[7] A. N. Kolmogorov, *La transformation de Laplace dans les espaces linéaires*, Compt. Rend. Acad. Sci. Paris 200 (1935), p. 1717.
[8] P. Lévy, *Theorie de l'addition des variables aléatoires*, Paris 1937.
[9] K. R. Parthasarathy, *Probability measures on metric spaces*, New York and London 1967.
[10] Ю. В. Прохоров, *Сходимость случайных процессов и предельные теоремы теории вероятностей*, Теория вероятностей и ее применения 1 (1956), pp. 177–238.
[11] S. R. S. Varadhan, *Limit theorems for sums of independent random variables with values in a Hilbert space*, Sankhyā, Indian J. Statist. 24 (3) (1962), pp. 213–238.

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On the trace of some operators

by

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Abstract. Let X and Y be Banach spaces and $1 \leqslant p < \infty$. We consider the space $\mathrm{TR}_p(Y, X)$ of all bounded linear operators A from Y to X, for which the functional $N \to \mathrm{trace}(AN)$ is continuous for the p-nuclear norm on the space of all finite-dimensional operators from X to Y. Each such A defines in a unique way a cont. linear functional Tr_A on $N_p(X, Y)$ — the space of all p-nuclear operators from X to Y. It is shown, that every p'-integral operator $(1/p+1/p' = 1)$ from Y to X belongs to $\mathrm{Tr}_p(Y, X)$, and that every element of this space is p'-absolutely summing. This result is used to prove that if $n \geqslant 3$ is such that $p \leqslant p'(n-1)$ and $A_1, A_2, \ldots, A_n$ are p-integral operators on X, then $\mathrm{Tr}_{A_1}(A_2, A_3, \ldots, A_n) = \mathrm{Tr}_{A_{i_1}}(A_{i_2}, \ldots, A_{i_n})$ for each cyclic permutation $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$. This trace formula was conjectured by R. Sikorski. It should be mentioned that X is not assumed to have the approximation property.

Introduction and background. Let X be a Banach space. If $N\colon X \to X$ is a finite dimensional operator, i.e. there are finite sets $\{x_1, x_2, \ldots, x_k\} \subseteq X$, $\{x_1^*, x_2^*, \ldots, x_k^*\} \subseteq X^*$ such that

$$Nx = \sum_{n=1}^{k} x_n^*(x)x_n \quad \text{for all } x \in X,$$

then it is well known that N has a uniquely determined trace $\mathrm{Tr}(N)$, defined by

$$\mathrm{Tr}(N) = \sum_{n=1}^{k} x_n^*(x_n),$$

i.e. $\mathrm{Tr}(N)$ does not depend on the actual finite dimensional representation of N.

If N is a nuclear operator on X with nuclear representation $N = \sum_{n=1}^{\infty} x_n^* \otimes x_n$, we shall say that N has a uniquely determined trace $\mathrm{Tr}(N)$, if the sum $\sum_{n=1}^{\infty} x_n^*(x_n)$ depends only on N and not on the actual representation, and in that case we put

$$\mathrm{Tr}(N) = \sum_{n=1}^{\infty} x_n^*(x_n).$$

It is a well-known theorem of Grothendieck [1] that every nuclear operator on X has a uniquely determined trace if and only if X has the approximation property.

If X and Y are Banach spaces and $1 \leqslant p \leqslant \infty$ it is natural to ask the following question:

Which operators $A\colon Y \to X$ have the property that for any p-nuclear operator $N\colon X \to Y$ the sum $\sum_{n=1}^{\infty} x_n^* A x_n$ is convergent and independent of the actual p-nuclear representation $N = \sum_{n=1}^{\infty} x_n^* \otimes x_n$ of N?

For operators $A\colon Y \to X$ with this property we can then define a trace with respect to A of any p-nuclear operator.

It is easy to see that this question is equivalent to asking for which operators $A\colon Y \to X$ the linear functional $N \to \mathrm{Tr}(AN)$ on the space of finite dimensional operators is continuous in the p-nuclear norm.

In the present paper we discuss the above question and prove some theorems on operators with the above property; it is for example proved that if $1/p + 1/p' = 1$, then every p'-integral operator from Y to X has the property. Finally this result is used to prove the main theorem of the paper, namely the so-called Sikorski trace formula for p-integral operators, a formula suggested by Sikorski [5].

1. Preliminaries and notation. Throughout this paper let X and Y denote Banach spaces and let us by $B(X, Y)$ denote the space of all bounded linear operators from X to Y. $B(X, X)$ will shortly be denoted by $B(X)$.

If $1 \leqslant p \leqslant \infty$ we shall by p' denote the dual number to p, i.e. $1/p + 1/p' = 1$.

In the sequel we shall make use of the following operator ideals for $1 \leqslant p \leqslant \infty$:

- $N_p(X, Y)$ — the space of all p-nuclear operators from X to Y with the p-nuclear norm ν_p.
- $I_p(X, Y)$ — the space of all p-integral operators from X to Y with the norm i_p.
- $I_p^Q(X, Y)$ — the space of all absolutely p-summing operators from X to Y with the norm i_p^Q.
- $B_0(X, Y)$ — the space of all finite dimensional operators from X to Y.

The reader may find the definitions and basic properties of these ideals in Persson and Pietsch [4]. We just note here that by definition $I_\infty^Q(X, Y) = B(X, Y)$ and i_∞^Q is the operator norm. In the sequel we shall use either of the two notations for $B(X, Y)$ depending on what is the most convenient.

2. The trace of p-nuclear operators with respect to a certain class of operators. If $A \in B(Y, X)$ we shall define the linear functional Tr_A on $B_0(X, Y)$ by

$$\mathrm{Tr}_A(N) = \mathrm{Tr}(AN) \quad \text{for all } N \in B_0(X, Y).$$

We consider for $1 \leqslant p \leqslant \infty$

$$\mathrm{TR}_p(Y, X) = \{A \in B(Y, X) \mid \mathrm{Tr}_A \text{ is cont. in the } \nu_p\text{-norm on } B_0(X, Y)\}.$$

If $A \in \mathrm{TR}_p(Y, X)$ we can extend Tr_A by continuity to $N_p(X, Y)$; the extended functional we shall still denote by Tr_A.

It is easy to see that the correspondence $A \to \mathrm{tr}_A$ is $1-1$ and from the proof of Satz 52 in [4] it follows that if $A \in \mathrm{TR}_p(Y, X)$, then $A \in I_{p'}^Q(Y, X)$ and $i_{p'}^Q(A) = \nu_p^*(\mathrm{Tr}_A)$, ν_p^* denoting the dual norm of ν_p.

The problem if $\mathrm{TR}_p(Y, X) = I_{p'}^Q(Y, X)$ is of course connected with the approximation property, in fact that $\mathrm{TR}_1(X, X) = B(X)$ if and only if X has the approximation property is just the theorem of Grothendieck, quoted in the introduction, and it is easy to see that if $\mathrm{TR}_1(X, X) = B(X)$, then $\mathrm{TR}_p(Y, X) = I_{p'}^Q(Y, X)$ for all Banach spaces Y and

$$\mathrm{Tr}_A(N) = \mathrm{Tr}_I(AN) = \mathrm{Tr}(AN) \quad \text{for all } N \in N_p(X, Y)$$

(the composition of a p-nuclear and a p'-absolutely summing operator is 1-nuclear).

The next proposition is an easy consequence of [4], Satz 52:

PROPOSITION 1. *If Y has the metric approximation property, then $\mathrm{TR}_p(Y, X) = I_{p'}^Q(Y, X)$ for all p, $1 \leqslant p \leqslant \infty$.*

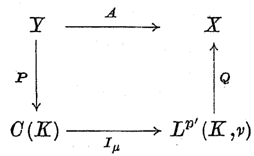
About the proof of Proposition 1 let us just remark, that the main point in the proof is that when Y has the metric approximation property, then the p-nuclear norm of any operator $N \in B_0(X, Y)$ can be computed only taking finite dimensional representations of N into consideration (Lemma 7 and Lemma 11 in [4]).

We are now able to prove

THEOREM 1. *If X and Y are arbitrary Banach spaces, then*

$$I_{p'}(Y, X) \subseteq \mathrm{TR}_p(Y, X) \quad \textit{for all } p, 1 \leqslant p \leqslant \infty.$$

Proof. Assume first that $1 < p \leqslant \infty$, let $A \in I_{p'}(Y, X)$ and $\varepsilon > 0$. We can then find a compact Hausdorff space K and a positive Radon measure μ on K, so that we have a factorization



of A, where $\|P\| \leqslant 1$, $\|Q\| \leqslant 1$, $\mu(K)^{1/p'} \leqslant i_{p'}(A)+\varepsilon$ and I_μ denoting the formal identity map of $C(K)$ into $L^{p'}(K, \mu)$.

If $N \in B(X, Y)$ is a finite dimensional operator, then

$$\mathrm{Tr}_A(N) = \mathrm{Tr}(AN) = \mathrm{Tr}(QI_\mu PN) = \mathrm{Tr}_{QI_\mu}(PN).$$

Since $C(K)$ has the metric approximation property and I_μ is p'-integral with $i_{p'}(I_\mu) = \mu(K)^{1/p'}$ we get from Proposition 1:

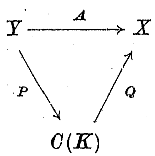
$$|\mathrm{Tr}_A(N)| = |\mathrm{Tr}_{QI_\mu}(PN)| \leqslant i_{p'}^Q(QI_\mu)\nu_p(PN) \leqslant i_{p'}^Q(I_\mu)\nu_p(N) \leqslant i_{p'}(I_\mu)\nu_p(N) \leqslant [i_{p'}(A)+\varepsilon]\nu_p(N).$$

Since $\varepsilon > 0$ was arbitrary this gives that $A \in \mathrm{TR}_p(Y, X)$.

If $p = 1$ we proceed as follows:

Let $A \in I_\infty(Y, X)$ and let $\varepsilon > 0$ be arbitrary.

We can then find a compact Hausdorff space K and a factorization



of A with $\|P\| \leqslant 1$ and $\|Q\| \leqslant i_\infty(A)+\varepsilon$.

If $N \in B_0(X, Y)$ we get as before using Proposition 1

$$|\mathrm{Tr}_A(N)| = |\mathrm{Tr}_Q(PN)| \leqslant \|Q\|\|P\|\nu_1(N) \leqslant [i_\infty(A)+\varepsilon]\nu_1(N)$$

and hence

$$|\mathrm{Tr}_A(N)| \leqslant i_\infty(A)\cdot\nu_1(N).$$

Corollary 1. *If X and Y are Banach spaces, then there is an isometric isomorphism φ of $I_2(Y, X^{**})$ onto $N_2(X, Y)^*$ so that*

$$\varphi(A)(N) = \mathrm{Tr}_A(N^{**}) \quad \textit{for all } N \in N_2(X, Y).$$

Proof. Let $A \in I_2(Y, X^{**})$. If $N \in N_2(X, Y)$, then we can consider N^{**} as element of $N_2(X^{**}, Y)$; Theorem 1 together with the fact that $\nu_2(N^{**}) \leqslant \nu_2(N)$ for $N \in N_2(X, Y)$ then gives that the mapping

$$N \to \mathrm{Tr}_A(N^{**}), \qquad N \in N_2(X, Y)$$

is a continuous linear functional on $N_2(X, Y)$. Let us denote this functional by $\varphi(A)$; clearly φ: $I_2(Y, X^{**}) \to N_2(X, Y^*)$ is linear and

(1) $$\nu_2^*(\varphi(A)) \leqslant i_2(A).$$

Let now $F \in N_2(X, Y)^*$. It is easy to see that there is a bounded linear operator A: $Y \to X^{**}$ such that

$$F(x^* \otimes y) = (Ay)(x^*) \quad \text{for } x^* \in X^* \text{ and } y \in Y$$

and hence

(2) $$F(N) = \sum_{n=1}^{\infty} (Ay_n)(x_n^*) \quad \text{for all } N \in N_2(X, Y),$$

where $N = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ is an arbitrary representation for N.

Using now the same argument as Persson and Pietsch [4], Satz 52, we get that $A \in I_2^Q(Y, X^{**}) = I_2(Y, X^{**})$ and that

(3) $$i_2(A) = i_2^Q(A) \leqslant \nu_2^*(F).$$

(2) now gives that $\varphi(A) = F$ and (1) and (3) gives

$$\nu_2^*(\varphi(A)) = i_2(A).$$

The following theorem was originally suggested by Sikorski [5] in case $p = 1$ and $n = 3$, and proved by Grothendieck [2]. Here we have generalized it to arbitrary $p \geqslant 1$.

Theorem 2 (Sikorski's trace formula). *Let X be a Banach space, p a real number with $1 \leqslant p < \infty$ and $n \geqslant 3$ a natural number such that $p \leqslant p'(n-1)$. If $\{A_1, A_2, \ldots, A_n\} \subseteq I_p(X)$, then for any cyclic permutation $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$ we have*

$$\mathrm{Tr}_{A_{i_1}}(A_{i_2}, \ldots, A_{i_n}) = \mathrm{Tr}_{A_1}(A_2, \ldots, A_n).$$

Remark. It follows easily from the final remarks of Persson [3] together with the multiplication theorems of [4] that any composition of $n-1$ of the operators $A_1, A_2, \ldots, A_n$ is p'-nuclear, and hence in view of Theorem 1 all the traces in Theorem 2 are well defined.

The theorem will be a direct consequence of the following two lemmas.

Lemma 1. *Let X be a Banach space and let B^* denote the unit ball of X^* equipped with the w^*-topology. If p, n and $A_1, A_2, \ldots, A_n$ are as in Theorem 2, then there exist p-integral operators $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n$ on $C(B^*)$ such that*

$$\mathrm{Tr}_{\hat{A}_{i_1}}(\hat{A}_{i_2} \circ \ldots \circ \hat{A}_{i_n}) = \mathrm{Tr}_{A_{i_1}}(A_{i_2} \circ A_{i_3} \circ \ldots \circ A_{i_n})$$

for all permutations $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$.

Proof. Let $P\colon X\to C(B^*)$ denote the canonical isometry. For each i, $1\leqslant i\leqslant n$ there is a Radon measure μ_i on B^* and a bounded linear operator $Q_i\colon L^p(B^*,\mu_i)\to X$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{A_i} & X \\ {\scriptstyle P}\downarrow & & \uparrow{\scriptstyle Q_i} \\ C(B^*) & \xrightarrow{I_{\mu_i}} & L^p(B^*,\mu_i) \end{array}$$

is commutative, I_{μ_i} denoting the formal identity map of $C(B^*)$ into $L^p(B^*,\mu_i)$.

Let us define

$$\hat{A}_i = PQ_iI_{\mu_i}\quad \text{for } 1\leqslant i\leqslant n.$$

Clearly $\hat{A}_i$ is a p-integral operator on $C(B^*)$. Let now $i_1, i_2, \ldots, i_n$ be a permutation of $1,2,\ldots,n$. The operator $N\colon C(B^*)\to X$ defined by

$$N = A_{i_2}\circ\ldots\circ A_{i_{n-1}}\circ Q_{i_n}\circ I_{\mu_{i_n}}$$

is p'-nuclear, hence it has a p'-nuclear representation

$$N=\sum_{k=1}^{\infty}\nu_k\otimes x_k,\qquad \{\nu_k\}\subseteq C(B^*)^*,\qquad \{x_k\}\subseteq X.$$

Since

$$NP = A_{i_2}\circ A_{i_3}\circ\ldots\circ A_{i_n}$$

and

$$PN = \hat{A}_{i_2}\circ\hat{A}_{i_3}\circ\ldots\circ\hat{A}_{i_n}$$

we get the p'-nuclear representations

$$\hat{A}_{i_2}\circ\ldots\circ\hat{A}_{i_n} = \sum_k \nu_k\otimes Px_k,$$

$$A_{i_2}\circ\ldots\circ A_{i_n} = \sum_k P^*\nu_k\otimes x_k$$

and hence

$$\begin{aligned}\operatorname{Tr}_{A_{i_1}}(A_{i_2}\circ\ldots\circ A_{i_n}) &= \sum_k (P^*\nu_k)A_{i_1}x_k = \sum_k \nu_k PA_{i_1}x_k\\ &= \sum_k \nu_k\hat{A}_{i_1}Px_k = \operatorname{Tr}_{\hat{A}_{i_1}}(\hat{A}_{i_2}\circ\ldots\circ\hat{A}_{i_n}).\end{aligned}$$

Lemma 2. *If X is Banach space having the approximation property and $1\leqslant p\leqslant\infty$, $A\in I_p^Q(X)$, $N\in N_{p'}(X)$, then*

$$\operatorname{Tr}(AN) = \operatorname{Tr}(NA).$$

Proof. Let $A\in I_p^Q(X)$. Since X has the approximation property both $\operatorname{Tr}(AN)$ and $\operatorname{Tr}(NA)$ are defined for all $N\in N_{p'}(X)$; futhermore the linear functionals $N\to\operatorname{Tr}(AN)$ and $N\to\operatorname{Tr}(NA)$ are continuous on $N_{p'}(X)$, and since they clearly agree on $B_0(X)$ they are equal on the whole of $N_{p'}(X)$. Q.E.D.

Proof of Theorem 2. Let p, n and $A_1, A_2, \ldots, A_n$ be as in Theorem 2, and let $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n$ denote the operators from Lemma 2. Since $C(B^*)$ has the approximation property we get from Lemma 3:

$$\begin{aligned}\operatorname{Tr}_{A_1}(A_2\circ\ldots\circ A_n) &= \operatorname{Tr}(\hat{A}_1\circ\hat{A}_2\circ\ldots\circ\hat{A}_n)\\ &= \operatorname{Tr}(\hat{A}_2\circ\hat{A}_3\circ\ldots\circ\hat{A}_n\circ\hat{A}_1) = \operatorname{Tr}_{\hat{A}_2}(\hat{A}_3\circ\ldots\circ\hat{A}_n\circ\hat{A}_1)\\ &= \operatorname{Tr}_{A_2}(A_3\circ\ldots\circ A_n\circ A_1).\end{aligned}$$

Continuing now, using Lemma 1 and Lemma 2 in that way, it is clear that we for any cyclic permutation $i_1, i_2, \ldots, i_n$ of $1,2,\ldots,n$ get that

$$\operatorname{Tr}_{A_{i_1}}(A_{i_2}\circ\ldots\circ A_{i_n}) = \operatorname{Tr}_{A_1}(A_2\circ\ldots\circ A_n).$$

This concludes the proof of the theorem.

References

[1] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[2] — *The trace of some operators*, Studia Math. 20 (1961), pp. 141–143.

[3] A. Persson, *On some properties of p-nuclear and p-integral operators*, ibidem 33 (1969), pp. 213–222.

[4] — and A. Pietsch, *p-nukleäre und p-integrale Abbildungen in Banachräumen*, ibidem 33 (1969), pp. 19–62.

[5] R. Sikorski, *The determinant theory in Banach spaces*, Colloq. Math. 8 (1961), pp. 141–198.

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