B_u^{-1} = B_{w-u} B_u B_u^{-1} = B_{w-u} \in \mathbf{H}$. Let $\mathbf{S}$ be the closure of $\mathbf{H}$ in $\mathbf{Aut}\, R^N$. It is clear that $\mathbf{S} \subset \mathbf{E}(\mu)$ and $\mathbf{G} = \mathbf{S} \cup \mathbf{S}^{-1}$ is a closed subgroup of the group $\mathbf{Aut}\, R^N$. Moreover, by (5.6),

(5.9) $$\det_{J_1} A = \det_{J_2} A = \ldots = \det_{J_q} A \quad (A \in \mathbf{S}),$$

and, by (5.7),

(5.10) $$0 < \det A \leqslant 1 \quad (A \in \mathbf{S}).$$

Since, by (5.8), $B_0 = I$, the set $\mathbf{S}_0 = \mathbf{S} \cap \mathbf{A}(\mu)$ is non-void. Moreover, being a closed subsemigroup of the compact group $\mathbf{A}(\mu)$ it is a compact group (see [9] p. 23). From (5.10) we obtain the equation

(5.11) $$\det A = 1 \quad \text{for } A \in \mathbf{S}_0.$$

The mapping $h(A) = \log\det A$ is a homomorphism of the topological group $\mathbf{G}$ onto the additive group R. We shall prove that $\mathbf{S}_0$ is the kernel of this homomorphism. By (5.11), $\mathbf{S}_0$ is contained in the kernel of h. Suppose that $A \in \mathbf{G}$ and $\det A = 1$. Of course, A or A^{-1} belongs to $\mathbf{S}$. Without loss of generality we may assume that $A \in \mathbf{S}$. Consequently, $A \in \mathbf{E}(\mu)$ and, by Proposition 1.4, $A \in \mathbf{A}(\mu)$ which implies the relation $A \in \mathbf{S}_0$. Thus $\mathbf{S}_0$ is the kernel of h. Hence it follows that the factor group $\mathbf{G}/\mathbf{S}_0$ is isomorphic to R. Since the group $\mathbf{G}$ is commutative and compactly generated, we infer, by Pontrjagin Theorem ([7], p. 187; [15], § 29), that $\mathbf{G}$ is isomorphic to the direct sum of R and $\mathbf{S}_0$. Let g: $\mathbf{G} \to R \times \mathbf{S}_0$ be such isomorphism. Since $R \times \mathbf{S}_0 = g(\mathbf{G}) = g(\mathbf{S}) \cup g(\mathbf{S})^{-1}$ and $g(\mathbf{S})$ is a closed semigroup, we infer that either $g(\mathbf{S}) = R^+ \times \mathbf{S}_0$ or $g(\mathbf{S}) = R^- \times \mathbf{S}_0$, where R^+ and R^- denote the right and left half-lines respectively. For $t \geqslant 0$ we put $T_t = g^{-1}(\langle t, I\rangle)$ in the first case and $T_t = g^{-1}(\langle -t, I\rangle)$ in the remaining one, where I is the unit of $\mathbf{S}_0$. It is clear that $T_t (t \geqslant 0)$ is a continuous one-parameter semigroup of operators from $\mathbf{S}$ satisfying the condition $\lim_{t\to 0} T_t = I$. By Theorem 8.4.2 in [3] it can be represented in an exponential form $T_t = \exp tQ$ $(t \geqslant 0)$. Moreover, $T_t \notin \mathbf{S}_0$ for $t > 0$. Consequently, by (5.10) and (5.11),

$$0 < \det T_t < 1 \quad \text{for } t > 0. \tag{5.12}$$

From the definition of the operators $S_{n,r}$, B_w and the semigroup $\mathbf{S}$ it follows that the idempotents $J_1, J_2, \ldots, J_q$ commute with the elements of $\mathbf{S}$. We note that the semigroup $T_t (t \geqslant 0)$ is precompact in $\mathbf{E}(\mu)$. Consequently, to prove the relation $\lim_{t\to\infty} T_t = 0$ it suffices to prove that 0 is a limit point of this semigroup. It is well-known that the set of limit points of the semigroup T_t $(t \geqslant 0)$ contains an idempotent K (see [9], p. 109). By (5.12), we have the equation $\det K = 0$. Consequently, by (5.9) and (4.3),

$$\det_{J_r} K = 0 \quad (r = 1, 2, \ldots, q). \tag{5.13}$$

Since J_r commutes with K, the operator $J_r K$ is an idempotent in $\mathbf{E}(\mu)$. Taking into account (4.1) and (5.13), we have the inequality $J_r K \neq J_r$. On the other hand $J_r(J_r K) = (J_r K) J_r = J_r K$, which by the definition of the idempotents $J_1, J_2, \ldots, J_q$ yields the equation $J_r K = 0$. Thus $K = J_1 K + J_2 K + \ldots + J_q K = 0$ and, consequently, 0 is a limit point of the semigroup $T_t (t \geqslant 0)$. The Proposition is thus proved.

PROPOSITION 5.2. *Suppose that a one-parameter semigroup* $\exp tQ$ $(t \geqslant 0)$ *fulfils the condition* $\lim_{t\to\infty} \exp tQ = 0$. *Then each* $\exp tQ$*-decomposable for* $t \geqslant 0$ *probability measure* μ *is a Lévy's measure. Moreover, for every* $t \geqslant 0$ $\mu = \exp tQ\mu * \lambda_t$, *where* λ_t *is an infinitely divisible measure.*

Proof. Setting $B_n = \exp \frac{1}{n} Q$ $(n = 1, 2, \ldots)$ we have the formula

$$\mu = B_n \mu * \mu_{B_n}, \tag{5.14}$$

where $\mu_{B_n} \in \mathscr{P}(R^N)$. It is easy to verify the relation

$$\lim_{n\to\infty} \mu_{B_n} = \delta_0. \tag{5.15}$$

Moreover, the operators B_n satisfy conditions of the Proposition 1.7. Thus

$$\hat{\mu}(y) \neq 0 \quad \text{for all} \quad y \in R^N. \tag{5.16}$$

Put

$$A_n = \prod_{k=1}^{n} B_k = \exp \sum_{k=1}^{n} \frac{1}{k} Q \quad (n = 1, 2, \ldots) \tag{5.17}$$

and

$$\mu_1 = A_1^{-1}\mu, \quad \mu_k = A_k^{-1}\mu_{B_k} \quad (k = 2, 3, \ldots). \tag{5.18}$$

From (5.17) it follows that the set $\{A_n A_k^{-1}\colon k = 1, 2, \ldots, n; n = 1, 2, \ldots\}$ is precompact in $\mathbf{End} R^N$. Moreover, for every k $\lim_{n\to\infty} A_n A_k^{-1} = 0$. Consequently, by (5.15), $\{A_n A_k^{-1} \mu_{B_k}\}$ $(k = 1, 2, \ldots, n; n = 1, 2, \ldots)$ form a uniformly infinitesimal triangular array. Consequently, by (5.18) $\{A_n \mu_k\}$ $(k = 1, 2, \ldots, n; n = 1, 2, \ldots)$ are uniformly infinitesimal too.

From (5.14) and (5.18), by virtue of (5.16), we get the formulae

$$\mu_1(y) = \hat{\mu}((A_1^{-1})^* y),$$

$$\hat{\mu}_k(y) = \hat{\mu}_{B_k}((A_k^{-1})^* y) = \hat{\mu}((A_k^{-1})^* y)/\hat{\mu}((A_{k-1}^{-1})^* y) \quad (k = 2, 3, \ldots).$$

Thus

$$\overline{A_n(\mu_1 * \mu_2 * \ldots * \mu_n)}(y) = \prod_{k=1}^{n} \hat{\mu}_k(A_n^* y) = \hat{\mu}(y)$$

and, consequently,

$$A_n(\mu_1 * \mu_2 * \ldots * \mu_n) = \mu \quad (n = 1, 2, \ldots). \tag{5.19}$$

which shows that μ is a Lévy's measure.

Given $t \geqslant 0$, we can choose a sequence of integers $k_n > n$ such that $\lim_{n\to\infty} \sum_{k=n+1}^{k_n} \frac{1}{k} = t$. Then, by (5.17),

$$\lim_{n\to\infty} A_{k_n} A_n^{-1} = \exp tQ.$$

Further, by (5.19),

$$\mu = A_{k_n} A_n^{-1} \mu * A_{k_n}(\mu_{n+1} * \mu_{n+2} * \ldots * \mu_{k_n}).$$

The characteristic functions of the measures in question are, according to (5.16), different from 0 everywhere on R^N. Thus the last equation yields the existence of the limit

$$\lambda_t = \lim_{n\to\infty} A_{k_n}(\mu_{n+1}*\mu_{n+2}*\ldots*\mu_{k_n})$$

and the equation $\mu = \exp t Q\mu * \lambda_t$. Since $\{A_{k_n}\mu_j\}$ $(j = n+1, n+2, \ldots, k_n;$ $n = 1, 2, \ldots)$ are uniformly infinitesimal, the limit distribution λ_t is infinitely divisible (see [10], p. 52) which completes the proof.

The class of infinitesimal generators Q which can occur in Propositions 5.1 and 5.2 is closed under similarity transformations and is simply describable through spectral properties. Namely, $\lim_{t\to\infty} \exp tQ = 0$ if and only if all eigenvalues of Q have negative real part. As a consequence of Propositions 5.1 and 5.2 we get a characterization of full Lévy's measures.

THEOREM 5.1. *A full probability measure on R^N is a Lévy's measure if and only if it is $\exp tQ$-decomposable for $t \geqslant 0$ where Q is an operator whose all eigenvalues have negative real part.*

6. An extreme point method. Our next aim is to give a representation of the characteristic functions of $\exp tQ$-decomposable for $t \geqslant 0$ measures in R^N. By Proposition 5.2 all such measures are Lévy's measures and, consequently, are infinitely divisible. The method of proof consists in finding the extreme points of a certain convex set formed by Khintchine measures corresponding to $\exp tQ$-decomposable measures. Once the extreme points are found one can apply a Theorem by Choquet on representation of the points of a compact convex set as barycentres of the extreme points.

First we introduce some auxiliary spaces. Let Q be an operator on R^N whose eigenvalues have negative real part. Let S^m be the m-dimensional unit sphere and $\bar{R}$ the compactified real line: $\bar{R} = R \cup \{-\infty\} \cup \{\infty\}$. Put $H^N = S^{N-1} \times \bar{R}$. Obviously, the space H^N is compact.

We define a congruence relation in H^N as follows: $\langle x, t\rangle \sim \langle y, u\rangle$ where $x, y \in S^{N-1}$ and $t, u \in \bar{R}$ if and only if there exists a real number s such that $\exp sQx = y$ and $u = t+s$. Suppose that $\langle x_n, t_n\rangle \sim \langle y_n, u_n\rangle$ $(n = 1, 2, \ldots)$ and the sequences $\{\langle x_n, t_n\rangle\}$ and $\{\langle y_n, u_n\rangle\}$ converge to $\langle x, t\rangle$ and $\langle y, u\rangle$ respectively. Then for some real numbers s_n $\exp s_n Q x_n = y_n$ $(n = 1, 2, \ldots)$. Since all eigenvalues of Q have negative real part, the last equations and the compactness of S^{N-1} imply that the sequence $\{s_n\}$ is bounded. If s is its limit point, then $\exp sQx = y$ and $u = t+s$. Thus $\langle x, t\rangle \sim \langle y, u\rangle$ and, consequently, the quotient space $H^N/\sim$ denoted by M^N is compact (see [1], p. 97).

The element of M^N, i.e. the equivalence class containing $\langle x, t\rangle$ from H^N will be denoted by $[x, t]$. We define a one-parameter group T_s $(s \in R)$ of transformations of M^N by assuming

$$T_s[x, t] = [x, s+t]. \tag{6.1}$$

Further, for every element $[x, t] \in M^N$ we put

$$|[x, t]| = \|\exp tQx\| \text{ if } t \in R, \quad |[x, \infty]| = 0 \text{ and } |[x, -\infty]| = \infty. \tag{6.2}$$

Since $\lim_{t\to\infty} \exp tQ = 0$ and for every $z \in R^N \setminus \{0\}$ $\lim_{t\to -\infty} \|\exp tQz\| = \infty$, each element $z \in R^N \setminus \{0\}$ can be represented in the form $z = \exp tQx$, where $x \in S^{N-1}$ and $t \in R$. In general this representation is not unique. But $z = \exp uQy$, where $y \in S^{N-1}$ and $u \in R$ if and only if $\langle x, t\rangle \sim \langle y, u\rangle$. Thus the mapping

$$\pi(\exp tQx) = [x, t] \quad (x \in S^{N-1}, t \in R) \tag{6.3}$$

is an embedding of $R^N \setminus \{0\}$ into M^N. Obviously,

$$\|y\| = |\pi(y)| \tag{6.4}$$

and

$$\pi(\exp sQy) = T_s \pi(y) \tag{6.5}$$

for all $y \in R^N \setminus \{0\}$ and $s \in R$.

We say that a subset E of M^N is *bounded from below* if $\inf\{|a|: a \in E\} > 0$. Let λ be a finite Borel measure on M^N. For any Borel subset E of M^N bounded from below we put

$$I_\lambda(E) = \int_E (1+|u|^{-2})\lambda(du), \tag{6.6}$$

where the integrand is assumed to be 1 if $|u| = \infty$.

Let $\mathscr{M}$ be the set of all finite Borel measures λ on M^N satisfying the condition

$$I_\lambda(E) - T_t I_\lambda(E) \geqslant 0 \tag{6.7}$$

for all $t \geqslant 0$ and all Borel subsets E bounded from below. It is clear that the set $\mathscr{M}$ is convex. Let $\mathscr{K}$ be the subset of $\mathscr{M}$ consisting of probability measures. The set $\mathscr{K}$ is convex and compact.

Suppose that a Borel subset F of M^N is T_t-invariant for all $t \in R$ and $\lambda \in \mathscr{M}$. Then the restriction $\lambda|F$ belongs to $\mathscr{M}$ too because of the equation

$$I_{\lambda|F}(E) - T_t I_{\lambda|F}(E) = I_\lambda(E\cap F) - T_t I_\lambda(E \cap E).$$

Hence it follows that the extreme points of the set $\mathscr{K}$ are measures concentrated on orbits of elements of M^N. In other words, we have the following proposition:

PROPOSITION 6.1. *The extreme points of $\mathscr{K}$ are measures concentrated on one of the following sets*: $\{[x, -\infty]\}$, $\{[x, \infty]\}$, $\{[x, t]: t \in R\}$ *where* $x \in S^{N-1}$.

We proceed now to the investigation of extreme points of $\mathscr{K}$ concentrated on the set $F_x = \{[x, t]: t \in R\}$. Let λ be a probability measure concentrated on F_x. Put

$$J_\lambda(u) = I_\lambda(\{[x, t]: t < u\}) \quad (u \in R). \tag{6.8}$$

It is easy to verify that $\lambda \in \mathscr{K}$ if and only if the inequality (6.7) holds for all $t \geqslant 0$ and all subsets E of the form $\{[x, t]: a \leqslant t < b\}$, where $a < b$ and $a, b \in R$. Taking into account the formulae

$$I_\lambda(\{[x, t]: a \leqslant t < b\}) = J_\lambda(b) - J_\lambda(a),$$

$$T_s I_\lambda(\{[x, t]: a \leqslant t < b\}) = I_\lambda(\{[x, t]: a - s \leqslant t < b - s\})$$

we infer that $\lambda \in \mathscr{K}$ if and only if for every triplet $a, b, t \in R$ satisfying the conditions $a < b$ and $t \geqslant 0$ the inequality

$$J_\lambda(b) - J_\lambda(a) - J_\lambda(b-t) + J_\lambda(a-t) \geqslant 0 \tag{6.9}$$

is fulfilled.

Now we shall give more convenient description in terms of the function J_λ of measures λ from $\mathscr{K}$. Let f be a continuous bounded function on F_x. By (6.6) and (6.8) we have the formula

$$\int_{F_x} f(z)\lambda(dz) = \int_{-\infty}^{\infty} f([x, u]) \frac{|[x, u]|^2}{1 + |[x, u]|^2} dJ_\lambda(u). \tag{6.10}$$

Substituting $b = a + t$ into (6.9) we get the inequality

$$J_\lambda(a) \leqslant \tfrac{1}{2}\big(J_\lambda(a-t) + J_\lambda(a+t)\big)$$

for all $a \in R$ and $t \geqslant 0$. Thus the function J_λ is convex. Moreover, by (6.8), it is also monotone non-decreasing with $J_\lambda(-\infty) = 0$. Consequently,

$$J_\lambda(t) = \int_{-\infty}^{t} q_\lambda(u)\,du \quad (t \in R),$$

where the function q_λ is non-negative and monotone non-decreasing. Of course, we may assume that q_λ is continuous from the left. In this case the function q_λ is uniquely determined by λ. Further, by (6.10), we have the equation

$$\int_{-\infty}^{\infty} \frac{|[x, u]|^2}{1 + |[x, u]|^2} q_\lambda(u)\,du = 1.$$

Suppose now that we have a non-negative monotone non-decreasing function q on R satisfying the condition

$$\int_{-\infty}^{\infty} \frac{|[x, u]|^2}{1 + |[x, u]|^2} q(u)\,du = 1. \tag{6.11}$$

We define a measure λ on F_x by means of the formula

$$\int_{F_x} f(z)\lambda(dz) = \int_{-\infty}^{\infty} f([x, u]) \frac{|[x, u]|^2}{1 + |[x, u]|^2} q(u)\,du, \tag{6.12}$$

for any bounded continuous function f on F_x. It is obvious, by (6.11) that λ is a probability measure on F_x. Moreover, $J_\lambda(t) = \int_{-\infty}^{t} q(u)\,du$. Since q is monotone non-decreasing, the function J_λ fulfils the condition (6.9). Consequently, $\lambda \in \mathscr{K}$. Thus we proved the following proposition.

PROPOSITION 6.2. *Equation* (6.12) *defines a one-to-one correspondence between all measures λ from $\mathscr{K}$ concentrated on F_x and all non-negative monotone non-decreasing continuous from the left functions q on R satisfying the condition* (6.11).

In the sequel we shall use the following Lemma.

LEMMA 6.1. *For every $a \in R$ and $x \in R^N$ the integral*

$$P_a(x) = \int_{a}^{\infty} \frac{\|\exp tQx\|^2}{1 + \|\exp tQx\|^2} dt$$

is finite. Moreover, for every $a \in R$ there exist positive constants b_1 and b_2 such that for all $x \in R^N$ the inequality

$$b_1 \log(1 + \|x\|^2) \leqslant P_a(x) \leqslant b_2 \log(1 + \|x\|^2)$$

is true.

Proof. We assumed that all eigenvalues, say $a_1, a_2, \ldots, a_p$, of Q have negative real part. Consequently, for $0 > b > \operatorname{Re} a_j$ and $c < \operatorname{Re} a_j$ $(j = 1, 2, \ldots, p)$ we have the relations

$$\lim_{t \to \infty} e^{-bt} \exp tQ = 0 = \lim_{t \to -\infty} e^{-ct} xp\, tQ.$$

Hence we get the inequalities

$$\sup_{t \geqslant a} \|e^{-bt} \exp tQ\| = c_1 < \infty$$

and

$$\sup_{t \leqslant -a} \|e^{-ct} \exp tQ\| = c_2 < \infty.$$

Thus for $x \in R^N$ and $t \geqslant a$

$$\|\exp tQx\| \leqslant c_1 e^{bt} \|x\|$$

and

$$e^{ct} \|x\| = \|e^{ct} \exp(-tQ)(\exp tQx)\| \leqslant c_2 \|\exp tQx\|.$$

Since the function $\frac{t^2}{1+t^2}$ is monotone non-decreasing on the right half-line we have the inequalities

$$\int_a^\infty \frac{c_2^{-2} e^{2ct} \|x\|^2}{1+c_2^{-2} e^{2ct} \|x\|^2} dt \leqslant P_a(x) \leqslant \int_a^\infty \frac{c_1^2 e^{2bt} \|x\|^2}{1+c_1^2 e^{2bt} \|x\|^2} dt.$$

Hence and from the formula for $m < 0$

$$\int_a^\infty \frac{e^{2mt} s^2}{1+s^2 e^{2mt}} dt = -\frac{1}{2m} \log(1+s^2 e^{2ma})$$

by a simple computation we get the assertion of the Lemma.

As a consequence of the definition (6.2) and Lemma 6.1 we get the following Corollary:

COROLLARY. *For every $a \in R$ and $x \in S^{N-1}$ the integral $\int_a^\infty \frac{|[x, u]|^2}{1+|[x, u]|^2} du$ is finite.*

We define a family $m_{(x,a)}$ $(a \in R)$ of probability measures on F_x as follows. Put

$$g_a(t) = \begin{cases} 0 & \text{if } t \leqslant a, \\ c_a & \text{if } t > a, \end{cases} \tag{6.13}$$

where

$$c_a^{-1} = \int_a^\infty \frac{|[x, u]|^2}{1+|[x, u]|^2} du.$$

By Corollary to Lemma 6.1, the constant c_a is finite. It is very easy to verify that the function g_a fulfils the conditions of the Proposition 6.2. Consequently, it determines, by formula (6.12), the measure $m_{(x,a)}$ belonging to $\mathscr{K}$ and concentrated on F_x.

PROPOSITION 6.3. *The set of measures $\{m_{(x,a)}\colon a \in R\}$ is identical with the set of extreme points of $\mathscr{K}$ concentrated on F_x.*

Proof. First we shall prove that each measure $m_{(x,a)}$ is an extreme point of $\mathscr{K}$. Suppose that

$$m_{(x,a)} = c, \mu_1 + (1-c)\mu_2,$$

where $0 < c < 1$ and $\mu_1, \mu_2 \in \mathscr{K}$. It is clear that both measures μ_1 and μ_2 are concentrated on the set F_x. Let q_1 and q_2 be the functions corresponding, by Proposition 6.2, to μ_1 and μ_2 respectively. Then

$$g_a = cq_1 + (1-c)q_2.$$

Since both functions q_1 and q_2 are non-negative monotone non-decreasing and continuous from the left, we infer that the last equation is possible if they are constant on the half-lines $(-\infty, a]$ and (a, ∞). Furthermore, by condition (6.11) $q_1 = q_2 = g_a$ which proves that the measures $m_{(x,a)}$ are extreme points of $\mathscr{K}$.

Now we shall prove that each extreme point λ of $\mathscr{K}$ concentrated of F_x is one of the measures $m_{(x,a)}$ $(a \in R)$. Let q_λ be the function corresponding to λ according to Proposition 6.2. Suppose that there exists a real number v_0 such that the function q_λ is not constant on both half-lines $(-\infty, v_0]$ and (v_0, ∞). Setting

$$c = q_\lambda(v_0) \int_{-\infty}^{v_0} \frac{|[x, u]|^2}{1+|[x, u]|^2} du + \int_{v_0}^\infty \frac{|[x, u]|^2}{1+|[x, u]|^2} q_\lambda(u) du$$

we have, by (6.11), the inequalities $0 < c < 1$. Further, the functions

$$h_1(t) = \begin{cases} c^{-1} q_\lambda(t) & \text{if } t \leqslant v_0, \\ c^{-1} q_\lambda(v_0) & \text{if } t > v_0, \end{cases}$$

and

$$h_2(t) = \begin{cases} 0 & \text{if } t > v_0, \\ (1-c)^{-1}(q_\lambda(t) - q_\lambda(v_0)) & \text{if } t \leqslant v_0, \end{cases}$$

satisfy the conditions of the Proposition 6.2 and, consequently, determine the probability measures, say λ_1 and λ_2 belonging to $\mathscr{K}$ and concentrated on F_x. Since $h_1 \neq h_2$ and $q_\lambda = ch_1 + (1-c)h_2$, we infer that $\lambda_1 \neq \lambda_2$ and $\lambda = c\lambda_1 + (1-c)\lambda_2$ which contradicts the assumption that λ is an extreme point. Thus for every real number v the function q_λ is constant on at least one half-line $(-\infty, v]$ and (v, ∞). But, according to (6.11), it is not constant on the whole line R. Let a be a point of increase of q_λ. Then the function q_λ is constant on both half-lines $(-\infty, a]$ and (a, ∞). Taking into account condition (6.11), we infer that q_λ is equal to 0 on the half-line $(-\infty, a]$ and is equal to c_a on the remaining half-line. Thus $q_\lambda = g_a$ and, consequently, $\lambda = m_{(x,a)}$ which completes the proof.

From the definition (6.13), in view of (6.2) and (6.12), we get the relation $m_{(x,a)} = m_{(y,b)}$ if and only if $[x, a] = [y, b]$. This fact permits us to introduce the notation

$$m_{[x,a]} = m_{(x,a)} \quad \text{if } x \in S^{N-1} \text{ and } a \in R. \tag{6.14}$$

By (6.1), (6.12) and (6.13), for any function f continuous on M^N we have the formula

$$(6.15)\qquad \int_{M^N} f(z)\,m_{[x,a]}(dz) = c_{[x,a]}\int_0^\infty f(T_t[x,a])\,\frac{|T_t[x,a]|^2}{1+|T_t[x,a]|^2}\,dt,$$

where

$$(6.16)\qquad c_{[x,a]}^{-1} = \int_0^\infty \frac{|T_t[x,a]|^2}{1+|T_t[x,a]|^2}\,dt.$$

Further, we put

$$(6.17)\qquad m_{[x,a]} = \delta_{[x,a]} \quad \text{if either } a=-\infty \text{ or } a=\infty.$$

The mapping $z\to m_z$ from M^N onto the set of extreme points of $\mathcal{K}$ is one-to-one. From (6.15) and (6.16) it follows that this mapping is continuous at every point $[x,a]$ with $a\in R$. Further, it is obvious that $m_{[x_n,a]}$ tends to $m_{[x,a]}$ whenever $x_n\to x$ in S^{N-1} and either $a=-\infty$ or $a=\infty$. Suppose that $x_n\to x$ in S^{N-1}, $a_n\in R$ and $a_n\to-\infty$, i.e. $[x_n,a_n]\to[x,-\infty]$. Then, by (6.1) and (6.2),

$$\lim_{n\to\infty}|T_t[x_n,a_n]| = \infty$$

uniformly in t in every finite interval. Hence and from (6.16) it follows that

$$(6.18)\qquad \lim_{n\to\infty} c_{[x,a_n]} = 0.$$

Given $\varepsilon>0$ and a continuous function f on M^N, we can choose a number t_0 and an integer n_0 such that for all $t<t_0$ and $n\geqslant n_0$ the inequality

$$|f([x_n,t])-f([x,-\infty])|<\varepsilon$$

holds. Consequently,

$$(6.19)\qquad |f(T_t[x_n,a_n])-f([x,-\infty))|<\varepsilon$$

whenever $n\geqslant n_0$ and $t<|a_n|+t_0$. Since, by (6.1),

$$(6.20)\qquad \int_{|a_n|+t_0}^\infty \frac{|T_t[x_n,a_n]|^2}{1+|T_t[x_n,a_n]|^2}\,dt = \int_{t_0}^\infty \frac{|[x_n,u]|^2}{1+|[x_n,u]|^2}\,du$$

and $\|x_n\|=1$, we infer, by virtue of Lemma 6.1, that the integrals (6.20) are bounded in common. Thus, by (6.18),

$$(6.21)\qquad \lim_{n\to\infty} c_{[x_n,a_n]}\int_{|a_n|+t_0}^\infty \big(f(T_t[x_n,a_n])-f([x,-\infty])\big)\frac{|T[x_n,a_n]|^2}{1+|T_t[x_n,a_n]|^2}\,dt = 0.$$

Further, by (6.19),

$$c_{[x_n,a_n]}\int_0^{|a_n|+t_0}\big(f(T_t[x_n,a_n])-f([x,-\infty])\big)\frac{|T_t[x_n,a_n]|^2}{1+|T_t[x_n,a_n]|^2}\,dt\leqslant\varepsilon$$

whenever $n\geqslant n_0$. The arbitrariness of ε and (6.21) show that $m_{[x_n,a_n]}\to m_{[x,-\infty]}$. Thus the mapping $z\to m_z$ is also continuous at the points z of the form $[x,-\infty]$.

Suppose now that $x_n\to x$ in S^{N-1}, $b_n\in R$ and $b_n\to\infty$, i.e. $[x_n,b_n]\to[x,\infty]$. Then, by (6.1),

$$\lim_{n\to\infty} T_t[x_n,b_n] = [x,\infty]$$

uniformly in t ($t\geqslant 0$) which, by (6.15), implies the relation $m_{[x_n,b_n]}\to m_{[x,\infty]}$. Thus the mapping $z\to m_z$ is continuous at the points z of the form $[x,\infty]$. This completes the proof of continuity of the mapping $z\to m_z$. Hence, by well known Theorem (see [4], p.11), we conclude that this mapping is a homeomorphism between M^N and the set of extreme points of $\mathcal{K}$. Thus we have the following Proposition:

PROPOSITION 6.4. *The set of measures m_z ($z\in M^N$) defined by formulae* (6.14) *and* (6.17) *coincides with the set of extreme points of $\mathcal{K}$. Moreover, the mapping $z\to m_z$ is a homeomorphism between M^N and the set of extreme points of $\mathcal{K}$.*

Once the extreme points of K are found we can apply a Theorem by Choquet ([2], see also [11], Chapter 3). Since each element of $\mathcal{M}$ is of the form $c\nu$, where $c\geqslant 0$ and $\nu\in\mathcal{K}$, we then get the following proposition:

PROPOSITION 6.5. *A measure μ belongs to $\mathcal{M}$ if and only if there exists a finite Borel measure λ on M^N such that for each continuous function f on M^N the equation*

$$\int_{M^N} f(u)\,\mu(du) = \int_{M^N}\int_{M^N} f(u)\,m_z(du)\,\lambda(dz)$$

holds.

7. A representation of characteristic functions. Suppose that all eigenvalues of Q have negative real part. By Proposition 5.2 each $\exp tQ$-decomposable for $t\geqslant 0$ probability measure μ is infinitely divisible. Consequently, the characteristic function $\hat{\mu}$ has a Lévy–Khintchine representation

$$(7.1)\qquad \hat{\mu}(y) = \exp\Big\{i(a,y)-\tfrac{1}{2}(Dy,y)+ \int_{R^N\setminus\{0\}}\Big(e^{i(y,u)}-1-\frac{i(y,u)}{1+\|u\|^2}\Big)\frac{1+\|u\|^2}{\|u\|^2}\gamma(du)\Big\},$$

where a is a vector from R^N, D is a symmetric non-negative operator in R^N and γ is a finite Borel measure on $R^N\setminus\{0\}$. The triplet a, D and γ is uniquely determined by μ. The operator D and the measure γ will be called the Lévy–Khintchine operator and measure of μ respectively. In what follows π will denote the embedding of $R^N\setminus\{0\}$ into M^N defined by the formula (6.3).

PROPOSITION 7.1. *A symmetric non-negative operator D and a finite Borel measure γ on $R^N\setminus\{0\}$ are Lévy–Khintchine operator and measure respectively of an $\exp tQ$-decomposable for $t\geqslant 0$ probability measure if and only if the operator $QD+DQ^*$ is non-positive and the induced measure $\pi\gamma$ belongs to $\mathscr{M}$.*

Proof. Put for simplicity of notation $T_t = \exp tQ$ $(t\in R)$. Suppose that the characteristic function of a measure μ is given by (7.1). By Proposition 5.2, μ is T_t-decomposable for $t\geqslant 0$ if and only if for any $t\geqslant 0$ $\mu = T_t\mu * \mu_t$ where μ_t is an infinitely divisible measure. This condition can be formulated in terms of the characteristic functions as follows: μ is T_t-decomposable for $t\geqslant 0$ if and only if for any $t\geqslant 0$ $\hat{\mu}/\widehat{T_t\mu}$ is the characteristic function of an infinitely divisible measure. From (7.1) by a simple computation we get the formula

$$\hat{\mu}(y)/\widehat{T_t\mu}(y) = \exp\Big\{i(a_t, y)-\tfrac{1}{2}(D_t y, y)+ \int\limits_{R^N\setminus\{0\}}\Big(e^{i(y,u)}-1-\frac{i(y,u)}{1+\|u\|^2}\Big)\frac{1+\|u\|^2}{\|u\|^2}\gamma_t(du)\Big\},$$

where $a_t\in R^N$,

$$D_t = D - T_t D T_t^* \tag{7.2}$$

and

$$\gamma_t(E) = \gamma(E) - \int\limits_E \frac{\|v\|^2(1+\|T_{-t}v\|^2)}{(1+\|v\|^2)\|T_{-t}v\|^2}\gamma(T_{-t}dv). \tag{7.3}$$

Hence if follows that μ is T_t-decomposable for $t\geqslant 0$ if and only if for any $t\geqslant 0$ D_t is non-negative and γ_t is a non-negative measure.

First we shall prove that the operator D_t is non-negative for $t\geqslant 0$ if and only if the operator $QD+DQ^*$ is non-positive. Suppose that D_t is non-negative for all $t\geqslant 0$. By (7.2), we have the expansion in a neighborhood of 0 $D_t = -t(QD+DQ^*)+o(t)$. Hence it follows that $QD+DQ^*$ is non-positive.

Assume now that $QD+DQ^*$ is non-positive. Given $x\in R^N$, we put $w_x(t) = (D_t x, x)$. By a simple computation we get the formula

$$\frac{d}{dt}w_x(t) = -\big((QD+DQ^*)T_t^*x, T_t^*x\big)$$

which implies the inequality $\frac{d}{dt}w_x(t)\geqslant 0$. Taking into account the initial condition $w_x(0) = 0$, we get the inequality $w_x(t)\geqslant 0$ for all $t\geqslant 0$ and all $x\in R^N$. Thus the operators D_t are for $t\geqslant 0$ non-negative.

Taking into account (6.2), (6.3), (6.4), (6.5) and (7.3) for each Borel subset E of M^N bounded from below we have the formula

$$\int\limits_{\pi^{-1}(E)}\frac{1+\|u\|^2}{\|u\|^2}\gamma_t(du) = I_{\pi\gamma}(E) - T_t I_{\pi\gamma}(E).$$

Consequently, γ_t is non-negative for $t\geqslant 0$ if and only if $\pi\gamma\in\mathscr{M}$. The Proposition is thus proved.

THEOREM 7.1. *Suppose that all eigenvalues of Q have negative real part. A function φ on R^N is the characteristic function of an $\exp tQ$-decomposable for $t\geqslant 0$ probability measure if and only if*

$$\varphi(y) = \exp\Big\{i(a,y)-\tfrac{1}{2}(Dy,y)+ \int\limits_{R^N\setminus\{0\}}\int\limits_0^\infty\Big(e^{i(y,\exp tQx)}-1-\frac{i(y\exp tQx)}{1+\|\exp tQx\|^2}\Big)dt\,\frac{\nu(dx)}{\log(1+\|x\|^2)}\Big\}, \tag{7.4}$$

where a is a vector from R^N, D is a symmetric non-negative operator in R^N for which the operator $QD+DQ^*$ is non-positive and ν is a finite Borel measure on $R^N\setminus\{0\}$. Moreover, the function φ determines the triplet a, D and ν uniquely.

Proof. The necessity. Suppose that μ is an $\exp tQ$-decomposable for $t\geqslant 0$ probability measure. By Proposition 5.2 μ is infinitely divisible and its characteristic function can be written in the form (7.1) with parameters a, D and γ. Moreover, by Proposition 7.1, the operator $QD+DQ^*$ is non-positive and the induced measure $\pi\gamma$ on M^N belongs to $\mathscr{M}$. By Proposition 6.5 there exists a finite Borel measure ω on M^N such that for every continuous function f on M^N the equation

$$\int\limits_{M^N} f(u)\pi\gamma(du) = \int\limits_{M^N}\int\limits_{M^N} f(u)m_z(du)\omega(dz) \tag{7.5}$$

holds. Here m_z $(z\in M^N)$ denote the extreme points of $\mathscr{K}$ defined by the formulae (6.14) and (6.17). It is clear that the measure $\pi\gamma$ is concentrated on the set $U_N = \pi(R^N\setminus\{0\})$. Consequently, by (7.5), the measure ω is also concentrated on U_N. Since for $z\in U_N$ the measures m_z are concentrated on U_N (see (6.15)), the formula (7.5) can be rewritten in the form

$$\int\limits_{U_N} f(u)\pi\gamma(du) = \int\limits_{U_N}\int\limits_{U_N} f(u)m_z(du)\omega(dz), \tag{7.6}$$

for any function f continuous and bounded on U_N. Let us introduce the notation $\lambda = \pi^{-1}\omega$ and

$$\nu(E) = \int_E c(x)\log(1+\|x\|^2)\lambda(dx)$$

where E are Borel subsets of $R^N\setminus\{0\}$ and

$$c(x)^{-1} = \int_0^\infty \frac{\|\exp tQx\|^2}{1+\|\exp tQx\|^2}\,dt.$$

By Lemma 6.1 ν is a finite measure on $R^N\setminus\{0\}$. By a simple computation, in view of (6.2), (6.3), (6.4), (6.5) and (7.6), for every continuous and bounded function g on $R^N\setminus\{0\}$ we get the formula

$$(7.7)\quad \int_{R^N\setminus\{0\}} g(x)\gamma(dx) = \int_{R^N\setminus\{0\}}\int_0^\infty g(\exp tQx)\frac{\|\exp tQ\|^2}{1+\|\exp tQx\|}\,dt\,\frac{\nu(dx)}{\log(1+\|x\|^2)}.$$

Setting

$$g(x) = \left(e^{i(y,x)}-1-\frac{i(y,x)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\qquad (y\in R^N)$$

into the last formula and taking into account (7.1) we get the representation (7.4). The necessity of the conditions is thus proved.

The sufficiency. Suppose that the function φ is given by formula (7.4). First we note that φ is a limit of products of a Gaussian characteristic function $\exp\left(i(a,y)-\frac{1}{2}(Dy,y)\right)$ and Poissonian characteristic functions of the form

$$\exp c\left(e^{i(y,b)}-1-\right)\frac{i(y,b)}{1+\|b\|^2},$$

where $c\geqslant 0$ and $b\in R^N\setminus\{0\}$. Thus φ is the characteristic function of an infinitely divisible measure, say μ (see [10], Theorems 4.1 and 4.10). For every $s\geqslant 0$ from (7.4) by a simple computation we get the formula

$$(7.8)\quad \hat{\mu}(y)/\widehat{\exp sQ\mu}(y) = \exp\Big\{i(a_s,y)-\tfrac{1}{2}(D_sy,y)+$$

$$+\int_{R_N\setminus\{0\}}\int_0^s\left(e^{i(y,\exp tQx)}-1-\frac{i(y,\exp tQx)}{1+\|\exp tQx\|^2}\right)dt\,\frac{\nu(dx)}{\log(1+\|x\|^2)}\Big\},$$

where $a_s\in R^N$ and $D_s = D-(\exp sQ)D(\exp sQ)^*$. From the assumption that the operator $QD+DQ^*$ is non-positive and from Proposition 7.1 it follows that the Gaussian probability measure with the characteristic function $\exp\left(-\frac{1}{2}(Dy,y)\right)$ is $\exp tQ$-decomposable for $t\geqslant 0$. Hence we infer that the operator D_s is non-negative. Consequently, the function (7.8) is a limit of products of Gaussian and Poissonian characteristic functions. Thus (7.8) is the characteristic function of a probability measure, say λ_s. Obviously, $\mu = \exp sQ\mu*\lambda_s$ for $s\geqslant 0$ which shows that the function φ is the characteristic function of an $\exp tQ$-decomposable for $t\geqslant 0$ probability measure. The sufficiency of the conditions is thus proved.

It remains to prove the uniqueness of the triplet a, D and ν in the representation (7.4). First we note that the formula (7.7) establishes a one-to-one correspondence between the Lévy–Khintchine measure γ and the measure ν. In fact, it is evident that ν determines γ uniquely. To prove the converse let us take an arbitrary continuously differentiable function f in R^N with a compact support and vanishing in a neighborhood of the origin. Then the function

$$g_f(x) = -\frac{1+\|x\|^2}{\|x\|^2}\frac{d}{dt}f(\exp tQx)\log(1+\|\exp tQx\|^2)_{t=0}$$

is continuous and bounded on R^N. Setting it into (7.7) we get the formula

$$\int_{R^N\setminus\{0\}} g_f(x)\gamma(dx) = \int_{R^N\setminus\{0\}} f(x)\nu(dx)$$

which shows that γ determines ν uniquely.

Suppose now that the function φ has two representations (7.4) with the triplets (a_1, D_1, ν_1) and (a_2, D_2, ν_2) respectively. Then denoting by γ_1 and γ_2 the measures corresponding in (7.7) to ν_1 and ν_2 respectively we have, by (7.4) and (7.7), the equations

$$\varphi(y) = \exp\Big\{i(a_1,y)-\tfrac{1}{2}(D_1y,y)+$$

$$+\int_{R^N\setminus\{0\}}\left(e^{i(y,x)}-1-\frac{i(y,x)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\gamma_1(dx)\Big\}$$

$$= \exp\Big\{i(a_2,y)-\tfrac{1}{2}(D_2y,y)+$$

$$+\int_{R^N\setminus\{0\}}\left(e^{i(y,x)}-1-\frac{i(y,x)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|_2}\gamma_2(dx)\Big\}.$$

Hence, by the uniqueness of the Lévy–Khintchine representation (see [10] Theorem 4.10), $a_1 = a_2$, $D_1 = D_2$ and $\gamma_1 = \gamma_2$. Since the measures γ_1 and γ_2 determine the measures ν_1 and ν_2 respectively, we have the equation $\nu_1 = \nu_2$ which completes the proof.

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On the Riesz–Fischer theorem for vector-valued functions

by

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Dedicated to Professor A. Zygmund
on the occasion of 50th anniversary
of his scientific research

Abstract. Let $\varphi\colon \langle 0, \infty) \to R_+$ be a nondecreasing continuous function satisfying conditions $\varphi(u)/u \to 0$ if $u \to 0$, $\varphi(u)/u \to \infty$ if $u \to \infty$, X let denote a Banach space, Ξ its dual space. Let, further, $X^\wedge$ denote a vector space consisting of sequences $x^\wedge = \{x_i\}$, $x_i \in X$.

Assuming that φ is a convex function on $X^\wedge$ one can define a modular $\varrho_\varphi(x) = \sup\sum_1^\infty \varphi(|\xi(x_i)|)$, where supremum is taken over the ball $\Xi_0 = \{\xi\colon \|\xi\| \leqslant 1\}$.

Investigated are the properties of the space $l^{*\varphi}(X)$, elements of which are the sequences $x^\wedge \in X^\wedge$ such that $\varrho_\varphi(\lambda x^\wedge) < \infty$ for some $\lambda > 0$. Section 2 of the paper deals with the spaces of vector functions $x(\cdot)\colon \langle a, b\rangle \to X$, of finite Riesz φ-variation (as defined in 2.1) and with the spaces $V^{*\varphi}(X)$.

In Section 3 certain remarks are made about orthogonal series of the form $(*)\ x_1\varphi_1 + x_2\varphi_2 + \ldots$ where $x_i \in X$, and $\{\varphi_i\}$ is an orthogonal system in $\langle a, b\rangle$.

If $x(\cdot)\colon \langle a, b\rangle \to X$ is a vector function absolutely continuous in $\langle a, b\rangle$, then its Fourier coefficients are represented by $x_n = \int_{\langle a,b\rangle} \varphi_n(t)\,dx$ where the integral in this formula is a (Dunford) integral $\langle a, b\rangle$ with respect to the vector measure $x(:)$ associated with $x(\cdot)$.

Using the spaces $l^{*\varphi}(X)$, $V^{*\varphi}(X)$, where $\varphi(u) = u^2$, authors obtain the analogue of Riesz–Fischer Theorem for series of the form $(*)$.

1. In this note X always stands for a real Banach space provided with a norm $\|\cdot\|$, Ξ for its conjugate space, $\Xi_0 = \{\xi \in \Xi;\ \|\xi\| \leqslant 1\}$. H denotes the class of all zero-one sequences $\{\eta_i\}$, $\mathscr{I}$ denotes the algebra of subsets of an interval $T = \langle a, b\rangle$ whose elements are finite unions of intervals $\langle c, d\rangle$, $a \leqslant c < d \leqslant b$, $\langle d, b\rangle$, $a \leqslant d < b$ and the empty set, $\mathscr{E}$ is the σ-algebra of Lebesgue measurable subsets of T and μ is the Lebesgue measure on $\mathscr{E}$. Measurability of sets and functions are always understood with respect to μ.

$x(\cdot)$, $y(\cdot), \ldots$ or $x, y, \ldots$ always denote vector-valued functions from T into X, $f(\cdot)$, $g(\cdot), \ldots$ or $f, g, \ldots$ real-valued function on T. A series $\sum_1^\infty x_i$ of elements belonging to a Banach space is said to be *perfectly con-*