An alternate proof of Pettis' Theorem. Let J be a minimal right ideal in $\bar{S}$. Since each element in J is affine, it follows from Rosen's Theorem [7], Theorem 1, and Day's fixed point theorem [1], Theorem 3, that K must contain a fixed point x for J.

Remark. If S has finite intersection property for right ideals (which is the case when S is commutative or when S is a group) and equicontinuous, then $\bar{S}$ has a unique minimal right ideal J. It follows that $\varphi(J) = J$ for all $\varphi \in \bar{S}$. Hence in this case, the element $x \in K$ chosen in the proof of our main theorem (and that of Pettis) is even a common fixed point for $\bar{S}$, and consequently for S (see [4] and [5]).

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Intertwining operators

by

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Abstract. This paper deals with operators which intertwine semi-spectral measures and subnormal operator representations of function algebras. It is shown thar such operators extend uniquely to those which intertwine dilations of measures or extensions of representations respectively. The function algebras in question are approximating in modulus. The extension mapping preserves several properties of operators.

The present paper deals with intertwining operators for representations of function algebras and for semi-spectral measures. We are interested mostly in subnormal representations and their intertwining operators. The principal question is when one can extend the intertwining operator so that the extension intertwines the extensions of representations. That this is not always possible is shown by an example given in [6]. However, if the function algebra satisfies a certain approximation property first defined by Glicksberg in [9], then intertwining operators for subnormal representations extend to interwining ones for minimal $*$-extensions of representations. But intertwining operators for $*$-representations are suitably decomposable. Consequently, we are able to describe some analytical properties of intertwining operators for subnormal representations at least in separable case.

We use the methods of dilation theory. As to this theory we refer to [1], [2], [13] and [23]. For references in $*$-representations of C^*-algebras see [3], for references in function algebras [8].

1. Let Z be a compact Hausdorff space. The uniformly closed subalgebra $A \subset C(Z)$ is called a *function algebra* on Z, if $1 \in A$ and the functions in A separate the points of Z. $\|\cdot\|$ is the sup-norm in $C(Z)$.

Suppose we are given two non-trivial complex Hilbert spaces S' and S''. The space of all linear bounded operator $X\colon S' \to S''$ is denoted by $L(S', S'')$. We write $L(S') = L(S', S')$. $I_{S'}, I_{S''}$ stand for the identity operators in S' and S'' respectively.

The algebra homomorphism $T\colon A \to L(S)$ (S — a complex Hilbert space) of the function algebra on Z is called a *representation* of A if:

$$T(1) = I_S, \tag{1.1}$$

$$\|T(u)\| \leqslant \|u\| \quad \text{for } u \in A. \tag{1.2}$$

In what follows we consider merely Borel regular measures on Z. The space of such measures is identified via Riesz–Kakutani theorem with the dual of $C(Z)$, with the total variation norm $\|p\|$. If $T\colon A\to L(S)$ is a representation of the function algebra A on Z, then there is a system $\{p(f,g)\colon f,g\in S\}$ of measures on Z such that

$$(1.3)\quad \big(T(u)f,g\big)=\int u\,dp(f,g),\qquad \|p(f,g)\|\leqslant\|f\|\cdot\|g\|\qquad \text{for } u\in A;\ f,g\in S.$$

The measures $p(f,g)$ which satisfy (1.3) are called *elementary measures* of T.

Let $T'\colon A\to L(S')$, $T''\colon A\to L(S'')$ be two representations of the function algebra A. We say that the operator $X\in L(S',S'')$ is (T',T'')-*intertwining* if $XT'(u)=T''(u)X$ for all $u\in A$. If $Q'\in L(S')$ and $Q''\in L(S'')$, then $X\in L(S',S'')$ is (Q',Q'')-intertwining if $XQ'=Q''X$. The ordered pair (T',T'') ((Q',Q'') resp.) is called *disjoint*, if the only (T',T'')-intertwining ((Q',Q'')-intertwining respectively) operator is the zero operator.

2. Suppose A is a function algebra on Z and let T',T'' be the representations of A into $L(S')$ and $L(S'')$ respectively. Let $\{G_a\}$ be the totality of all Gleason parts of A. Then T' and T'' have decompositions

$$(2.1)\qquad T'=T_0'\oplus(\oplus T_a'),\qquad T''=T_0''\oplus(\oplus T_a'')$$

(see [14] and [20] for definitions and proofs), where T_a', T_a'' are representations of A having G_a-continuous elementary measures and T_0',T_0'' have completely singular elementary measures. We call for convenience the T_0',T_0'' parts the G_0-continuous ones. Let $P_0'^{('')}$, $P_a'^{('')}$ be the orthogonal projections on representation subspaces of $T_0'^{('')}$ and $T_a'^{('')}$ respectively. We will prove that

(2.2) *If X is an (T',T'')-intertwining operator, then $P_\beta''XP_a'=0$ if $a\neq\beta$.*

Proof. Let p',p'' be elementary measures of T' and T'' respectively and let $f'\in S'$, $f''\in S''$. Since P_β'' and P_a' commute with T'' and T' respectively, we have

$$\begin{aligned}\big(T''(u)P_\beta''XP_a'f',f''\big)&=\big(P_\beta''T''(u)XP_a'f',f''\big)\\&=\big(P_\beta''XT'(u)P_a'f',f''\big)=\big(P_\beta''XP_a'T'(u)f',f''\big).\end{aligned}$$

Denote by p_β'' the G_β-part of the measure $p''(XP_a'f',f'')$ and by p_a' the G_a-part of $p'(f',X^*P_\beta''f'')$. It follows that the measure $p_\beta''-p_a'$ is orthogonal to A. Since $a\neq\beta$, the abstract M. and F. Riesz theorem, yields $(P_\beta''XP_a'f',f'')=\int 1\,dp_\beta''=0$, q.e.d.

Next we have

THEOREM 2.1. *Let* (2.1) *be the decompositions of T' and T'' into G_a-continuous parts. Then every (T',T'')-intertwining operator X is uniquely represented as $X=\oplus X_\beta$, where $X_\beta\in L(P_\beta'S',P_\beta''S'')$ is an (T_a',T_a'')-intertwining operator. Every operator X of the above form is (T',T'')-intertwining.*

Proof. The last part of the asserion is trivial. Let be X be (T',T'')-intertwining. It follows from 2.2 that $X=\oplus P_a''XP_a'$ ($a=0$ included). Write $X_a=P_a''XP_a'$ and regard X as an element of $L(P_a'S',P_a''S'')$. That X_a is (T_a',T_a'')-intertwining follows easily.

A theorem similar to Theorem 2.1 can be formulated for Bishop's decompositions of T',T'' — see [15] for basic properties. Even more, a general technique may developed to obtain abstract generalizations of Theorem 2.1 based on an "axiomatic" version of the M. and F. Riesz theorem — the property (R) of [16] and [22]. In what follows we will present some examples and illustrations related to Theorem 2.1 and spectral sets. We refer here to [12], [18], [19], [20] and [23] for other references on spectral sets.

Suppose Z is a compact subset of the complex plane. Let Z be a spectral set for $T'\in L(S')$ and $T''\in L(S'')$. We denote by $\{G_n\}$; $n\geqslant 1$, the sequence of all non-trivial Gleason parts of $A=R(Z)$ — the uniform closure on Z of the algebra of rational functions having poles off Z. T' and T'' can be written as sums

$$T'=T_0'\oplus(\oplus T_n),\qquad T''=T_0''\oplus(\oplus T_n''),$$

where T_0' and T_0'' are normal operators with spectrum carried by ∂Z, T_n' and T_n'' have $\bar{G}_n$ as spectral sets; the representations of $R(\bar{G}_n)$ generated by T_n' and T_n'' ($n\geqslant 1$) have elementary G_n-continuous measures. Moreover, T_0' and T_0'' have G_n-singular spectral measures for all $n\geqslant 1$.

Every (T',T'')-intertwining operator X can be written in the form

$$X=\bigoplus_{n\geqslant 0}X_n,$$

where X_n intertwines T_n' and T_n''. It follows that (T',T'') is a disjoint pair if and only if (T_n',T_n'') is disjoint for all $n\geqslant 0$. Since the unitary dilation of a completely non-unitary contraction has a Lebesgue spectrum — [23], Chapter II, Theorem 6.4, we get

COROLLARY 2.1. *If T is a c.n.u. contraction and U a singular unitary operator, then (T,U) and (U,T) are disjoint.*

This is the result of Sz.-Nagy and Foiaş — [23], Theorem 2.3, Chapter II, which extended a result of Sarason [21] that if X intertwines the contractions Q' and Q'', then $X=\operatorname{pr}Y$, where Y intertwines the minimal isometric dilations of Q' and Q'' respectively. Moreover, Y can be chosen so that $\|X\|=\|Y\|$. We need an equivalent formulation:

(*) *If U' and U'' are minimal unitary dilations of Q' and Q'' respectively, then every (Q',Q'')-intertwining operator X is a projection of an (U',U'')-intertwining operator Y; Y can be chosen so that $\|X\|=\|Y\|$.*

Suppose Z is a compact subset of C having a connected complement. Let $T' \in L(S')$, $T'' \in L(S'')$ have Z as a spectral set. Then the following refinements of decompositions of T' and T'' are avaible — Theorem 4 of [20] (G_n $(n = 1, 2, \ldots)$ stand for components of the interior of Z):

$$T' = T'_0 \oplus \big(\bigoplus_{n\geq 1} \varphi_n(Q'_n)\big), \qquad T'' = T''_0 \oplus \big(\bigoplus_{n\geq 1} \varphi_n(Q''_n)\big), \tag{2.3}$$

where T'_0 and T''_0 are normal with spectrum on ∂Z, Q'_n and Q''_n are c.n.u. contractions and $\varphi_n \in H^\infty(D)$. Both T'_0, T''_0 have decompositions

$$T'_0 = \oplus T'_{0,a} \oplus \tilde{T}'_0, \qquad T''_0 = \oplus T''_{0,a} \oplus \tilde{T}''_0, \tag{2.4}$$

where $T'_{0,a}$, $T''_{0,a}$ are G_a-parts of the ∂Z-normal part of T'_0, T''_0 respectively (G_a ranges over the totality of all Gleason parts) and $\tilde{T}'_0$, $\tilde{T}''_0$ are normal operators with completely singular spectral measures carried by ∂Z.

Let us fix $n \geq 1$ and write φ, G, Q', Q'' $\tilde{T}'$ $\tilde{T}''$ in place of φ_n, G_n etc. It is shown in [20] that Q' and Q'' can be chosen so that $Q' = w(\tilde{T}')$, $Q'' = w(\tilde{T}'')$, where $w \in H^\infty(m)$; $H^\infty(m)$ is an algebra corresponding to the representing measure m for some point of G. There is also a bounded (in Z) sequence of polynomials $\{w_v\}$ such that $w_v \to w$ a.e.m. Now, if X intertwines $\tilde{T}'$ and $\tilde{T}''$, then $Xw_v(\tilde{T}') = w_v(\tilde{T}'')X$. Consequently, $\int w_v dp'(f', X^*f'') = \int w_v dp''(Xf', f'')$, where $p' << m$, $p'' << m$. By dominated convergence and arbitrarness of f' and of f'' we infer that X intertwines Q' and Q''. By (*) X is a projection of an (U', U'') intertwining operator Y. It follows that Y is $(\varphi(U'), \varphi(U''))$-intertwining. But $\varphi(U')$ and $\varphi(U'')$ are minimal normal ∂Z-dilations of T' and T'' respectively. Since the minimal dilation of a direct sum equals to a sum of minimal dilations, (2.3) and (2.4) together with the above proved property prove via Theorem 2.1 the following

Corollary 2.2. *Let $Z \subset C$ be a compact with connected complement. Suppose Z is a spectral set for ∂Z-pure T', T''. Then, every (T', T'')-intertwining operator X is a projection of an operator Y which interwines the minimal normal ∂Z-dilations of T' and T'' respectively. The operator Y may be chosen so that $\|X\| = \|Y\|$.*

3. The semi-spectral measure is a mapping $F\colon \mathscr{B} \to L(S)$ of the σ-field of subsets of some space Z, such that $p_f\colon \sigma \to (F(\sigma)f, f)$ is a positive measure for every $f \in S$. F is called *normalized* if $F(Z) = I_S$. The semi-spectral measure F is called *spectral* if $F(\sigma_1 \cap \sigma_2) = F(\sigma_1)F(\sigma_2)$ for σ_1, $\sigma_2 \in \mathscr{B}$. We say that the operator X intertwines the s.s.m. F' and F'' if $XF'(\sigma) = F''(\sigma)X$ for all σ.

The ordered pair (F', F'') of semi-spectral measures is called *disjoint* if the only (F', F'')-intertwining operator is zero. (F', F'') is disjoint if and only if (F'', F') is disjoint. We say that F' and F'' are *mutually singular* if $(F'f', f') \perp (F''f'', f'')$ for all f', f''.

The following proposition is a simple extension of Theorem 2.4 of [7].

Proposition 3.1. *Let F', F'' be spectral measures. Then (F', F'') is disjoint if and only if F' and F'' are mutually singular.*

We next prove propositions related to Lemma 4.1 of [6]. We suppose that Z is a compact Hausdorff space. Let T', T'' be two involution preserving — simply $*$-representations of $C(Z)$ into $L(S')$ and $L(S'')$ respectively. Let A be a function algebra on Z. We denote by T'_A and T''_A the restrictions to A of T' and T'' respectively. Let E' and E'' be regular spectral measures on the Borel sets of Z corresponding to T' and T'' respectively.

Proposition 3.2. *Let $X \in L(S', S'')$. Then the following conditions are equivalent:*

(a) *X is (E', E'')-intertwining.*

(b) *X is (T', T'')-intertwining.*

(c) *X is (T'_A, T''_A)-intertwining for every function algebra A on Z.*

(d) *X is (T'_A, T''_A)-intertwining for some function algebra A on Z.*

Proof. The equivalence of (a) with (b) is obvious. The implications (b) ⇒ (c) ⇒ (d) are trivial. Assume (d) holds true. Then $XT'(\bar{v}) = T''(\bar{v})X$ for all $v \in A$ by Fuglede–Putnam theorem. It follows that $XT'(\bar{v}u) = T''(\bar{v}u)X$ for all $u, v \in A$. Now, by Stone–Weierstrass theorem $XT'(z) = T''(z)X$ for all $z \in C(Z)$ which completes the proof.

Proposition 3.3. *Let B a C^*-algebra and let T', T'' be two $*$-representations of B into $L(S')$ and $L(S'')$ respectively. If $XT'(u) = T''(u)X$ for $X \in L(S', S'')$ and all $u \in B$, then $\tilde{S}' = (\operatorname{Ker} X)^\perp$ is invariant for T', $\tilde{S}'' = \overline{\mathscr{R}(X)}$ is invariant for T'' and the parts of T' and T'' in $\tilde{S}'$ and $\tilde{S}''$ respectively are equivalent.*

The above proposition extends Lemma 4.1 of [6]. Indeed, by Fuglede–Putnam theorem, any (T', T'')-intertwining operator (T', T'' normal) intertwines suitable $*$-representations of $C(\operatorname{Spec} T' \cup \operatorname{Spec} T'')$.

4. Let F be a semi-spectral measure on the σ-field $\mathscr{B}$ with values in $L(S)$. This is the dilation theorem of Naimark that:

(*) *There exists a space R, an operator $V\colon S \to R$ and a normalized spectral measure $E\colon \mathscr{B} \to L(R)$ such that $F(\sigma) = V^*E(\sigma)V$ for all $\sigma \in \mathscr{B}$. The minimality condition $R = \bigvee_{\sigma \in \mathscr{B}} E(\sigma)VS$ determines R, V and E uniquely up to unitary isomorphism.*

The above spectral measure E is called a *spectral dilation* of F; it is minimal, if R is minimal. The minimal E is essentially unique and the expression $F = V^*EV$ is called the *canonical form* of F. If F is normalized, then V is an isometric embedding of S into R and after identifying VS with S, the adjoint V^* will be interpreted as the orthogonal projection of R onto S.

It is not difficult to show that:

PROPOSITION 4.1. *Let F_i be a semi-spectral measure on $\mathscr{B}$ with values in $L(S_i)$ $(i=1,2)$ and let $F_i = V_i^* E_i V_i$ $(i=1,2)$ be the canonical expression of F_i. Then E_1 and E_2 are mutually singular if and only if F_1 and F_2 are mutually singular.*

The Lemmas below are basic for our purposes. They generalize a result of Lebow [12] Theorem on p. 68, and are in fact particular cases of Lemma 1.4.1 of [1]. They are closely related to Theorem 1.3.1 of [1]. We present the proofs for the sake of clarity.

LEMMA 4.1. *Let F', F'' be semi-spectral measures on $\mathscr{B}$ with values in $L(S')$ and LS'' respectively. Suppose E' and E'' are minimal spectral dilations of F' and F'' and let $F' = V'^*EV'$, $F'' = V''^*E''V''$ be their canonical expressions. Let R' and R'' be the corresponding dilation spaces.*

Suppose we are given an operator $X \in L(S', S'')$. Then the following conditions are equivalent:

(a) *There is constant $c \geqslant 0$ such that $\big(F''(\sigma)Xf, Xf\big) \leqslant c^2\big(F'(\sigma)f, f\big)$ for $f' \in S'$ and $\sigma \in \mathscr{B}$.*

(b) *There is an (E', E'')-intertwining operator Y such that $YV' = V''X$.*

The operator Y is determined by X satisfying (a) *in a unique way and $\|Y\| \leqslant c$. If F' and F'' are normalized, then Y is simply an extension of X.*

Proof. Suppose (b) holds true. Then

$$\begin{aligned}\big(F''(\sigma)Xf, Xf\big) &= \big(E''(\sigma)V''Xf, V''Xf\big) = \big(E''(\sigma)YV'f, YV'f\big)\\ &= \|E''(\sigma)YV'f\|^2 = \|YE'(\sigma)V'f\|^2 \leqslant \|Y\|^2\big(E'(\sigma)V'f, V'f\big)\end{aligned}$$

which proves (a) with $c = \|Y\|$.

We assume (a). Let $\sigma_1 \ldots \sigma_n$ be a partition of the measure space and define $Z = \dfrac{1}{c}X$. Then by (a)

$$\begin{aligned}\Big\|\sum E''(\sigma_j)V''Zf_j\Big\|^2 &= \sum\|E''(\sigma_j)V''Zf_j\|^2 = \sum\big(F''(\sigma_j)Zf_j, Zf_j\big)\\ &\leqslant \sum\big(F'(\sigma_j)f_j, f_j\big) = \Big\|\sum E'(\sigma_j)V'f_j\Big\|^2\end{aligned}$$

which implies that there is a unique contraction T_0 such that $T_0\sum E'(\sigma_j)V'f_j = \sum E''(\sigma_j)V''Zf_j$. By minimality conditions T_0 has a unique extension to a contraction $T \in L(R', R'')$ such that $TE'(\sigma)V' = E''(\sigma)V''Z$ for $\sigma \in \mathscr{B}$. Hence $TV' = V''Z$ and $TE'(\sigma)E'(\gamma)V' = E''(\sigma)E''(\gamma)V''Z = E''(\sigma)TE'(\gamma)V$ which again by minimality of R' proves that T is (E', E'')-intertwining. The operator $Y = cT$ does the trick. The final part of assertion follows easily.

Notice that if X intertwines F' and F'', then X satisfies (a) with $c = \|X\|$. Indeed we then have $\big(F''(\sigma)Xf, Xf\big) = \big(X^*XF'(\sigma)f, f\big) \geqslant 0$. It follows that X^*X and $F'(\sigma)$ commute and consequently

$$\big(X^*XF'(\sigma)f, f\big) = \big(X^*XF'(\sigma)^{1/2}f, F'(\sigma)^{1/2}f\big) \leqslant \|X\|^2\big(F'(\sigma)f, f\big) \qquad \text{q.e.d.}$$

We get therefore the following:

COROLLARY 4.1. *If X is (F', F'')-intertwining, then Y related to X as in Lemma 4.1 satisfies $\|Y\| \leqslant \|X\|$. In particular, if both F', F'' are normalized, then $\|Y\| = \|X\|$.*

Next we have:

COROLLARY 4.2. *If X intertwines F' and F'' and X is isometric, then Y related to X as in Lemma 4.1 is isometric. Consequently, if X is (F', F'')-intertwining, then $X^*X = I_{S'}$ if and only if $Y^*Y = I_{R'}$, provided F' is normalized.*

In what follows we use notation of Lemma 4.1. We will prove the following:

COROLLARY 4.3. *Let F' and F'' be arbitrary semi-spectral measures. Assume that X satisfies* (a)*. Then, if $\mathscr{R}(X)$ is dense in S'', then the corresponding operator Y has a dense range.*

Proof. Let $g \in R''$, $g \perp \mathscr{R}(Y)$. If $\overline{\mathscr{R}(X)} = S''$, then $V''\mathscr{R}(X)$ spans $V''S''$ and consequently $g \perp \bigvee_\sigma E''(\sigma)V''S'' = R''$. Hence $g = 0$, q.e.d.

COROLLARY 4.4. *Let F', F'' be semi-spectral measures and let X be a (F', F'')-intertwining operator. Let Y be the corresponding (E', E'')-intertwining operator. Then Y is strictly invertible if X is strictly invertible. If X is strictly invertible, then the corresponding (E'', E')-intertwining operator for X^{-1} equals Y^{-1}.*

Proof. Suppose X is strictly invertible. Then X^{-1} intertwines F'' and F'. Let W be the corresponding (E'', E')-intertwining operator for X^{-1}. We have for $f \in S'$

$$\begin{aligned}WYE'(\sigma)V'f &= WE''(\sigma)YV'f = WE''(\sigma)V''Xf\\ &= E'(\sigma)WV''Xf = E'(\sigma)V'X^{-1}Xf = E'(\sigma)V'f\end{aligned}$$

and by similar token, for $g \in S''$, $YWE''(\sigma)V''g = E''(\sigma)V''g$. By minimality of E' and E'', $WY = I_{R'}$, $YW = I_{R''}$, q.e.d.

COROLLARY 4.5. *Let E be the minimal spectral dilation of the semi-spectral measure F. Denote by Y the operator which commutes with E and which corresponds to an operator X which commutes with F. The correspond-*

ence $X \to Y$ is an involution preserving isometric homomorphism. Consequently:

(a) $X = X^*$ *if and only if* $Y = Y^*$.

(b) Y *is normal iff* X *is normal.*

(c) $X \geqslant 0$ *if and only if* $Y \geqslant 0$.

If F' and F'' are normalized, then $V'S'$ and $V''S''$ are identified with S' and S'' respectively. In this case V'^* and V''^* are projections on S' and S'' in R' and R'' respectively. We write $P' = V'^*$, $P'' = V''^*$. Using this notation we have:

LEMMA 4.2. *Suppose F' and F'' are normalized. If X is (F', F'')-intertwining, then its (E', E'')-intertwining extension satisfies $YP' = P''Y$: If X satisfies* (a) *of Lemma* 4.1 *and its (E', E'')-intertwining extension Y satisfies $YP' = P''Y$, then X is (F', F'')-intertwining.*

Proof. If $XF'(\sigma) = F''(\sigma)X$, then for $f \in S'$ $YP'E'(\sigma)f = P''E''(\sigma)Yf = P''YE'(\sigma)f$. Since E' is minimal $YP' = P''Y$. Conversely, if $YP' = P''Y$, then $XF'(\sigma)f = YP'E'(\sigma)f = P''YE'(\sigma)f = P''E''(\sigma)Xf = F''(\sigma)Xf$ for $f \in S'$ which completes the proof.

5. Let Z be a compact Hausdorff space and let A be a function algebra on Z. We say that the representation $T\colon A \to L(S)$ is *subnormal* if there is a Hilbert space R which contains S as a Hilbert space and an involution preserving representation $\tilde{T}\colon C(Z) \to L(R)$ such that $T(u) \subset \tilde{T}(u)$ for every $u \in A$. $\tilde{T}$ is called a **-extension* of T. It is called *minimal*, if $R = \bigvee_{u \in C(Z)} \tilde{T}(u)S$. The minimal *-extension is determined uniquely up to equivalence. Indeed, let T_1, T_2 be two minimal *-extensions of subnormal representation $T\colon A \to L(S)$. We denote by E_1, E_2 the regular spectral measures of T_1 and T_2 respectively. Then for $f, g \in S$ and $u, v \in A$

$$\int u\bar{v}\,d(E_1 f, g) = (T(u)f, T(v)g) = \int u\bar{v}\,d(E_2 f, g)$$

which by Stone–Weierstrass theorem yields that $(E_1 f, g) = (E_2 f, g)$. Since minimal spaces of T_1 and T_2 respectively are spanned by $E_1(\sigma)S$ and $E_2(\sigma)S$, E_1 and E_2 are equivalent. It follows that for every subnormal $T\colon A \to L(S)$ there is a unique (regular!) normalized semi-spectral measure F on $\mathscr{B}(Z)$ — the σ-field of Borel sets in Z, such that $T(u) = \int_Z u\,dF$ and $\|T(u)f\|^2 = \int_Z |u|^2 d(Ff, f)$ for $u \in A$, $f \in H$. F is then called the *semi-spectral measure of* T.

In all what follows all operator measures are Borel regular ones.

The study of subnormal operators has been initiated by Halmos [10] and later developed by Bram [4] and Ito [11] and other authors. From our point of view the study of commutative families of operators which have commutative algebraically coherent simultaneous normal extensions reduces to the study of suitable subnormal representations. The point is that having in mind the Gelfand–Naimark theorem about abelian C^*-algebras one proves that to every uniformly closed abelian algebra R with unit I whose elements form a positive definite family there is a unique (up to homeomorphism) compact space Z and a unique subnormal representation (up to equivalence) T of the function algebra A on Z such that $R = T(A)$, with Z — spectrum of C^*-algebra generated by the minimal normal extension of R. For details see [11]. Avoiding a rather standard proofs of the above statements we present a direct proof of a proposition which is a reformulation of Theorem 2 [11].

PROPOSITION 5.1. *Let T be a subnormal representation of the function algebra A on Z. Then T is equivalent to a subnormal representation of A_G, where G — closed support of the spectral measure of the minimal *-extension of T. Moreover, T is isometric that is $\|T(u)\| = \sup_G |u|$ for $u \in A$.*

Proof. The first part of the assertion follows immediately from the well-known properties of spectral integrals. It is also obvious that $\|T(u)\| \leqslant \sup_G |u|$ for $u \in A$. Denote $\sigma = \{z \in G \mid \|T(u)\| < |u(z)|\}$ and let $f \in S$. Then for $p_f = (Ef, f)$ (E — the spectral measure of the minimal *-extension of T — the minimal dilation of semi-spectral F of T)

$$\int_\sigma \left\| \frac{u}{T(u)} \right\|^{2n} dp_f \leqslant \|f\|^2 \quad \text{for all } n = 1, 2, \ldots$$

provided $T(u) \neq 0$.

It follows that $p_f(\sigma) = 0$. Consequently $F(\sigma) = 0$. Since E and F are mutually absolutely continuous, $E(\sigma) = 0$ which is in contradiction with the definition of G provided $\sigma \neq \emptyset$. If $T(u) = 0$, then $\int |u|^2 d(Ef, f) = 0$ for all $f \in S$.

It follows that $p_f(\gamma) = 0$ for $\gamma = \{z \in Z \mid |u(z)| > 0\}$ for all $f \in S$ which by minimality of E completes the proof.

In what follows we deal with operators which intertwine subnormal representations of some special class of function algebras. This class first defined by Glicksberg in [9] is the class of algebras approximating in modulus.

We say that the function algebra A on Z is *approximating in modulus*, shortly a.i.m., if every positive continuous function on Z can be approximated uniformly on Z by moduli of functions in A. Every log modular and consequently every Dirichlet algebra is a.i.m. This is the result of Glicksberg [9] that A is a.i.m. if the unimodular functions in A separate the points of Z. In particular, every polydisc algebra $A(D^n)$ — the uniform closure of analytic polynomials on the n-dimensional torus Γ^n is a.i.m.

The following theorem generalizes a theorem of [17]:

THEOREM 5.1. *Let T', T'' be subnormal representations of the function algebra A on Z. Suppose $\tilde{T}'$, $\tilde{T}''$ are minimal $*$-extensions of T', T'' respectively. If A is approximating in modulus, then every (T', T'')-intertwining operator X extends uniquely to a $(\tilde{T}', \tilde{T}'')$-intertwining operator $\tilde{X}$. Moreover, $\|\tilde{X}\| = \|X\|$.*

Proof. The proof is almost trivial in view of Lemma 4.1. Indeed, since $XT'(u) = T''(u)X$, then

$$\int |u|^2 d(F''Xf, Xf) = \|T''(u)Xf\|^2 \leqslant \|X\|^2 \int |u|^2 d(F'f, f),$$

where F', F'' are semi-spectral measures of T', T'' respectively, $u \in A$ and $f \in S'$. Since A is a.i.m. the above inequality proves that

$$(F''(\sigma)Xf, Xf) \leqslant \|X\|^2 (F'(\sigma)f, f)$$

for Borel sets σ. The minimal spectral dilations of F' and F'' are spectral measures of $\tilde{T}'$ and of $\tilde{T}''$ respectively. The assertion follows now from Lemma 4.1.

A simple application of Theorem 5.1 is now in order. It concerns an extension of Corollary 5.1 of [6]. For the sake of simplicity we deal with finite families of operators. As to existence of suitable unitary extension see [11].

PROPOSITION 5.2. *Suppose we are given commuting isometries $V'_k \in L(S')$, $V''_k \in L(S'')$ $(k = 1, \ldots, n)$. Then every operator $X \in L(S', S'')$ such that $XV'_k = V''_k X$ for $k = 1, \ldots, n$ has a unique extension which intertwines the minimal unitary extensions of $V'_1, \ldots, V'_n$ and of $V''_1, \ldots, V''_n$.*

Proof. Let $A = A(D^n)$ and define

$$T'(u) = V'^{k_1}_1, \ldots, V'^{k_n}_n, \quad T''(u) = V''^{k_1}_1, \ldots, V''^{k_n}_n$$

for $u(z_1, \ldots, z_n) = z_1^{k_1}, \ldots, z_n^{k_n}$. Both T', T'' extend to subnormal representations of A. Their minimal $*$-extensions are generated by minimal unitary extensions of isometries in question. The assertion follows now from Theorem 5.1.

The mapping $X \to \tilde{X}$ established in Theorem 5.1 is linear and isometric. It is natural to ask which properties of X are invariant under this mapping, in analogy of a series of corollaries of section 4. One could try to apply this corollaries. However, this would lead in some cases to rather restricted theorems. The point is that if X intertwines subnormal representations, then in general it does not intertwine their semi-spectral measures. A simple example is at hand, namely the uniteral shift which commutes with representation of $A(D)$ which it generates, but does not commute with the semi-spectral measure of this representation.

However, under the assumptions of Theorem 5.1 the mapping $X \to \tilde{X}$ preserves quite a lot of properties.

THEOREM 5.2. *Suppose X, T', T'' and $\tilde{X}$, $\tilde{T}'$, $\tilde{T}''$ and A are such as in Theorem 5.1. Then:*

(a) *$\tilde{X}$ is isometric if and only if X is isometric.*

(b) *If X has a dense range, then $\tilde{X}$ has a dense range.*

(c) *If X is strictly invertible, then $\tilde{X}$ is strictly invertible and $\widetilde{X^{-1}} = \tilde{X}^{-1}$.*

Proof. The non-trivial part of (a) can be proved as follows: if X is isometric, then for $f \in S'$

$$\|XT'(u)f\|^2 = \int |u|^2 d(F'f, f) = \|T''(u)Xf\|^2$$
$$= \int |u|^2 d(F''Xf, Xf).$$

Since A is a.i.m. we infer therefore

$$(F'f, f) = (X^*F''Xf, f)$$

and consequently $F'(\sigma) = X^*F''(\sigma)X$ for Borel sets σ. It follows that

$$\Big\|\tilde{X}\sum E'(\sigma_j)f_j\Big\|^2 = \Big\|E''(\sigma_j)Xf_j\Big\|^2$$
$$= \sum (X^*F''(\sigma_j)Xf_j, f_j) = \Big\|\sum E'(\sigma_j)f_j\Big\|^2$$

for partition $\sigma_1, \ldots, \sigma_n$ and $f_j \in S'$. E' and E'' are as usually the minimal spectral dilations of F' and F''. The required assertion follows from the minimality of E' i.e. that of $\tilde{T}'$.

The statement (b) follows from Corollary 4.3 and from the proof of Theorem 5.1.

The part (c) is simple. Indeed if X intertwines T' and T'', then X^{-1} interwines T'' and T'. Since $\widetilde{X^{-1}}$ is (E'', E')-interwining, $\widetilde{X^{-1}} = \tilde{X}^{-1}$ (see the proof of Corollary 4.4) q.e.d.

Combining Proposition 3.3 with (c) of Theorem 5.2 we get

COROLLARY 5.1. *The minimal $*$-extensions of similar subnormal representations of an a.i.m. algebra are equivalent.*

Theorem 5.2 can be completed in case $T' = T''$ i.e. if the things are going about operators commuting with subnormal representation. The theorem below deals with such operators. It completes also a Theorem of [17].

THEOREM 5.3. *Let T: $A \to L(S)$ be a subnormal representation of an a.i.m. algebra. If X commutes with T, then:*

(a) *$X = X^*$ iff $\tilde{X} = \tilde{X}^*$.*

(b) *If X is normal, then $\tilde{X}$ is normal.*

(c) *$X \geqslant 0$ iff $\tilde{X} \geqslant 0$.*

The proof of the above theorem is based on the following lemmas:

LEMMA 5.1. *Suppose the assumptions of Theorem* 5.3 *hold true. If* X *is an orthogonal projection so is* $\tilde{X}$.

Proof. It is obvious by Theorem 5.1 that in our case the mapping $X \to \tilde{X}$ is an algebra homomorphism.

It follows that if X is a projection so is $\tilde{X}$. If additionally $X = X^*$, then $\|X\| \leqslant 1$ and consequently $\|\tilde{X}\| \leqslant 1$. But a contractive projection is necessarilly orthogonal q.e.d.

LEMMA 5.2. *Suppose* X_n *commute with* T, T *being a subnormal representation of an a.i.m. algebra. If* $X_n \to X$ *weakly, then* $\tilde{X}_n \to \tilde{X}$ *weakly.*

Proof. The norms $\|\tilde{X}_n\|$ are equibounded because $\|X_n\|$ are. Let E be the spectral measure of the minimal $*$-extension of T, and R the dilation space. We have for $f \in S$, $g \in R$

$$\big(\tilde{X}_n E(\sigma)f, g\big) = \big(E(\sigma)X_n f, g\big) = \big(X_n f, P_S E(\sigma)g\big) \to \big(Xf, P_S E(\sigma)g\big), = \big(\tilde{X}E(\sigma)f, g\big).$$

It follows that $\tilde{X}_n h \to \tilde{X}h$ weakly for h in a dense set of R. Consequently by boundedness of norms $\|\tilde{X}_n\|$, $\tilde{X}_n \to \tilde{X}$ weakly, q.e.d.

Proof of Theorem 5.3. Let $G(\cdot)$ be the spectral measure of $X = X^*$. Since X commutes with $T(u)$, the values of $G(\cdot)$ commute with $T(u)$. It follows from Lemmas 5.1, 5.2 that $\tilde{G}(\cdot)$ is a spectral measure. Hence $Y = \int_{-\infty}^{+\infty} \lambda d\tilde{G}_\lambda$ is selfadjoint. But the value $\widetilde{G(\sigma)}$ of $\tilde{G}(\cdot)$ is an extension of $G(a)$. It follows that Y is an extension of $\tilde{X}$. Since $\tilde{G}$ commutes with the spectral measure of the minimal $*$-extension of T, Y shares this property. By the uniqueness of the extension $Y = \tilde{X}$. We just proved that $X = X^*$ implies $\tilde{X} = \tilde{X}^*$. Suppose now that $\tilde{X} = \tilde{X}^*$. If $f \in S$, then $(Xf, f) = (\tilde{X}f, f)$ is real, q.e.d.

Suppose now that X is normal. By Fuglede–Putnam theorem X^* commutes with T. Hence $\operatorname{Re} X$ and $\operatorname{Im} X$ commute with T. Consequently $\widetilde{\operatorname{Re} X}$ and $\widetilde{\operatorname{Im} X}$ are self adjoint and commute. Hence $\tilde{X} = \widetilde{\operatorname{Re} X} + i\widetilde{\operatorname{Im} X}$, and by uniqueness of extension we have that $\widetilde{\operatorname{Re} X} = \operatorname{Re}\tilde{X}$, $\widetilde{\operatorname{Im} X} = \operatorname{Im}\tilde{X}$ which completes the proof of (b).

Assume now that $X \geqslant 0$. Then $\tilde{X}$ is selfadjoint by (a). Since the measures G and $\tilde{G}$ have equal closed supports $\tilde{X} \geqslant 0$, q.e.d.

6. This section deals with what one can call more or less precisely the functional models of intertwining operators. Such models are in [23] for operators which intertwine contractions in separable spaces. In what follows we restrict ourselves to models of operators intertwining some simple subnormal representations of a.i.m. algebras. We notice only that when using the theorems of Section 4 one can prove for models of operators intertwining semi-spectral measures more precise theorems, than those avaible for interwining subnormal representations. We apply usual methods of reduction theory for which we refer to [3] and [5].

Let Z be a compact metric space, and p the positive finite measure on Z. Suppose we are given two direct integrals of measurable fields $H'(z)$, $H''(z)$ of separable Hilbert spaces:

$$H' = \int \oplus H'(z)\,dp, \quad H'' = \int \oplus H''(z)\,dp.$$

Denote by T', T'' the natural representations of $C(Z)$ into H' and H'' respectively i.e. representations defined by

$$\big(T'(u)f'\big)(z) = u(z)f'(z), \quad \big(T''(u)f''\big)(z) = u(z)f''(z);$$

$u \in C(Z)$, $f' \in H'$, $f'' \in H''$. This is the basic theorem on representations that every $*$-representation of $C(Z)$ is equivalent to a natural one. An operator $Y \in L(H', H'')$ intertwines T' and T'' if and only if it is decomposable i.e.

$$Y = \int \oplus Y(z)\,dp,$$

where $Y(z)$ is a measurable operator function such that

$$\sup_p \operatorname{ess}\|Y(z)\| = \|Y\|. \tag{6.1}$$

Suppose we are given two separable Hilbert spaces S' and S''. We define

$$L^2(S'^{('')}, p) = \int \oplus H'^{('')}(z)\,dp,$$

where $H'(z) = S'$, $H''(z) = S''$ for all z; p normalized.

For a function algebra A on Z we define $H^2(S'^{('')}, A, p)$ as the $L^2(S'^{('')}, p)$ closure of the space of functions $f'^{('')}(z) = \sum u_i(z) f_i'^{('')}$ (finite sum), where $u_i \in A$ and $f_i'^{('')} \in S'^{('')}$. The natural representation in $L^2(S'^{('')}, p)$ restricted to $H^2(S'^{('')}, A, p)$ is a subnormal representation $T'^{('')}$ of A; the minimal $*$-extension of $T'^{('')}$ is just equal to the natural representation $\tilde{T}'^{('')}$ in $L^2(S'^{('')}, p)$.

Using all the above notation we have:

THEOREM 6.1. *Suppose* X *is an* (T', T'')*-intertwining operator. Assume that the algebra* A *is approximating in modulus. Then there exists a unique decomposable operator* $\int \oplus \tilde{X}(z)\,dp$ *which is the* $(\tilde{T}', \tilde{T}'')$*-intertwining extension* $\tilde{X}$ *of* X. *The following properties hold true:*

(a) X *is isometric if and only if* $\tilde{X}(z)$ *is isometric a.e.p.*

(b) *If* X *is strictly invertible and* $\|X^{-1}\| \leqslant k$, *then* $\tilde{X}(z)$ *is strictly invertible a.e.p. and* $\|\tilde{X}^{-1}(z)\| \leqslant k$ *a.e.p.*

(c) *If* X *is unitary, then* $\tilde{X}(z)$ *is unitary a.e.p.*

Proof. That the extension $\tilde{X}$ is decomposable follows from the property, that every $(\tilde{T}', \tilde{T}'')$-intertwining operator is decomposable. The point (a) follows from Theorem 5.2 (a) and from the well-known properties of decomposable operators. The property (b) follows easily from Theorem 5.2 (c) and from the property that the extension mapping is isometric, via (6.1).

To prove (c) we assume that X is unitary. Then $\overline{\mathscr{R}(X)} = H^2(S'', A, p)$ which by Theorem 5.2 (b) proves that $\tilde{X}$ is unitary. Consequently $\tilde{X}(z)$ is unitary, a.e.p.

The above Theorem is a certain kind generalization of Lemmas 3.1, 3.2 Chapter V of [23]. Using Theorem 5.3 one proves easily the following:

THEOREM 6.2. *Let $S' = S''$, $T = T' = T''$, A be as in Theorem* 6.1 *and let $\int \oplus \tilde{X}(z)\,dp$ be the decomposition of the extension $\tilde{X}$ of X commuting with T. Then*:

(a) *X is selfadjoint if and only if $\tilde{X}(z)$ is selfadjoint a.e.p.*

(b) *If X is normal, then $\tilde{X}(z)$ is normal a.e.p.*

(c) *$X \geqslant 0$ if and only if $\tilde{X}(z) \geqslant 0$ a.e.p.*

Theorem 6.1 and Theorem 6.2 hold true if L^2 and H^2 spaces are replaced by sums

$$\oplus L^2(S_n^{'('')}, p_n), \quad \oplus H^2(S_n^{'('')}, A, p_n),$$

where p_n are mutually singular. The involved representations should be replaced by suitable direct sums of representations.

Conditions (a), (b) and (c) of both Theorems 6.1 and 6.2 hold true for operators which commute with semi-spectral measures. The L^2 space should be then replaced by a suitable integral in which the involved representation is interpreted as a natural representation. The decomposition of the extension of the operator in question is taken with respect to this direct integral.

Final remarks. Some of the properties discussed in Section 5 and Section 6 remain true under more general assumptions.

Suppose namely:

(a) A is a.i.m.

(b) T' is linear and dilatable, i.e. $T'(u) = \int u\,dF$ for some semi-spectral F.

(c) T'' is a subnormal representation of A.

Then, if $XT'(u) = T''(u)X$ for $u \in A$, then X extends to an operator $\tilde{X}$ which intertwines the minimal dilation of T' which corresponds to F and the minimal $*$-extension of T''. Notice, that $\tilde{X}$ depends on F; T' may have non-equivalent dilations. However, the methods based on Lemma 4.2 apply with suitable changes.

Since the disc algebra when considered as a function algebra on the unit circle is a.i.m., the desired extension $\tilde{X}$ (unique!) exists if $XT' = T''X$, where T' is a contraction and T'' isometric. $\tilde{X}$ intertwines the minimal unitary dilation of T' and the minimal unitary extension of T''.

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