

$f \in \Gamma(A)$ which has an extension \bar{f} not belonging to $\Gamma(B)$, or even $\text{cor } B$ for some superalgebra $B \supset A$. To see this take as B the sup-norm disc algebra of all continuous functions on the unit disc of the complex plane, holomorphic in its interior and let $A = \{x \in B: x(0) = x(1)\}$. The maximal ideal space of A is the closed unit disc with identified 0 and 1 and the Šilov boundary of A is the unit circle (with 1 identified with 0). So the functional $f(x) = x(0) = x(1)$ is in $\Gamma(A)$ and it has two extensions onto B : $f_1(x) = x(1)$ and $f_0(x) = x(0)$ such that $f_1 \in \Gamma(B)$ but $f_0 \notin \text{cor } B$.

The following purely algebraic result can support the conjecture that $\mathcal{U}(A)$ coincides with the family of all non-removable closed ideals of A . Let R and P be arbitrary rings with unit elements. P is an extension of R if there is an isomorphic imbedding of R into P sending the unit of R into unit of P . Call an ideal I of R non-removable if in any extension P of R the ideal I is contained in a proper ideal of P . A subset S of R consists of joint divisors of zero if for any finite subset $\{x_1, \dots, x_n\} \subset R$ there is a non zero element $y \in R$ such that $x_i y = 0$ for $i = 1, 2, \dots, n$.

PROPOSITION 4. *An ideal I of a commutative ring R is a non-removable ideal if and only if it consists of joint divisors of zero.*

The proof can be obtained from a reasoning in [2].

References

- [1] R. Arens, *Extensions of Banach algebras*, Pacific J. Math. 10 (1960), pp. 1-16.
- [2] — *Ideals in Banach algebra extensions*, Studia Math. 31 (1968), pp. 29-34.
- [3] Г. Е. Шялов, *О расширении максимальных идеалов*, Докл. А. Н. СССР 29 (1940) pp. 83-85.
- [4] — *О нормированных кольцах с одной образующей*, Мат. сб. 21 (63) (1947), pp. 25-37.
- [5] W. Żelazko, *A characterization of Šilov boundary in function algebras*, Prace Mat. 14 (1970), pp. 59-64.
- [6] B. Bollobás, *Banach algebras and theory of numerical ranges*, Ph. D. Thesis, Cambridge 1971.
- [7] J. A. Lindberg, *Extension of algebra norms and applications*, Studia Math. 40 (1971), pp. 35-39.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Received June 15, 1971

(354)

On cosine operator functions and one-parameter groups of operators

by

J. KISYŃSKI (Warszawa)

Dedicated to Professor Antoni Zygmund

Abstract. If A is a complex number then

$$(*) \quad \exp \left(t \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \right) = \begin{pmatrix} \cos(-A)^{1/2} t & \int_0^t \cos(-A)^{1/2} \tau d\tau \\ \frac{d}{dt} \cos(-A)^{1/2} t & \cos(-A)^{1/2} t \end{pmatrix}, \quad -\infty < t < \infty.$$

The paper gives a generalization of this formula to the case, when A is an unbounded linear operator in a Banach space.

1. Preliminaries.

1.1. If E and F are Banach spaces over the same, real or complex, field of scalars then $\mathcal{L}(E; F)$ denotes the space of all linear bounded operators from E to F . Let $\mathcal{L}_s(E; F)$ denote $\mathcal{L}(E; F)$ equipped with the topology of pointwise convergence (called also the strong topology). An $\mathcal{L}(E; F)$ -valued function of a real variable is called *strongly continuous*, or *strongly continuously differentiable*, if it is continuous or continuously differentiable, when regarded as a mapping from $(-\infty, \infty)$ to $\mathcal{L}_s(E; F)$. For instance, by an application of the Banach-Steinhaus theorem, it follows that a function $K: (-\infty, \infty) \rightarrow \mathcal{L}(E; F)$ is strongly continuously differentiable on $(-\infty, \infty)$ if and only if for any fixed $x \in E$ the F -valued function $t \rightarrow K(t)x$ is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in F .

1.2. Let E be a Banach space. A strongly continuous mapping $G: (-\infty, \infty) \rightarrow \mathcal{L}(E; E)$ is called a *one-parameter strongly continuous group of operators* if $G(0) = 1$ and

$$G(t)G(s) = G(t+s) \quad \text{for every } s, t \in (-\infty, \infty).$$

The infinitesimal generator of the one parameter group G is the

linear operator A from E to E , with the domain $D(A)$ defined by the conditions

$$D(A) = \left\{ x: x \in E, \lim_{t \rightarrow 0} \frac{1}{t} (G(t)x - x) \text{ exists} \right\},$$

$$Ax = \lim_{t \rightarrow 0} \frac{1}{t} (G(t)x - x) \quad \text{for } x \in D(A),$$

where the limit is taken in the sense of the norm in E .

It is known (see e.g. [3], chapter IX) that if G is a strongly continuous one-parameter group of bounded linear operators in a Banach space E and if A is the infinitesimal generator of G , then

(1.2.1) there are constants $M \geq 1$ and $k \geq 0$ such that

$$\|G(t)\| \leq M e^{kt} \quad \text{for every } t \in (-\infty, \infty);$$

(1.2.2) for every $n = 1, 2, \dots$ the domain $D(A^n)$ of A^n is dense in E and A^n is a closed operator from E to E ;

(1.2.3) $G(t)D(A) = D(A)$ for every $t \in (-\infty, \infty)$ and, for every fixed $x \in D(A)$, the E -valued function $t \rightarrow G(t)x$ is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E and

$$\frac{dG(t)x}{dt} = AG(t)x = G(t)Ax, \quad t \in (-\infty, \infty).$$

1.3. Let E be a Banach space. A mapping $\mathcal{C}: (-\infty, \infty) \rightarrow \mathcal{L}(E; E)$ is called cosine operator function if it satisfies the d'Alembert functional equation

$$\mathcal{C}(t+s) + \mathcal{C}(t-s) = 2\mathcal{C}(t)\mathcal{C}(s)$$

for $s, t \in (-\infty, \infty)$, and if, moreover, $\mathcal{C}(0) = 1$. As it is easy to see, any cosine operator function is a pair function on $(-\infty, \infty)$, its range being a commutative family of operators.

The theory of $\mathcal{L}(E; E)$ -valued strongly continuous cosine functions was developed by M. Sova [2].

It should be remarked, that in [2] a cosine operator function is defined only on $[0, \infty)$. However, as Sova proved, the range of any strongly continuous cosine operator function defined on $[0, \infty)$ is a commutative family of operators, and from this it follows easily, that the pair extension onto $(-\infty, \infty)$ of such a cosine function satisfies the d'Alembert's equation on whole $(-\infty, \infty)$. According to [2], the infinitesimal generator of an

$\mathcal{L}(E; E)$ -valued cosine function \mathcal{C} is the linear operator A from E to E , with the domain $D(A)$, defined by the conditions

$$D(A) = \left\{ x: x \in E, \lim_{t \rightarrow 0} \frac{2}{t^2} (\mathcal{C}(t)x - x) \text{ exists} \right\},$$

$$Ax = \lim_{t \rightarrow 0} \frac{2}{t^2} (\mathcal{C}(t)x - x) \quad \text{for } x \in D(A),$$

the limit taken in the sense of the norm in E .

As proved by Sova [2], if E is a Banach space and if \mathcal{C} is an $\mathcal{L}(E; E)$ -valued strongly continuous cosine function with the infinitesimal generator A , then

(1.3.1.) the domain $D(A)$ of A is dense in E and A is a closed operator from E to E ;

(1.3.2.) $\mathcal{C}(t)D(A) \subset D(A)$ for every $t \in (-\infty, \infty)$ and, for every fixed $x \in D(A)$, the E -valued function $t \rightarrow \mathcal{C}(t)x$ is twice continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E and

$$\frac{d^2 \mathcal{C}(t)x}{dt^2} = A\mathcal{C}(t)x = \mathcal{C}(t)Ax, \quad -\infty < t < \infty.$$

The following lemma will be also useful in our further reasonings.

(1.3.3.) **LEMMA.** Let E be a Banach space, let \mathcal{C} be an $\mathcal{L}(E; E)$ -valued function strongly continuous on $(-\infty, \infty)$ and let A be a closed linear operator from E to E with a domain $D(A)$ dense in E . If

(a) $\mathcal{C}(t)D(A) \subset D(A)$ for every $t \in (-\infty, \infty)$

and

(b) for any fixed $x \in D(A)$ the E -valued function $t \rightarrow \mathcal{C}(t)x$ is twice strongly continuously differentiable on $(-\infty, \infty)$ and we have

$$\frac{d^2 \mathcal{C}(t)x}{dt^2} = A\mathcal{C}(t)x = \mathcal{C}(t)Ax \quad \text{for } t \in (-\infty, \infty),$$

$$\mathcal{C}(0)x = x \quad \text{and} \quad \left. \frac{d\mathcal{C}(t)x}{dt} \right|_{t=0} = 0,$$

then $\mathcal{C}(t)$ is a cosine function and A is its infinitesimal generator.

Proof. In order to prove that \mathcal{C} satisfies the d'Alembert functional equation it is sufficient to show that for any fixed $x_0 \in D(A)$ and any fixed s the E -valued function

$$x(t) = \mathcal{C}(t+s)x_0 + \mathcal{C}(s-t)x_0 - 2\mathcal{C}(s)\mathcal{C}(t)x_0$$

vanishes identically on $(-\infty, \infty)$. As it is easy to see, the function $t \rightarrow \omega(t)$ has the following properties:

- (α) $x(t) \in D(A)$ for every $t \in (-\infty, \infty)$,
- (β) $t \rightarrow x(t)$ is twice continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E and $x''(t) = Ax(t)$ for $t \in (-\infty, \infty)$,
- (γ) $x(0) = 0$,
- (δ) $x'(0) = \mathcal{C}'(s)x - \mathcal{C}'(s)x - 2\mathcal{C}(s)\mathcal{C}'(0)x = 0$.

If we define a norm on $D(A)$ by

$$\|x\|_{D(A)} = \|x\|_E + \|Ax\|_E$$

then, by the closedness of A , $D(A)$ with the norm $\|\cdot\|_{D(A)}$ is a Banach space. Moreover, by (α) and (β), $t \rightarrow x(t)$ is a $D(A)$ -valued function continuous on $(-\infty, \infty)$ in the sense of the norm $\|\cdot\|_{D(A)}$ and, by (b), $t \rightarrow \mathcal{C}(t)|_{D(A)}$ is an $\mathcal{L}(D(A); E)$ -valued function twice strongly continuously differentiable on $(-\infty, \infty)$. For any $t \in (-\infty, \infty)$ let $\mathcal{C}'(t) \in \mathcal{L}(D(A); E)$ and $\mathcal{C}''(t) \in \mathcal{L}(D(A); E)$ denote the corresponding derivatives at the point t . It follows that for any fixed $t \in (-\infty, \infty)$ the E -valued function $\tau \rightarrow y(\tau) = \mathcal{C}(t-\tau)x(\tau)$ is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E and that

$$y'(\tau) = \mathcal{C}(t-\tau)x'(\tau) - \mathcal{C}'(t-\tau)x(\tau), \quad -\infty < \tau < \infty.$$

Since $y(t) = \mathcal{C}(0)x(t) = x(t)$ by (b), and $y(0) = \mathcal{C}(t)x(0) = 0$ by (γ), we have $x(t) = \int_0^t y'(\tau) d\tau$, i.e.

$$x(t) = \int_0^t \{\mathcal{C}(t-\tau)x'(\tau) - \mathcal{C}'(t-\tau)x(\tau)\} d\tau, \quad -\infty < t < \infty.$$

By (δ) and (β) we have $x'(\tau) = \int_0^\tau Ax(\varrho) d\varrho$. By (b) we have $\mathcal{C}'(t-\tau)x(\tau) = \int_0^{t-\tau} \mathcal{C}(\sigma)Ax(\tau) d\sigma$. Therefore

$$x(t) = \int_0^t \left[\int_0^\tau \mathcal{C}(t-\tau)Ax(\varrho) d\varrho \right] d\tau - \int_0^t \left[\int_0^{t-\tau} \mathcal{C}(\sigma)Ax(\tau) d\sigma \right] d\tau = 0,$$

by the Fubini theorem.

So we proved that \mathcal{C} is a cosine function and now it follows at once from (b) that if A_0 denotes the infinitesimal generator of \mathcal{C} then $A \subset A_0$, i.e. $D(A) \subset D(A_0)$ and $Ax = A_0x$ for $x \in D(A)$. In order to prove that $A = A_0$ it is sufficient to apply Lemma 2 from Section 3 of the author's paper [1].

However, the reasoning in [1] is complicated by the fact that the general case of a locally convex sequentially complete space is treated

there. Therefore we shall repeat the argumentation in a simplified version for a Banach space. We have to prove that $D(A_0) \subset D(A)$.

If $x \in D(A)$ then, by (b) and by the closedness of A , we have $\mathcal{C}(t)x - x = \int_0^t \int_0^\tau A\mathcal{C}(\sigma)x d\sigma d\tau = A \int_0^t \int_0^\tau \mathcal{C}(\sigma)x d\sigma d\tau$. Again by the closedness of A and by the fact that $D(A)$ is dense in E it follows that

$$(*) \quad \int_0^t \int_0^\tau \mathcal{C}(\sigma)x d\sigma d\tau \in D(A) \quad \text{and} \quad A \int_0^t \int_0^\tau \mathcal{C}(\sigma)x d\sigma d\tau = \mathcal{C}(t)x - x$$

for every $x \in E$ and $t \in (-\infty, \infty)$. If $x \in D(A_0)$ then $\lim_{t \rightarrow 0} \frac{2}{t^2} (\mathcal{C}(t)x - x) = A_0x$ and $\lim_{t \rightarrow 0} \frac{2}{t^2} \int_0^t \int_0^\tau \mathcal{C}(\sigma)x d\sigma d\tau = x$, so that, by (*) and by the closedness of A , $x \in D(A)$ and $Ax = A_0x$. Consequently $D(A_0) \subset D(A)$ and the proof is complete.

1.4. In the sequel we shall consider one-parameter groups of operators belonging to $\mathcal{L}(E_1 \times E_0; E_1 \times E_0)$, where E_0 and E_1 are Banach spaces. It will be convenient to write elements of $E_1 \times E_0$ in the form of columns $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x \in E_1$, $y \in E_0$, and to represent any operator $B \in \mathcal{L}(E_1 \times E_0; E_1 \times E_0)$ as a matrix

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

whose elements are operators $B_{ik} \in \mathcal{L}(E_{2-k}; E_{2-i})$ defined by the condition that, for every column $\begin{pmatrix} x \\ y \end{pmatrix} \in E_1 \times E_0$,

$$B \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} B_{11}x + B_{12}y \\ B_{21}x + B_{22}y \end{pmatrix},$$

according to the common rule of multiplication of matrices. A similar matricial representation may be used for any linear operator A from $E_1 \times E_0$ into $E_1 \times E_0$, having domain of the form $D(A) = D_1 \times D_0$, where D_i is a linear subset of E_i .

2. Generalization of the formula (*).

THEOREM. Let E_0 be a Banach space and let A be a linear operator from E_0 to E_0 with domain $D(A)$. If E_1 is a Banach space such that $D(A) \subset E_1 \subset E_0$ and that the Banach space topology of E_1 is not weaker than the topology induced in E_1 by E_0 , and if the operator \mathcal{A} from $E_1 \times E_0$ into $E_1 \times E_0$, with domain $D(\mathcal{A})$, defined by the conditions

$$(2.1) \quad D(\mathcal{A}) = D(A) \times E_1, \quad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

is the infinitesimal generator of a strongly continuous one-parameter group $\mathcal{G}: (-\infty, \infty) \rightarrow \mathcal{L}(E_1 \times E_0; E_1 \times E_0)$, then A is the infinitesimal generator of a strongly continuous $\mathcal{L}(E_0; E_0)$ -valued cosine function $\mathcal{C}(t)$,

(2.2) $E_1 = \{x: x \in E_0, \text{ the } E_0\text{-valued function } t \rightarrow \mathcal{C}(t)x \text{ is continuously differentiable on } (-\infty, \infty) \text{ in the sense of the norm in } E_0\}$,

and

$$(2.3) \quad \mathcal{G}(t) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathcal{C}(t)x + \int_0^t \mathcal{C}(\tau)y d\tau \\ \frac{d\mathcal{C}(t)x}{dt} + \mathcal{C}(t)y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \in E_1 \times E_0, -\infty < t < \infty.$$

On the other hand, if A is the infinitesimal generator of a strongly continuous $\mathcal{L}(E_0; E_0)$ -valued cosine function $\mathcal{C}(t)$ and if E_1 is defined by (2.2), then E_1 under the norm

$$(2.4) \quad \|x\|_{E_1} = \|x\|_{E_0} + \sup_{0 \leq t \leq 1} \left\| \frac{d\mathcal{C}(t)x}{dt} \right\|_{E_0}, \quad x \in E_1,$$

becomes a Banach space and the formula (2.3) defines a strongly continuous one-parameter group $\mathcal{G}: (-\infty, \infty) \rightarrow \mathcal{L}(E_1 \times E_0; E_1 \times E_0)$, whose infinitesimal generator is the operator \mathcal{A} defined by the conditions (2.1).

3. Proof of the part "from \mathcal{C} to \mathcal{G} ". Everywhere throughout this section it is assumed that E_0 is a Banach space, \mathcal{C} is a strongly continuous $\mathcal{L}(E_0; E_0)$ -valued cosine function and A is the infinitesimal generator of \mathcal{C} . We define the linear subset E_1 of E_0 by (2.2) and define the norm $\|\cdot\|_{E_1}$ on E_1 by (2.4).

LEMMA 3.1. E_1 under the norm $\|\cdot\|_{E_1}$ is a Banach space. We have $\mathcal{C}(t)|_{E_1} \in \mathcal{L}(E_1; E_1)$, $\int_0^t \mathcal{C}(\tau) d\tau \in \mathcal{L}(E_0; E_1)$ and $\frac{d}{dt} \mathcal{C}(t) \Big|_{E_1} \in \mathcal{L}(E_1; E_0)$ for every $t \in (-\infty, \infty)$, where the integral is the Riemann integral of a strongly continuous $\mathcal{L}(E_0; E_0)$ -valued function and the derivative is taken in the sense of the strong topology in $\mathcal{L}(E_1; E_0)$. Moreover, the mappings

$$(a) \quad (-\infty, \infty) \ni t \rightarrow \mathcal{C}(t)|_{E_1} \in \mathcal{L}(E_1; E_1),$$

$$(b) \quad (-\infty, \infty) \ni t \rightarrow \int_0^t \mathcal{C}(\tau) d\tau \in \mathcal{L}(E_0; E_1),$$

$$(c) \quad (-\infty, \infty) \ni t \rightarrow \frac{d}{dt} \mathcal{C}(t) \Big|_{E_1} \in \mathcal{L}(E_1; E_0)$$

are strongly continuous.

Proof. It follows from the d'Alembert functional equation, that if the function $t \rightarrow \mathcal{C}(t)x$ is continuously differentiable on $[0, 1]$ in the sense of the norm in E_0 , then $x \in E_1$. Therefore, by the theorem on term by term

differentiation, E_1 is a complete space under the norm $\|\cdot\|_{E_1}$. If $x \in E_1$ and $s \in (-\infty, \infty)$ are fixed then the E_0 -valued function $t \rightarrow \mathcal{C}(t)\mathcal{C}(s)x = \mathcal{C}(s)\mathcal{C}(t)x$ is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E_0 , so that $\mathcal{C}(s)E_1 \subset E_1$. Since, for $x \in E_1$,

$$\begin{aligned} \|\mathcal{C}(t)x\|_{E_1} &= \|\mathcal{C}(t)x\|_{E_0} + \sup_{0 \leq s \leq 1} \left\| \frac{d}{ds} (\mathcal{C}(s)\mathcal{C}(t)x) \right\|_{E_0} \\ &= \|\mathcal{C}(t)x\|_{E_0} + \sup_{0 \leq s \leq 1} \left\| \mathcal{C}(t) \frac{d\mathcal{C}(s)x}{ds} \right\|_{E_0} \\ &\leq \|\mathcal{C}(t)\|_{\mathcal{L}(E_0; E_0)} \|x\|_{E_1}, \end{aligned}$$

we see that $\mathcal{C}(t)|_{E_1} \in \mathcal{L}(E_1; E_1)$. Moreover, if $x \in E_1$, then, by the d'Alembert equation,

$$\begin{aligned} \|\mathcal{C}(t+h)x - \mathcal{C}(t)x\|_{E_1} &= \|\mathcal{C}(t+h)x - \mathcal{C}(t)x\|_{E_0} \\ &+ \frac{1}{2} \sup_{0 \leq s \leq 1} \left\| \frac{d\mathcal{C}(t+h+s)x}{dt} - \frac{d\mathcal{C}(t+h-s)x}{dt} - \frac{d\mathcal{C}(t+s)x}{dt} + \frac{d\mathcal{C}(t-s)x}{dt} \right\|_{E_0}, \end{aligned}$$

whence it follows that the mapping (a) is strongly continuous. This implies that if $x \in E_1$ then also $\int_0^t \mathcal{C}(\tau)x d\tau \in E_1$ and

$$\begin{aligned} \frac{d}{ds} \left(\mathcal{C}(s) \int_0^t \mathcal{C}(\tau)x d\tau \right) &= \frac{d}{ds} \int_0^t \mathcal{C}(s)\mathcal{C}(\tau)x d\tau = \frac{1}{2} \frac{d}{ds} \int_0^t \{ \mathcal{C}(\tau+s)x + \mathcal{C}(\tau-s)x \} d\tau \\ &= \frac{1}{2} \int_0^t \left\{ \frac{d\mathcal{C}(\tau+s)x}{d\tau} - \frac{d\mathcal{C}(\tau-s)x}{d\tau} \right\} d\tau = \frac{1}{2} \mathcal{C}(t+s)x - \frac{1}{2} \mathcal{C}(t-s)x. \end{aligned}$$

Now let $x \in E_0$. It follows from 1.3.1 and 1.3.2 that E_1 is dense in E_0 and so there is a sequence x_n , $n = 1, 2, \dots$, of elements of E_1 , such that

$\lim_{n \rightarrow \infty} \|x_n - x\|_{E_0} = 0$. If, for a fixed t , we put $y_n = \int_0^t \mathcal{C}(\tau)x_n d\tau$, then

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq 1} \left\| \mathcal{C}(s)y_n - \mathcal{C}(s) \int_0^t \mathcal{C}(\tau)x d\tau \right\|_{E_0} = 0$$

and

$$\frac{d\mathcal{C}(s)y_n}{ds} = \frac{1}{2} \mathcal{C}(t+s)x_n - \frac{1}{2} \mathcal{C}(t-s)x_n,$$

so that $\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq 1} \left\| \frac{d\mathcal{G}(s)y_n}{ds} - \frac{1}{2}\mathcal{G}(t+s)x + \frac{1}{2}\mathcal{G}(t-s)x \right\|_{E_0} = 0$. Hence, by the theorem on term by term differentiation,

(3.1)

$$\int_0^t \mathcal{G}(\tau)x d\tau \in E_1 \quad \text{and} \quad \frac{d}{ds} \left(\mathcal{G}(s) \int_0^t \mathcal{G}(\tau)x d\tau \right) = \frac{1}{2}\mathcal{G}(t+s)x - \frac{1}{2}\mathcal{G}(t-s)x$$

for every $x \in E_0$ and $s, t \in (-\infty, \infty)$. From (3.1) it follows immediately that $\int_0^t \mathcal{G}(\tau) d\tau \in \mathcal{L}(E_0; E_1)$ and that the mapping (b) is strongly continuous. The statements that $\frac{d}{dt} \mathcal{G}(t)|_{E_1} \in \mathcal{L}(E_1; E_0)$ and that the mapping (c) is strongly continuous are trivial consequences of the definitions of E_1 and $\|\cdot\|_{E_1}$.

LEMMA 3.2. *The formula (2.3) defines a one-parameter strongly continuous group $(-\infty, \infty) \ni t \rightarrow \mathcal{G}(t) \in \mathcal{L}(E_1 \times E_0; E_1 \times E_0)$, whose infinitesimal generator is the operator \mathcal{A} defined by (2.1).*

Proof. It follows from Lemma 3.1 that $\mathcal{G}(t) \in \mathcal{L}(E_1 \times E_0; E_1 \times E_0)$ for every $t \in (-\infty, \infty)$, and that the mapping $(-\infty, \infty) \ni t \rightarrow \mathcal{G}(t) \in \mathcal{L}(E_1 \times E_0; E_1 \times E_0)$ is strongly continuous. For any $\begin{pmatrix} x \\ y \end{pmatrix} \in E_1 \times E_0$ and $t, s \in (-\infty, \infty)$ we have

$$\begin{aligned} & \mathcal{G}(t)\mathcal{G}(s) \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{G}(t)\mathcal{G}(s)x + \int_0^t \mathcal{G}(\tau) d\tau \frac{d\mathcal{G}(s)x}{ds} + \mathcal{G}(t) \int_0^s \mathcal{G}(\sigma)y d\sigma + \int_0^t \mathcal{G}(\tau) d\tau \mathcal{G}(s)y \\ \frac{d}{dt} (\mathcal{G}(t)\mathcal{G}(s)x) + \mathcal{G}(t) \frac{d\mathcal{G}(s)x}{ds} + \frac{d}{dt} \left(\mathcal{G}(t) \int_0^s \mathcal{G}(\sigma)y d\sigma \right) + \mathcal{G}(t)\mathcal{G}(s)y \end{pmatrix} \end{aligned}$$

and so, in order to prove that \mathcal{G} is a group, we have to show that the following equalities hold for any $s, t \in (-\infty, \infty)$, $x \in E_1$ and $y \in E_0$,

$$1^\circ \quad \mathcal{G}(t)\mathcal{G}(s)x + \int_0^t \mathcal{G}(\tau) d\tau \frac{d\mathcal{G}(s)x}{ds} = \mathcal{G}(t+s)x,$$

$$2^\circ \quad \mathcal{G}(t) \int_0^s \mathcal{G}(\sigma)y d\sigma + \int_0^t \mathcal{G}(\tau) d\tau \mathcal{G}(s)y = \int_0^{t+s} \mathcal{G}(\tau)y d\tau,$$

$$3^\circ \quad \frac{d}{dt} (\mathcal{G}(t)\mathcal{G}(s)x) + \mathcal{G}(t) \frac{d\mathcal{G}(s)x}{ds} = \frac{d\mathcal{G}(t+s)x}{dt},$$

$$4^\circ \quad \frac{d}{dt} \left(\mathcal{G}(t) \int_0^s \mathcal{G}(\sigma)y d\sigma \right) + \mathcal{G}(t)\mathcal{G}(s)y = \mathcal{G}(t+s)y.$$

It is easy to see, 3° and 4° follow from 1° and 2° by differentiation. If $x \in E_1$, then

$$\begin{aligned} \int_0^t \mathcal{G}(\tau) \frac{d\mathcal{G}(s)x}{ds} d\tau &= \int_0^t \frac{d}{ds} (\mathcal{G}(\tau)\mathcal{G}(s)x) d\tau = \frac{1}{2} \frac{d}{ds} \int_0^t (\mathcal{G}(\tau+s)x + \mathcal{G}(s-\tau)x) d\tau \\ &= \frac{1}{2} \int_0^t \frac{d}{d\tau} (\mathcal{G}(\tau+s)x - \mathcal{G}(s-\tau)x) d\tau = \frac{1}{2} \mathcal{G}(t+s)x - \frac{1}{2} \mathcal{G}(s-t)x \\ &= \mathcal{G}(t+s)x - \mathcal{G}(t)\mathcal{G}(s)x \end{aligned}$$

and so 1° is proved. Recalling that $\mathcal{G}(t)$ is a pair function of t we have

$$\begin{aligned} \mathcal{G}(t) \int_0^s \mathcal{G}(\sigma) d\sigma + \int_0^t \mathcal{G}(\tau) d\tau \mathcal{G}(s) &= \frac{1}{2} \int_0^s [\mathcal{G}(t+\sigma) + \mathcal{G}(t-\sigma)] d\sigma + \frac{1}{2} \int_0^t [\mathcal{G}(\tau-s) + \\ &+ \mathcal{G}(\tau+s)] d\tau = \frac{1}{2} \left[\int_t^{t+s} + \int_{t-s}^t + \int_{-s}^{t-s} + \int_{-t-s}^{-s} \right] \mathcal{G}(\tau) d\tau = \frac{1}{2} \int_{-t-s}^{t+s} \mathcal{G}(\tau) d\tau = \int_0^{t+s} \mathcal{G}(\tau) d\tau \end{aligned}$$

and so 2° is proved. Therefore \mathcal{G} is a one-parameter group. Let \mathcal{A}_0 be its infinitesimal generator. If $x \in D(\mathcal{A})$ then, by 1.3.2 and by Lemma 3.1,

$t \rightarrow \int_0^t \mathcal{G}(\tau) \mathcal{A} x d\tau = \frac{d\mathcal{G}(t)x}{dt}$ is an E_1 -valued function continuous on $(-\infty, \infty)$ in the sense of the norm $\|\cdot\|_{E_1}$. Since $\mathcal{G}(t)$ is a pair function of t , it follows that $\lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{G}(t)x - x) = 0$ in the sense of the norm $\|\cdot\|_{E_1}$. Moreover

$$\lim_{t \rightarrow 0} \frac{1}{t} \frac{d\mathcal{G}(t)x}{dt} = \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \mathcal{G}(\tau) \mathcal{A} x d\tau = \mathcal{A} x$$

in the sense of the norm $\|\cdot\|_{E_0}$. If $y \in E_1$ then, by Lemma 3.1, $t \rightarrow \mathcal{G}(t)y$ is an E_1 -valued function continuous on $(-\infty, \infty)$ in the sense of the norm $\|\cdot\|_{E_1}$, so that

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \mathcal{G}(\tau)y d\tau = \mathcal{G}(0)y = y$$

in the sense of the norm $\|\cdot\|_{E_1}$. Moreover, if $y \in E_1$ then, since $\mathcal{G}(t)$ is pair,

$$\lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{G}(t)y - y) = 0$$

in the sense of the norm $\|\cdot\|_{E_0}$. It follows, that if $\begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathcal{A}) = D(\mathcal{A}) \times E_1$ then

$$\lim_{t \rightarrow 0} \frac{1}{t} \left(\mathcal{G}(t) \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} y \\ \mathcal{A}x \end{pmatrix} = \mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

in the sense of the norm in $E_1 \times E_0$. This means that $\mathcal{A} \subset \mathcal{A}_0$.

On the other hand, if $\begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathcal{A}_0)$ then, by 1.2.3, the $(E_1 \times E_0)$ -valued function

$$t \rightarrow \mathcal{G}(t) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathcal{G}(t)x + \int_0^t \mathcal{G}(\tau)y d\tau \\ \frac{d\mathcal{G}(t)x}{dt} + \mathcal{G}(t)y \end{pmatrix}$$

is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in $E_1 \times E_0$. It follows that $\frac{d\mathcal{G}(t)x}{dt} + \mathcal{G}(t)y \in E_1$ for every $t \in (-\infty, \infty)$ and so $y = \left(\frac{d\mathcal{G}(t)x}{dt} + \mathcal{G}(t)y \right) \Big|_{t=0} \in E_1$. Therefore $t \rightarrow \frac{d\mathcal{G}(t)x}{dt} = \left(\frac{d\mathcal{G}(t)x}{dt} + \mathcal{G}(t)y \right) - \mathcal{G}(t)y$ is an E_0 -valued function continuously differentiable on $(-\infty, \infty)$ in the sense of the norm $\| \cdot \|_{E_0}$, and so $x \in D(A)$. Hence $D(\mathcal{A}_0) \subset D(A) \times E_1 = D(\mathcal{A})$. Since we already know, that $\mathcal{A} \subset \mathcal{A}_0$, it follows that $\mathcal{A} = \mathcal{A}_0$ and the proof is complete.

4. Proof of the part "from \mathcal{G} to \mathcal{C} ". Everywhere throughout this section it is assumed that E_1 and E_0 are Banach spaces such that $E_1 \subset E_0$ and the Banach space topology of E_1 is not weaker than the topology induced in E_1 by E_0 .

Moreover, it is assumed that A is a linear operator from E_1 to E_0 and that the operator \mathcal{A} defined by (2.1) is the infinitesimal generator of a strongly continuous one parameter group $\mathcal{G}: (-\infty, \infty) \rightarrow \mathcal{L}(E_1 \times E_0; E_1 \times E_0)$. We have to prove that A is the infinitesimal generator of a strongly continuous $\mathcal{L}(E_0; E_0)$ -valued cosine function $\mathcal{C}(t)$ and that (2.2) and (2.3) hold.

LEMMA 4.1. *The operator A is closed as an operator from E_0 to E_0 and its domain $D(A)$ is dense in E_0 .*

Proof. We have

$$D(\mathcal{A}^2) = \{x: x \in D(A), Ax \in E_1\} \times D(A),$$

$$\mathcal{A}^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax \\ Ay \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathcal{A}^2),$$

and, according to 1.2.2, \mathcal{A}^2 is a closed operator from $E_1 \times E_0$ to $E_1 \times E_0$, with the domain $D(\mathcal{A}^2)$ dense in $E_1 \times E_0$.

LEMMA 4.2. *The operator A is the infinitesimal generator of an $\mathcal{L}(E_0; E_0)$ -valued strongly continuous cosine function \mathcal{C} such that $\mathcal{C}(t)E_1 \subset E_1$ for every $t \in (-\infty, \infty)$, and for any fixed $x \in E_1$ the E_1 -valued function $t \rightarrow \mathcal{C}(t)x$ is continuous on $(-\infty, \infty)$ in the sense of the norm in E_1 and is continuously*

differentiable on $(-\infty, \infty)$ in the sense of the norm in E_0 . Moreover, the formula (2.3) holds.

Proof. Represent $\mathcal{G}(t)$ as a matrix

$$\mathcal{G}(t) = \begin{pmatrix} G_{11}(t) & G_{12}(t) \\ G_{21}(t) & G_{22}(t) \end{pmatrix}.$$

Then any $t \rightarrow G_{ik}(t)$ is a strongly continuous $\mathcal{L}(E_{2-k}; E_{2-i})$ -valued function and it follows from 1.2.3 that for any $\begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathcal{A})$, i.e. for $x \in D(A)$ and $y \in E_1$, and for any $t \in (-\infty, \infty)$ we have

$$(4.1) \quad \begin{pmatrix} \frac{dG_{11}(t)x}{dt} + \frac{dG_{12}(t)y}{dt} \\ \frac{dG_{21}(t)x}{dt} + \frac{dG_{22}(t)y}{dt} \end{pmatrix} = \begin{pmatrix} G_{21}(t)x + G_{22}(t)y \\ AG_{11}(t)x + AG_{12}(t)y \end{pmatrix} = \begin{pmatrix} G_{12}(t)Ax + G_{11}(t)y \\ G_{22}(t)Ax + G_{21}(t)y \end{pmatrix},$$

where the derivatives in the first row are taken in the sense of the norm in E_1 and the derivatives in the second row are taken in the sense of the norm in E_0 .

From these equalities and from the fact that $D(\mathcal{A})$ is dense in $E_1 \times E_0$ it is easy to see that for $\mathcal{C}(t) = G_{22}(t)$ all the continuity and differentiability properties stated in the lemma are valid and moreover the formula (2.2) holds. Therefore it remains only to prove that $\mathcal{C}(t)$ is an $\mathcal{L}(E_0; E_0)$ -valued cosine function and that A is its infinitesimal generator.

To that end we shall apply Lemma 1.3.3.

Let operators $\pi_0 \in \mathcal{L}(E_1 \times E_0; E_0)$ and $J_0 \in \mathcal{L}(E_0; E_1 \times E_0)$ be defined by the formulae

$$\pi_0 \begin{pmatrix} x \\ y \end{pmatrix} = y, \quad J_0 y = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Then $\mathcal{C}(t) = \pi_0 \mathcal{G}(t) J_0$, $J_0 D(A) \subset D(\mathcal{A}^2)$, $\pi_0 D(\mathcal{A}^2) = D(A)$, $J_0 A = \mathcal{A}^2 J_0$, $\pi_0 \mathcal{A}^2 = A \pi_0$, so that, by 1.2.2 and 1.2.3,

$$(i) \quad \mathcal{C}(t)D(A) = \pi_0 \mathcal{G}(t) J_0 D(A) \subset \pi_0 \mathcal{G}(t) D(\mathcal{A}^2) = \pi_0 D(\mathcal{A}^2) = D(A)$$

and

$$(ii) \quad \frac{d^2 \mathcal{C}(t)x}{dt^2} = \pi_0 \frac{d^2 \mathcal{G}(t) J_0 x}{dt^2} = \pi_0 \mathcal{A}^2 \mathcal{G}(t) J_0 x = \pi_0 \mathcal{G}(t) \mathcal{A}^2 J_0 x \\ = A \pi_0 \mathcal{G}(t) J_0 x = \pi_0 \mathcal{G}(t) J_0 Ax = A \mathcal{C}(t)x = \mathcal{C}(t)Ax$$

for any $t \in (-\infty, \infty)$ and $x \in D(A)$. We know from Lemma 4.1 that A is closed and $D(A)$ is dense in E_0 . Therefore all the assumptions of Lemma 1.3.3 are satisfied and consequently \mathcal{C} is a cosine function and A is its infinitesimal generator.

LEMMA 4.3. *The equality (2.2) is true.*

Proof. Let

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{E_1 \times E_0} = \|x\|_{E_1} + \|y\|_{E_0}$$

and put

$E_1^0 = \{x: x \in E_0, \text{ the } E_0\text{-valued function } t \rightarrow \mathcal{G}(t)x \text{ is continuously differentiable on } (-\infty, \infty) \text{ in the sense of the norm in } E_0\}$

and

$$\|x\|_{E_1^0} = \|x\|_{E_0} + \sup_{0 \leq t \leq 1} \left\| \frac{d\mathcal{G}(t)x}{dt} \right\|_{E_0}$$

for every $x \in E_1^0$. According to Lemma 3.1, E_1^0 under the norm $\|\cdot\|_{E_1^0}$ is a Banach space. According to 1.2.1, there are constants $\lambda > 0$ and $M \geq 1$ such that

$$(4.2) \quad \left\| \mathcal{G}(t) \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{E_1 \times E_0} \leq M e^{\lambda|t|} \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{E_1 \times E_0}$$

for every $\begin{pmatrix} x \\ y \end{pmatrix} \in E_1 \times E_0$ and $t \in (-\infty, \infty)$. From 1.2.2 and from Lemma 3.2 it follows that $D(A)$ is dense in E_1^0 in the sense of the norm $\|\cdot\|_{E_1^0}$. From Lemma 4.2 it follows, that $E_1 \subset E_1^0$. Since $D(A) \subset E_1$, it follows, that E_1 is dense in E_1^0 in the sense of the norm $\|\cdot\|_{E_1^0}$. Therefore the equality $E_1^0 = E_1$ will follow, if we shall show, that there is a constant C , such that

$$(4.3) \quad \|x\|_{E_1} \leq C \|x\|_{E_1^0} \quad \text{for every } x \in E_1.$$

If $x \in E_1$, then, by Lemma 4.2, $\mathcal{G}(t)x$ is an E_1 -valued function of t , continuous on $(-\infty, \infty)$ in the sense of the norm $\|\cdot\|_{E_1}$ and so, by the Lemma 4.2, by (2.3) and by (3.1), we have

$$\begin{aligned} \|x\|_{E_1} &\leq \frac{1}{2} \left\| \int_0^2 \mathcal{G}(t)x dt \right\|_{E_1} + \frac{1}{2} \left\| \int_0^2 (x - \mathcal{G}(t)x) dt \right\|_{E_1} \\ &\leq D \|x\|_{E_0} + \sup_{s, t \in [0, 1]} \|\mathcal{G}(t+s)x - \mathcal{G}(t-s)x\|_{E_1} \\ &= D \|x\|_{E_0} + 2 \sup_{s, t \in [0, 1]} \left\| \frac{d}{ds} \left(\mathcal{G}(s) \int_0^t \mathcal{G}(\tau)x d\tau \right) \right\|_{E_1} \end{aligned}$$

for every $x \in E_1$, where $D = \frac{1}{2} \left\| \int_0^2 \mathcal{G}(t) dt \right\|_{\mathcal{L}(E_0; E_1)}$. Therefore, inequality

(4.3) will be proved, if we shall show, that

$$(4.4) \quad \left\| \frac{d}{ds} \left(\mathcal{G}(s) \int_0^t \mathcal{G}(\tau)x d\tau \right) \right\|_{E_1} \leq M e^{\lambda|s|} \left\| \frac{d\mathcal{G}(t)x}{dt} \right\|_{E_0}$$

for every $x \in E_1$ and $s, t \in (-\infty, \infty)$.

If $x \in D(A)$, then $\begin{pmatrix} x \\ 0 \end{pmatrix} \in D(\mathcal{A})$, so that, by 1.2.3, $\mathcal{G}(t) \begin{pmatrix} x \\ 0 \end{pmatrix} \in D(\mathcal{A})$ and consequently $\frac{d\mathcal{G}(t)x}{dt} = \pi_0 \mathcal{G}(t) \begin{pmatrix} x \\ 0 \end{pmatrix} \in E_1$. Let the operator $\pi_1 \in \mathcal{L}(E_1 \times E_0; E_1)$ be defined by the formula $\pi_1 \begin{pmatrix} x \\ y \end{pmatrix} = x$. If $x \in D(A)$, then, by the Lemma 4.2, by (2.3) and 1.2.3, and by inequality (4.2), we have

$$\left\| \frac{d\mathcal{G}(s)x}{ds} \right\|_{E_1} = \left\| \pi_1 \mathcal{G}(s) \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|_{E_1} = \left\| \pi_1 \mathcal{G}(s) \begin{pmatrix} 0 \\ Ax \end{pmatrix} \right\|_{E_1} \leq M e^{\lambda|s|} \|Ax\|_{E_0}.$$

If $x \in E_1$, then $\begin{pmatrix} 0 \\ x \end{pmatrix} \in D(\mathcal{A})$, so that also $\mathcal{G}(t) \begin{pmatrix} 0 \\ x \end{pmatrix} \in D(\mathcal{A})$ and hence $\int_0^t \mathcal{G}(\tau)x d\tau = \pi_1 \mathcal{G}(t) \begin{pmatrix} 0 \\ x \end{pmatrix} \in D(A)$. It follows, that if $x \in E_1$, then

$$\left\| \frac{d}{ds} \left(\mathcal{G}(s) \int_0^t \mathcal{G}(\tau)x d\tau \right) \right\|_{E_1} \leq M e^{\lambda|s|} \left\| A \int_0^t \mathcal{G}(\tau)x d\tau \right\|_{E_0}.$$

But from equality (4.1) we have, that if $x \in E_1$, then $A \int_0^t \mathcal{G}(\tau)x d\tau = AG_{12}(t)x = G_{21}(t)x = \frac{d\mathcal{G}(t)x}{dt}$ and so, inequality (4.4) is proved. This completes the proof of Lemma 4.3 and, at the same time, the whole proof of the theorem from Section 2.

References

- [1] J. Kiszyński, *On operator valued solutions of d'Alembert's functional equation*, II, *Studia Math.* 42 (1971), pp. 43-66.
- [2] M. Sova, *Cosine operator functions*, *Dissertationes Math.* 49 (1966), pp. 1-47.
- [3] K. Yosida, *Functional Analysis*, 1965.

Received June 19, 1971

(351)