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On convergence of Fourier series of functions of generalized bounded variation

by

DANIEL WATERMAN (Syracuse, N. Y.)

Abstract. Various classes of functions of generalized bounded variation are introduced by assuming the finiteness of $\sum|f(I_n)|/\lambda_n$, $\lambda_n \nearrow \infty$, $\{I_n\}$ nonoverlapping intervals. The Fourier series of functions of one class, the functions of *harmonic bounded variation* (HBV), converge everywhere and converge uniformly on closed intervals of continuity. This result is best possible in that each larger class contains a continuous function whose Fourier series diverges at a point. The functions of Φ-bounded variation with complementary function Ψ satisfying $\sum\Psi(1/n) < \infty$, as considered by Salem, are contained in HBV, as are the functions with logarithmically integrable Banach indicatrix considered by Garsia and Sawyer. However, all these conditions are contained in the test of Lebesgue. An example of an application to absolute convergence of Fourier series is given.

In this note we introduce various classes of functions of generalized bounded variation. We show that the Fourier series of functions of one class, the functions of *harmonic bounded variation*, converge everywhere and converge uniformly on each closed interval of continuity. This result is best possible in the sense that each larger class contains a continuous function whose Fourier series diverges at a point. The functions of Φ-bounded variation in the sense of L. C. Young satisfying the Salem condition, $\Sigma\Psi(1/n) < \infty$, where Ψ is the complementary function, are included in this class of functions of harmonic bounded variation, as are the functions with logarithmically integrable Banach indicatrix considered by Garsia and Sawyer. We show, however, that all these conditions are contained in the test of Lebesgue. The notions of generalized bounded variation considered here have other applications, and an indication of such an application to absolute convergence of Fourier series is given.

1. Let us suppose that f is a real function defined on an interval $[a, b]$. $\{I_n\}$ will denote a sequence of non-overlapping intervals $I_n = [a_n, b_n] \subset [a, b]$ and we write $f(I_n) = f(b_n) - f(a_n)$. We let Λ denote a non-decreasing sequence of real numbers $\lambda_n > 0$ such that $\Sigma 1/\lambda_n$ diverges.

DEFINITION. A function f is said to be of *Λ-bounded variation* (ΛBV) if for every $\{I_n\}$ we have

$$\sum_1^\infty |f(I_n)|/\lambda_n < \infty;$$

for $\Lambda = \{n\}$, i.e.,

$$\sum_1^\infty |f(I_n)|/n < \infty,$$

we say that f is of *harmonic bounded variation* (HBV).

We note that ΛBV functions share many of the properties of BV functions as, for example,

1. ΛBV functions are bounded;

2. The discontinuities of a ΛBV function are simple and, therefore, at most denumerable:

3. The Helly selection theorem holds for ΛBV functions.

It may be shown readily that the following are equivalent:

(i) f is a ΛBV function;

(ii) There is an $M < \infty$ such that for every $\{I_n\}$, $\sum_1^\infty |f(I_n)|/\lambda_n < M$;

(iii) There is an $M < \infty$ such that for every finite collection $\{I_n\}$, $n = 1, 2, \ldots, N$, $\sum_1^N |f(I_n)|/\lambda_n < M$.

If $f \in \Lambda$BV, we may now define the *Λ-variation*,

$$V(x) = \sup\Big\{\sum_1^\infty |f(I_n)|/\lambda_n : \{I_n\} \text{ such that } \bigcup I_n \subset [a, x]\Big\},$$

where $x \in [a, b]$. Clearly $V(x)$ is a non-decreasing function, and we may show that V *is continuous at a point if and only if f is continuous there.*

It is easily seen that ΛBV is a Banach space with the norm

$$\|f\| = |f(a)| + V(b).$$

Let us now suppose that Φ and Ψ are two functions complementary in the sense of W. H. Young, i.e., $\Phi(x) = \int_0^x \varphi(t)\,dt$, where φ is continuous, strictly increasing, and $\varphi(0) = 0$, and $\Psi(x) = \int_0^x \varphi^{-1}(t)\,dt$. A function f on $[a, b]$ was said by L. C. Young [8] to be of Φ-bounded variation if there is an $M < \infty$ such that, for every partition $a = x_0 < x_1 < \ldots < x_n = b$, we have

$$\sum_1^{n-1} \Phi\big(|f(x_{i+1}) - f(x_i)|\big) < M.$$

The least such M is known as the Φ-variation of f.

Salem [7] showed that if a continuous function f is of Φ-BV and $\sum_1^\infty \Psi(1/n) < \infty$, then the Fourier series of f converges uniformly. It has been shown by Baernstein [1] that if $\sum \Psi(1/n) = \infty$, then there is a continuous Φ-BV function whose Fourier series diverges at a point, answering a question raised by Goffman and Waterman [5].

We note that the above definition of Φ-BV is unnecessarily restrictive. Suppose we assume merely that Φ is convex, $\Phi(x) = o(x)$ as $x \to 0+$, $\Phi(x)/x \to \infty$ as $x \to \infty$, and $\Phi(0) = 0$. Let

$$\Psi(x) = \sup\{xy - \Phi(y) \colon y \geqslant 0\}.$$

Then we have

$$xy \leqslant \Phi(x) + \Psi(y),$$

which is Young's inequality.

With Φ satisfying these requirements, we may now define the Φ-variation as above. In the following, a function f will be said to be of Φ-BV if for some $k > 0$, the function kf has finite Φ-variation. This class has been thoroughly studied by Musielak and Orlicz [6].

It is clear that if $\{I_n\}$ is as before and $f \in \Phi$-BV with Ψ satisfying Salem's condition, then for some $k > 0$, by Young's inequality,

$$k\sum_1^N |f(I_n)|/n \leqslant \sum_1^N \Phi(k|f(I_n)|) + \sum_1^N \Psi(1/n) < M + \sum_1^\infty \Psi(1/n) < \infty,$$

and so $f \in$HBV.

Another generalization of bounded variation was given by Garsia and Sawyer [2]. They restricted themselves to continuous functions with range $[0, 1]$. If $n(y)$ is the Banach indicatrix of f, that is, the cardinality of $\{x\colon f(x) = y\}$ if this is finite and ∞ otherwise, they consider those functions such that

$$\int_0^1 \log n(y)\,dy < \infty.$$

They showed that these functions too have uniformly convergent Fourier series.

Goffman [3] considered regulated functions, i.e., functions whose discontinuities are simple and for which $f(x) = [f(x+) + f(x-)]$. He showed that if we replace the continuous functions by these, the conditions imposed by Salem, as well as those of Garsia and Sawyer with a suitable modification of the definition of $n(y)$, then imply everywhere convergence of the Fourier series. The principal tool here was a result of Goffman and Waterman [4] on the everywhere convergence of Fourier series of continuous functions. Using this result, which he observed held for regulated functions as well, he showed that if the Fourier series of a regulated function diverged at a point, then the function was not of harmonic bounded variation (in our terminology) and showed further that this implied that the conditions of Salem and of Garsia and Sawyer would not be satisfied.

We have indicated above why HBV includes Φ-BV with the restriction $\sum\limits_1^\infty \Psi(1/n) < \infty$. We turn now to the relation between HBV and the condition of Garsia and Sawyer.

Let $\bar{f}(x) = \limsup\limits_{t\to x} f(t)$ and $\underline{f}(x) = \liminf\limits_{t\to x} f(t)$. A function is said to have an *external saltus* if for some x, $f(x) \notin [\underline{f}(x), \bar{f}(x)]$. It is well-known that the set of points at which an arbitrary function has an external saltus is at most countable. Thus if our only interest was in the convergence properties of Fourier series, there would be no loss in generality in restricting ourselves to functions with no external saltus. It is easily seen, in this case, that the property that $f \in \Lambda\mathrm{BV}$ is independent of its values at points of discontinuity. We note, however, that the demonstration of the fact that the condition of Garsia and Sawyer implies inclusion in HBV does *not* require the assumption of no external saltus.

We shall use the following result.

LEMMA. *Let $\{E_i\}$ be a sequence of μ-measurable sets of a measure space $(\mathscr{X}, \mathscr{A}, \mu)$ and let $S_\infty = \overline{\lim}\, E_i$ and S_n, $n = 1, 2, 3, \ldots$, be the set of points belonging to exactly n of the sets E_i. If $\{a_i\}$ is a decreasing sequence of nonnegative real numbers, then*

$$\sum_1^\infty a_i \mu(E_i) \leqslant \sum_1^\infty \mu(S_n)\Big(\sum_1^n a_i\Big) + \sum_1^\infty a_i \mu(S_\infty).$$

COROLLARY. *If $\mu(\mathscr{X}) < \infty$, then $\sum\limits_1^\infty \mu(E_i)/i = \infty$ implies that $\mu(S_\infty) \neq 0$ or $\sum\limits_1^\infty \mu(S_n)\log n = \infty$.*

Proof. If χ_i denotes the characteristic function of E_i, we have, for each $x \in S_n$,

$$\sum_1^\infty a_i \chi_i(x) \leqslant \sum_1^n a_i.$$

Hence

$$\begin{aligned}\sum_1^\infty a_i\mu(E_i) &= \int\Big(\sum_1^\infty a_i\chi_i(x)\Big)d\mu(x)\\ &= \sum_1^\infty \int_{S_n}\Big(\sum_1^\infty a_i\chi_i(x)\Big)d\mu(x) + \sum_1^\infty a_i\mu(S_\infty)\\ &\leqslant \sum_1^\infty \mu(S_n)\Big(\sum_1^n a_i\Big) + \sum_1^\infty a_i\mu(S_\infty).\end{aligned}$$

Letting $a_i = 1/i$, we have $\sum\limits_1^n a_i \sim \log n$; $\mu(\mathscr{X}) < \infty$ implies $\mu(S_n) < \infty$ for every n. If $\sum\limits_1^\infty \mu(E_i)/i = \infty$ and $\mu(S_n) = 0$, then $\sum\limits_1^n \mu(S_n)(\sum\limits_1^n 1/i) = \infty$, from which the corollary follows immediately.

Let us now consider a bounded function f on the interval $[a, b]$. For each x at which f is discontinuous, we adjoin to the graph of f the smallest segment containing $(x, f(x))$, $(x, \bar{f}(x))$, and $(x, \underline{f}(x))$, and call the resulting set $\mathscr{I}$. It is not difficult to see that $\mathscr{I}$ has the following

GENERALIZED DARBOUX PROPERTY. *If $a \leqslant x_0 < x_1 \leqslant b$ and (x_0, y_0) and (x_1, y_1) are in $\mathscr{I}$, then for each y between y_0 and y_1 there is an $x \in [x_0, x_1]$ such that $(x, y) \in \mathscr{I}$.*

We may now define a generalized Banach indicatrix $n(y) = n_f(y)$ to be the cardinality of $\{x\colon (x, y) \in \mathscr{I}\}$ if finite and ∞ otherwise. Let I be the interval $\{y\colon (x, y) \in \mathscr{I}\}$. We will say that f *satisfies the Garsia-Sawyer condition* if $\int\limits_I \log n(y)\,dy < \infty$.

If $f \notin \mathrm{HBV}$, but is bounded, there is $\{I_n\}$, a sequence of non-overlapping in $[a, b]$ such that $\sum\limits_1^\infty |f(I_n)|/n = \infty$. Let $I_n = [a_n, b_n]$ and let E_n be the interval with endpoints $f(a_n)$ and $f(b_n)$. If $y \in S_k$, then the generalized Darboux property implies that $n(y) \geqslant k$. Thus

$$\int_I \log n(y)\,dy \geqslant \sum_1^\infty m(S_n)\cdot\log n + \sum_1^\infty \frac{1}{n} m(S_n) = \infty$$

since $\sum\limits_1^\infty m(E_n)/n = \infty$ and $m(I) < \infty$.

The corollary above (proved otherwise) is due to Goffman [3], who used it in a similar manner to show that a bounded regulated function satisfying the Garsia–Sawyer condition satisfies the Goffman–Waterman condition [4] for everywhere convergence of its Fourier series.

A review of the above considerations shows that the following result may be established.

THEOREM 1. (i) *If* $f \in \Phi\text{-}BV$ *and* $\sum \Psi(1/\lambda_n) < \infty$, *then* $f \in \Lambda BV$.

(ii) *If the range of* f *is contained in a finite interval* I *and* $L(x)$ *is a finite-valued increasing function such that* $L(n) \sim \sum_{1}^{n} 1/\lambda_k$ *as* $n \to \infty$, *then*

$$\int_I L(n(y))\,dy < \infty$$

implies that $f \in \Lambda BV$.

2. In this section we establish our result on the relation between HBV and the Lebesgue test. We use the conventional notation

$$\varphi(t) = \varphi_x(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2f(x)\},\ \Phi(h) = \Phi_x(h) = \int_0^h |\varphi_x(t)|\,dt.$$

The Lebesgue test can be stated as follows ([9], vol. I, p. 65): *The Fourier series of* f *converges to* $f(x)$ *at every point* x *at which*

$$\Phi(h) = o(h), \qquad \int_\eta^\pi \frac{|\varphi(t)-\varphi(t+\eta)|}{t}\,dt \to 0 \qquad \text{as } \eta = \pi/n \to 0,$$

and the convergence is uniform over any closed interval of continuity where the second condition is satisfied uniformly.

When we consider the case of convergence at a point of simple discontinuity, it is clear that we must take $f(x) = \frac{1}{2}\{f(x+0)+f(x-0)\}$ in the test and in the definition of $\varphi_x(t)$.

Our principal result is the following.

THEOREM 2. *If* f *is a function of harmonic bounded variation, then* f *satisfies the condition of Lebesgue at each point and satisfies the second condition uniformly over any closed interval of continuity.*

Before we pass to the proof of this result, we turn to a simpler question, in what sense is this best possible, for which we have the following answer.

THEOREM 3. *If* $\Lambda BV \supset HBV$ *properly, then there is a continuous* $f \in \Lambda BV$ *whose Fourier series diverges at a point.*

Proof. There exists $a_n \searrow 0$ such that $\sum_1^\infty a_n/\lambda_n$ converges, but $\sum_1^\infty a_n/n$ diverges. Let $b_n = \sum_1^n a_i/\lambda_i$. Let $f_n(x)$ be defined in $[0, 2\pi]$ to be a_i for $(2i-2)\pi < (n+\frac{1}{2})x < (2i-1)\pi$, $i = 1, \ldots, n+1$, and 0 elsewhere. Then in ΛBV, $\|f_n\| = b_{n+1}$. If $S_n(f)$ denotes the nth partial sum of the Fourier series of a function f at the point 0, then

$$2\pi S_n(f_n) \geqslant 2\sum_{k=0}^{2n} \int_{k\pi/(n+\frac{1}{2})}^{(k+1)\pi/(n+\frac{1}{2})} f_n(t)\,\frac{\sin(n+\frac{1}{2})t}{t}\,dt$$

$$\geqslant \sum_{i=1}^{n+1} \frac{(2n+1)a_i}{(2i-1)\pi} \int_{(2i-2)\pi/(n+\frac{1}{2})}^{(2i-1)\pi/(n+\frac{1}{2})} \sin(n+\tfrac{1}{2})t\,dt$$

$$= \frac{4}{\pi}\sum_1^{n+1} a_i/(2i-1) > \frac{2}{\pi}\sum_1^{n+1} a_i/i.$$

Thus $\|S_n\| \geqslant \dfrac{1}{\pi^2 b_{n+1}} \sum_1^{n+1} a_i/i \to \infty$ as $n \to \infty$, which implies that there is an $f \in \Lambda BV$ such that $\{S_n(f)\}$ does not converge. Note that the continuous functions of Λ-bounded variation themselves form a Banach space, since convergence in ΛBV implies uniform convergence. The functions f_n used above can be modified so as to be continuous without excessively complicating the argument, thus yielding the desired result.

We now return to the previous result.

Proof of theorem 2. We consider first the case of uniform convergence and suppose that f is continuous at each point of $[c, d]$. Let

$$L(n, x) = \int_{\pi/n}^{\pi} |\varphi_x(t) - \varphi_x(t+\pi/n)|\,\frac{1}{t}\,dt.$$

If we assume that f does not satisfy the second condition of Lebesgue's test uniformly on $[c, d]$, then without loss of generality we may assume that $0 \in [c, d]$ and that there exists $x_k \searrow 0$, $n_k \nearrow \infty$, and $a > 0$ such that $L(n_k, x_k) > a$ for every k.

Now

$$L(n, x) = \int_{\pi/n}^{\delta} \ldots + \int_{\delta}^{\pi} \ldots = L_1(n, x, \delta) + L_2(n, x, \delta),$$

and

$$L_2(n, x, \delta) \leqslant \frac{1}{\delta}\cdot\frac{1}{2}\Big\{\int_\delta^\pi |f(x+t) - f(x+t+\pi/n)|\,dt$$

$$+\int_\delta^\pi |f(x-t) - f(x-t-\pi/n)|\,dt\Big\}$$

$$\leqslant \frac{1}{\delta}\int_{-\pi}^{\pi} |f(t) - f(t+\pi/n)|\,dt = o(1) \quad \text{as } n \to \infty.$$

Hence for a sequence $\delta_i \searrow 0$ there exists a subsequence n_{k_i} with $n_{k_i}\delta_i \nearrow \infty$ such that $L_2(n_{k_i}, x_{k_i}, \delta_i) \to 0$.

We suppose that $\{\delta_i\}$ is fixed and that $\{n_{k_i}\}$ and $\{x_{k_i}\}$ are our original sequences. Let $m_k = [n_k\delta_k/\pi+1]$. Then $m_k\pi/n_k > \delta_k$, $m_k \nearrow \infty$, and

$$L_2(n_k, x_k, m_k\pi/n_k) \to 0 \quad \text{as } k\to\infty.$$

For k sufficiently large, say $k > \bar{k}$, we have

$$3a/4 < L_1(n_k, x_k, m_k\pi/n_k)$$
$$= \int_0^{\pi/n_k} \sum_{i=1}^{m_k-1} |\varphi_{x_k}(t+i\pi/n_k) - \varphi_{x_k}(t+(i+1)\pi/n_k)| \frac{1}{(t+i\pi/n_k)}\, dt.$$

Thus for some $\theta\epsilon(0, \pi/n_k)$, we have

$$3a/4 < \frac{\pi}{n_k}\sum_{i=1}^{m_k-1} |\varphi_{x_k}(\theta+i\pi/n_k) - \varphi_{x_k}(\theta+(i+1)\pi/n_k)| \frac{1}{(\theta+i\pi/n_k)}$$
$$\leqq \sum_{i=1}^{m_k-1} \frac{1}{i} |\varphi_{x_k}(\theta+i\pi/n_k) - \varphi_{x_k}(\theta+(i+1)\pi/n_k)|.$$

Let $\Delta(k) = \sup\left\{|f(x+h)-f(x)|:\ |h|\leqslant \frac{\pi}{n_k}, x\epsilon\left[c-m_k\frac{\pi}{n_k}, d+m_k\frac{\pi}{n_k}\right]\right\}$.

Clearly $\Delta(k)\searrow 0$ as $k\to\infty$. Choose $k_1 > \bar{k}$ such that $\Delta(k_1) < a/8$. For each $j = 2, 3, \ldots$, choose $k_j > k_{j-1}$ such that

(i) $2(m_{k_j}+1)/n_{k_j} < 1/n_{k_{j-1}}$, (ii) $\left(\sum_{i=1}^{m_{k_{j-1}}} \frac{1}{i}\right)\Delta(k_j) < a/8$.

For each $j = 1, 2, 3, \ldots$, there is $\theta_j\epsilon(0, \pi/n_{k_j})$ such that, if $\varphi(i, j)$ denotes $|\varphi_{x_{k_j}}(\theta_j+i\pi/n_{k_j}) - \varphi_{x_{k_j}}(\theta_j+(i+1)\pi/n_{k_j})|$, we have

$$\sum_{i=1}^{m_{k_j}-1} \varphi(i, j)/i > 3a/4.$$

From (ii) we have

$$\sum_{i=1}^{m_{k_{j-1}}-1} \varphi(i, j)/i < a/8 \quad \text{for } j > 1.$$

Thus

$$\sum_{m_{k_{j-1}}}^{m_{k_j}-1} \varphi(i, j)/i > 5a/8 \quad \text{for } j > 1.$$

To $\varphi(i, j)$ correspond the intervals

$$I_{ij}^+ = [x_{k_j}+\theta_j+i\pi/n_{k_j},\ x_{k_j}+\theta_j+(i+1)\pi/n_{k_j}]$$

and

$$I_{ij}^- = [x_{k_j}-\theta_j-(i+1)\pi/n_{k_j},\ x_{k_j}-\theta_j-i\pi/n_{k_j}];$$

we have

$$\varphi(i, j) \leqslant \tfrac{1}{2}(|f(I_{ij}^+)| + |f(I_{ij}^-)|).$$

All the intervals corresponding to $\varphi(i, j)$, $1\leqslant i\leqslant m_{k_j}-1$, are contained in an interval of radius $(m_{k_j}+1)\pi/n_{k_j}$ and center x_{k_j}. Condition (i) implies that for each $j = 2, 3, \ldots$, these intervals overlap at most two intervals corresponding to $\varphi(i, j-1)$, $1\leqslant i\leqslant m_{k_{j-1}}-1$. The monotonicity of $\{x_{k_j}\}$ and condition (i) imply that these (at most) two intervals are of the form $I_{i,j-1}^-$, $I_{i+1,j-1}^-$ and also that I_{ij}^+, $j = 2, 3, \ldots$, overlaps with none of the $I_{i'j'}^+$, $j' < j$.

Let $m_{k_0} = 1$. For each $N = 1, 2, \ldots$, we have

$$S_N^+ + S_N^- = \sum_{j=1}^{N}\sum_{i=m_{k_{j-1}}}^{m_{k_j}-1} \frac{1}{i}|f(I_{ij}^+)| + \sum_{j=1}^{N}\sum_{i=m_{k_{j-1}}}^{m_{k_j}-1} \frac{1}{i}|f(I_{ij}^-)| > \frac{5}{4}aN.$$

As noted above, the intervals I_{ij}^+ in the sum S_N^+ are non-overlapping. If for each $j = 1, \ldots, N-1$, we eliminate from S_N^- the at most two terms containing the intervals overlapping with intervals of the $(j+1)$-st stage, we make at most $2(N-1)$ deletions, and the sum after these deletions will exceed

$$S_N^- - 2N\frac{a}{8}$$

since $\Delta(k_1) < a/8$. In the above sums denote I_{ij}^+ by I_i^+ and I_{ij}^- by I_i^-. If $\{i_n\}$, $n = 1, \ldots, m(N)$, denotes the indices of the terms remaining in S_N^- after the deletions, then the intervals $I_{i_n}^-$ are non-overlapping. Since $i_n \geqslant n$, we have

$$\sum_{i=1}^{m_{k_N}} \frac{1}{i}|f(I_i^+)| + \sum_{n=1}^{m(N)} \frac{1}{n}|f(I_{i_N}^-)| > Na,$$

which implies that one of these sums exceeds $Na/2$.

This argument is easily modified to include the pointwise case. The first condition of Lebesgue is immediate for HBV functions. If we assume that the second condition does not hold at $x = 0$ and that $f(0) = 0$, then we proceed as above, setting $x_k = 0$ for all k and

$$\Delta(k) = \sup\{|f(t)-f(t+h)|:\ 0 < h\leqslant \pi/n_k,$$
$$[t, t+h] \subset [-m_k\pi/n_k, 0) \cup (0, m_k\pi/n_k]\}.$$

Condition (i) now implies that the intervals corresponding to $\varphi(i, j)$ do not overlap with any of the intervals corresponding to $\varphi(i', j')$ with

$j' < j$. If we set $I_i^+ = I_{ij}^+$ and $I_i^- = I_{ij}^-$ for $m_{k_{j-1}} \leqslant i \leqslant m_{k_j} - 1$, we have two sequences of non-overlapping intervals with the property

$$\sum_1^\infty |f(I_n^+)|/n + \sum_1^\infty |f(I_n^-)|/n = \infty.$$

3. We now give an indication of other applications of generalized bounded variation by establishing a result on absolute convergence of Fourier series which generalizes a theorem of Zygmund ([9], vol. I, p. 241). We note that in the following, ΛBV may be replaced by a less restrictive requirement obtained by allowing $\{I_n\}$ in the definition of variation to be replaced by a partition in the usual sense, numbered in either direction.

Suppose now that f is continuous and of period 2π. As usual, we set $\varrho_n = (a_n^2 + b_n^2)^{\frac{1}{2}}$ and let ω denote the modulus of continuity of f.

If $f \in \Lambda$BV and V denotes its Λ-variation, then, letting $I_k = [x + (k-1)\pi/n, x + k\pi/n]$, we have

$$\sum_{k=1}^{2N} |f(I_k)|^2 = \sum_{k=1}^{2N} \lambda_k |f(I_k)| \cdot |f(I_k)|/\lambda_k \leqslant V\omega(\pi/N)\lambda_{2N},$$

Thus

$$2N \int_0^{2\pi} |f(x+\pi/n) - f(x-\pi/n)|^2 \, dx \leqslant 2\pi V \lambda_{2N} \omega(\pi/N)$$

or

$$\sum_1^\infty \varrho_n^2 \sin^2 n\pi/2N \leqslant \tfrac{1}{4} V\omega(\pi/N)\lambda_{2N}/N.$$

Setting $N = 2^\nu$,

$$\tfrac{1}{2} \sum_{2^{\nu-1}+1}^{2^\nu} \varrho_n^2 \leqslant \tfrac{1}{4} V\omega(\pi/2^\nu) 2^{-\nu} \lambda_{2^\nu+1}.$$

Thus

$$\sum_{2^{\nu-1}+1}^{2^\nu} \varrho_n \leqslant 2^{\nu/2} \Big(\sum_{2^{\nu-1}+1}^{2^\nu} \varrho_n^2 \Big)^{\frac{1}{2}} \leqslant 2^{-\frac{1}{2}} V^{\frac{1}{2}} \omega^{\frac{1}{2}}(\pi/2^\nu) \lambda_{2^\nu+1}^{\frac{1}{2}}$$

and so

$$\sum_2^\infty \varrho_n \leqslant 2^{-\frac{1}{2}} V^{\frac{1}{2}} \sum_1^\infty \omega^{\frac{1}{2}}(\pi/2^\nu) \lambda_{2^\nu+1}^{\frac{1}{2}}.$$

The convergence of the series on the right is equivalent to that of

$$\sum_1^\infty \lambda_n^{\frac{1}{2}} n^{-1} \omega^{\frac{1}{2}}(2\pi/n)$$

if the terms of this series are monotone decreasing from some point on. If $f \in$ HBV, the convergence of that series is equivalent to that of

$$\sum_1^\infty n^{-\frac{1}{2}} \omega^{\frac{1}{2}}(\pi/n).$$

We have established the following result for continuous functions of period 2π.

THEOREM 4. *If $f \in \Lambda BV$, then the Fourier series of f converges absolutely if $\sum \lambda_n^{\frac{1}{2}} n^{-1} \omega^{\frac{1}{2}}(2\pi/n)$ is a convergent monotone series. If $f \in HBV$, then the Fourier series of f converges absolutely if $\sum n^{-\frac{1}{2}} \omega^{\frac{1}{2}}(\pi/n)$ converges.*

References

[1] A. Baernstein, *Personal communication.*

[2] A. M. Garsia and S. Sawyer, *On some classes of continuous functions with convergent Fourier series*, J. of Math. and Mech. 13 (1964), pp. 586–601.

[3] C. Goffman, *Everywhere convergence of Fourier series*, Indiana Univ. Math. J. 20 (1970), pp. 107–113.

[4] C. Goffman and D. Waterman, *Functions whose Fourier series converge for every change of variable*, Proc.A.M.S. 19 (1968), pp. 80–86.

[5] — *Some aspects of Fourier series*, Amer. Math. Monthly 77 (1970), pp. 119–133.

[6] J. Musielak and W. Orlicz, *On generalized variations* I, Studia Math. 18 (1959), pp. 11–41.

[7] R. Salem, *Essais sur les séries trigonométriques*, Actualités Sci. Ind. No. 862, Paris, 1940.

[8] L. C. Young, *Sur une généralisation de la notion de variation de puissance p-ième bornée au sense de M. Wiener, et sur la convergence des séries de Fourier*, C. R. Acad. Sci. Paris 204 (1937), pp. 470–472.

[9] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1959.

DEPARTMENT OF MATHEMATICS
SYRACUSE UNIVERSITY

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