normal for each n) without the decomposition being monotone (cf. Remark 3.2). We also note that if the decomposition for X is monotone, then the decomposition for A $(k^n - X_s)$ is also monotone.

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The domain of attraction of a normal distribution in a Hilbert space

by

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Abstract. Let H be a separable real Hilbert space. Denote by $\Pi^{(nd)}$ the domain of attraction of normal non-degenerate probability distributions in H. If $p \in \Pi^{(nd)}$, then

$$\int_H \|x\|^\delta p(dx) < +\infty \quad \text{for } 0 \leqslant \delta < 2.$$

Assign to a distribution p in H the family of S-operators S defined by the bilinear form

$$(S_X g, h) = \int_{\|x\|<X} (x, g)(x, h) p*^{-}p(dx) \quad \text{for every } g, h \in H.$$

In terms of operators S_X we give necessary and sufficient conditions in order that $p \in \Pi^{(nd)}$.

Introduction. The paper is an attempt to extent the known results of A. J. Khinchin and P. Lévy concerning the domain of attraction of a normal distribution on a straight line to Hilbert spaces (see [6] and [8]).

Section 1 of the paper contains the basic definitions and theorems of the theory of probability distributions in a Hilbert space.

Section 2 includes the theorems concerning the shift-convergence of a sequence of distributions $\mu_n = \prod_{k=1}^{k_n} \mu_{n,k}$ with $\mu_{n,k}$ uniformly asymptotically negligible to a normal distribution. These theorems follow from the results formulated in the papers by Varadhan [11] and Jajte [3]. In Section 3 we give theorems which are the basic aim of the paper, viz. we formulate some properties of distributions belonging to the domain of attraction of a normal distribution in a Hilbert space and also the necessary and sufficient conditions in order that a distribution belong to the domain of attraction of a normal distribution in a Hilber space.

1. Let H be a separable real Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Let $\mathfrak{M}$ denote the set of all probability distributions in H, i.e. the set of normed regular measures defined in a σ-field $\mathscr{B}$ of Borelian subsets of H. $\mathfrak{M}$ is a complete space with the Lévy–Prochorov metric (see [10], p. 188). Convergence in this metric space is equivalent to the weak convergence of distributions (1).

The convolution $p*q$ (2) is a continuous operation in $\mathfrak{M}$. The convolution of n distributions $p_1, \ldots, p_n$ will be denoted by $\prod_{k=1}^{n}{}^{*} p_k$ while the convolution of n identical distributions will be denoted by p^{n*}.

Let $p \in \mathfrak{M}$ and $f \in H$. By p^f we denote the distribution on a straight line induced by the element f, i.e.

$$(1) \qquad p^f(B) = p\{x \in H\colon (x, f) \in B\}$$

for every B Borelian set on a straight line.

The characteristic functional $\hat{p}(f)$ of the distribution $p \in \mathfrak{M}$ is defined by the formula

$$(2) \qquad \hat{p}(f) = \int_H e^{i(x,f)} p(dx) \quad \text{(see [7])}.$$

This functional determines the distribution uniquely.

A linear operator in H is called an *S-operator* if it is non-negative, self-adjoint and has a finite trace (see [10], p. 193).

A distribution $\mu \in \mathfrak{M}$ is called *normal* if

$$(3) \qquad \hat{\mu}(f) = \exp[i(x_0, f) - \tfrac{1}{2}(Sf, f)],$$

where $x_0 \in H$ and S is an S-operator.

Denote by δ_x a degenerate distribution concentrated at a point $x \in H$, i.e. $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$ for $A \in \mathscr{B}$. A sequence of distributions p_n is called *shift-compact* (*shift-convergent*) *in* $\mathfrak{M}$ if there exists a $\{x_n\}$ of elements of H such that the sequence of distributions $p_n * \delta_{x_n}$ is compact (convergent).

A distribution μ is called *infinitely divisible* if for every natural number n there exists a distribution μ_n such that

$$\mu = \mu_n^{n*}.$$

The distributions $\mu_{n,k}$ $(k = 1, 2, \ldots, k_n)$ are *uniformly asymptotically negligible* if

$$(4) \qquad \lim_{n\to\infty} \inf_{1 \leqslant k \leqslant k_n} \mu_{n,k}(U) = 1,$$

where U is an arbitrary neighbourhood of zero in H.

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(1) A sequence of distributions $p_n \in \mathfrak{M}$ is called *weakly convergent to the distribution* p if for every bounded and continuous function defined in H we have

$$\lim_{n\to\infty} \int_H f(x) p_n(dx) = \int_H f(x) p(dx).$$

(2) $p*q(A) = \int_H p(A-x) q(dx)$ for every $A \in \mathscr{B}$.

Let $p \in \mathfrak{M}$, define the distribution $e(p)$ by the formula

$$(5) \qquad \widehat{e(p)}(f) = \exp\Big\{\int_H [e^{i(x,f)} - 1] p(dx)\Big\} \quad \text{for every } f \in H \text{ (3)}.$$

The distribution $e(p)$ is infinitely divisible (see [9], p. 79). We introduce the following notation:

$$T_a p(A) = p\{x \in H\colon ax \in A\}$$

for every $A \in \mathscr{B}$, where a is an arbitrary real number.

$$^{-}p = T_{-1}p, \qquad ^{\circ}p = p * {}^{-}p.$$

In the paper we make use of the following theorems:

1.1. *If the sequence of distributions $p_n = q_n * r_n$ is compact in $\mathfrak{M}$, then the sequences of distributions q_n and r_n are shift-compact in $\mathfrak{M}$* (see [9], Theorem 2.2).

1.2. *If the sequence of distributions p_n is compact in $\mathfrak{M}$ and $\lim_{n\to\infty} \hat{p}_n(f) = g(f)$ for every $f \in H$, then the sequence p_n converges weakly to the distribution p and $\hat{p}(f) = g(f)$* (see [10], Lemma 1.6).

1.3. *The set of distributions p_t, $t \in T$ is compact in $\mathfrak{M}$ if and only if for every $\varepsilon > 0$ there exists a compact set $Z_\varepsilon \subset H$ such that for every $t \in T$, $p_t(H - Z_\varepsilon) < \varepsilon$* (see [10], Theorem 1.12).

2. Let $\mu_{n,k}$ $(k = 1, 2, \ldots, k_n)$ be uniformly asymptotically negligible. Write

$$(1) \qquad \mu_n = \prod_{k=1}^{k_n}{}^{*} \mu_{n,k},$$

$$(2) \qquad (x_{n,k}, f) = \int_{\|x\| \leqslant 1} (x, f) \mu_{n,k}(dx) \quad \text{for every } f \in H,$$

$$(3) \qquad M_n = \sum_{k=1}^{k_n} (\mu_{n,k} * \delta_{-x_{n,k}}),$$

$$(4) \qquad (T_n^{\varepsilon} g, h) = \int_{\|x\| \leqslant \varepsilon} (x, g)(x, h) M_n(dx) \quad \text{with an arbitrary positive } \varepsilon.$$

It follows from the infinite divisibility of the distribution $\prod_{k=1}^{k_n}{}^{*} e(\mu_{n,k} * \delta_{-x_{n,k}})$ and from Theorem 5.10 in [11] that $\int_{\|x\| \leqslant \varepsilon} \|x\|^2 M_n(dx) < +\infty$, and hence it follows that the operator T_n^{ε} defined by bilinear form (4) is an S-operator.

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(3) Then $e(p) = e^{-1} \sum_{i=0}^{\infty} \frac{p^{i*}}{i!}$, where $p^{0*} = \delta_0$.

Corollary 2.1. *The sequence of distributions* (1) *is shift-convergent to a normal distribution if and only if*

1. $\lim\limits_{n\to\infty} \int\limits_{\|x\|\geqslant\varepsilon} M_n(dx) = 0$,

2. $\lim\limits_{N\to\infty} \sup\limits_{n} \sum\limits_{i=N}^{\infty} (T_n^{\varepsilon} e_i, e_i) = 0$, $\{e_i\}$ *is a basis in* H,

3. $\lim\limits_{n\to\infty} (T_n^{\varepsilon} f, f) = (Bf, f)$ *for every* $f \in H$,

for an arbitrary $\varepsilon > 0$ (see Theorem 6.3 in [9]).

Then the sequence (1) *is shift-convergent to the normal distribution determined by the S-operator B.*

Proof. To prove this it suffices to employ the corollary in [3] and Theorem 6.4 in [11] and also the fact that if condition 1 is satisfied, then $\overline{\lim\limits_{n\to\infty}} (T_n^{\varepsilon} f, f)$ and $\underline{\lim\limits_{n\to\infty}} (T_n^{\varepsilon} f, f)$ do not depend on ε. It follows from conditions 2 and 3 that B is an S-operator and hence that the assumption of compactness of the operators T_n is reduced to condition 2.

Define now by bilinear forms the S-operators:

$$(5)\qquad (B_n^{\varepsilon} g, h) = \sum_{k=1}^{k_n} \Big\{ \int\limits_{\|x\|\leqslant\varepsilon} (x, g)(x, h)\mu_{n,k}(dx) - \Big[\int\limits_{\|x\|\leqslant\varepsilon} (x, g)\mu_{n,k}(dx) \cdot \int\limits_{\|x\|\leqslant\varepsilon} (x, h)\mu_{n,k}(dx) \Big] \Big\},$$

$$(6)\qquad (\bar{B}_n^{\varepsilon} g, h) = \sum_{k=1}^{k_n} \int\limits_{\|x\|\leqslant\varepsilon} (x - x_{n,k}, g)(x - x_{n,k}, h)\mu_{n,k}(dx),$$

where ε is an arbitrary positive number and define the measure

$$(7)\qquad M_n' = \sum_{k=1}^{k_n} \mu_{n,k}.$$

Theorem 2.1. *The sequence of distributions* (1) *is shift-convergent to a normal distribution in* H *if and only if*

1′. $\lim\limits_{n\to\infty} \int\limits_{\|x\|\geqslant\varepsilon} M_n'(dx) = 0$,

2′. $\lim\limits_{N\to\infty} \sup\limits_{n} \sum\limits_{i=N}^{\infty} (B_n^{\varepsilon} e_i, e_i) = 0$, $\{e_i\}$ *is a basis in* H,

3′. $\lim\limits_{n\to\infty} (B_n^{\varepsilon} f, f) = (Bf, f)$ *for every* $f \in H$,

with an arbitrary $\varepsilon > 0$.

Proof. Write $l_n = \sup\limits_{1\leqslant k\leqslant k_n} \|x_{n,k}\|$. By Lemma 7.1 in [11]

$$(8)\qquad \lim_{n\to\infty} l_n = 0.$$

Simultaneously

$$(9)\qquad \int\limits_{\|x\|\geqslant\varepsilon+l_n} M_n(dx) = \sum_{k=1}^{k_n} \int\limits_{\|x-x_{n,k}\|\geqslant\varepsilon+l_n} \mu_{n,k}(dx) \leqslant \int\limits_{\|x\|\geqslant\varepsilon} M_n'(dx) \leqslant \sum_{k=1}^{k_n} \int\limits_{\|x-x_{n,k}\|\geqslant\varepsilon-l_n} \mu_{n,k}(dx) = \int\limits_{\|x\|\geqslant\varepsilon-l_n} M_n(dx).$$

From (8) and (9) we find that conditions 1 and 1′ are equivalent.

From (8) it also follows that condition

$$(10)\qquad \lim_{N\to\infty} \sup_{n} \sum_{i=N}^{\infty} (\bar{B}_n^{\varepsilon} e_i, e_i) = 0 \quad \text{for every } \varepsilon > 0$$

is equivalent to condition 2 and that condition

$$(11)\qquad \lim_{n\to\infty} (\bar{B}_n^{\varepsilon} f, f) = (Bf, f) \quad \text{for every } f \in H \text{ and } \varepsilon > 0$$

is equivalent to condition 3 (see Corollary 2.1).

Let $0 < \varepsilon < 1$. Find

$$(\bar{B}_n^{\varepsilon} f, f) - (B_n^{\varepsilon} f, f) = \sum_{k=1}^{k_n} (g_{n,k}^{\varepsilon}, f)^2 - \sum_{k=1}^{k_n} (x_{n,k}, f)^2 \int\limits_{\|x\|\geqslant\varepsilon} \mu_{n,k}(dx),$$

where $(g_{n,k}^{\varepsilon}, f) = \int\limits_{\varepsilon<\|x\|\leqslant 1} (x, f)\mu_{n,k}(dx)$ for every $f \in H$. Introduce the notation

$$(12)\qquad (Q_n^{\varepsilon} g, h) = \sum_{k=1}^{k_n} (g_{n,k}^{\varepsilon}, g)(g_{n,k}^{\varepsilon}, h) + \sum_{k=1}^{k_n} (x_{n,k}, g)(x_{n,k}, h) \int\limits_{\|x\|\geqslant\varepsilon} \mu_{n,k}(dx).$$

Bilinear form (12) defines the S-operator Q_n^{ε}.

Suppose condition 1′ is satisfied. It follows from (8), (12) and from the fact that $\mu_{n,k}$ are uniformly asymptotically negligible that

$$(13)\qquad \lim_{n\to\infty} \sum_{i=1}^{\infty} (Q_n^{\varepsilon} e_i, e_i) = 0,$$

$$(14)\qquad \lim_{n\to\infty} (Q_n^{\varepsilon} f, f) = 0 \quad \text{for every } f \in H,$$

(13) means that (10) is equivalent to condition 2′, (14) means that (11) is equivalent to condition 3′. The assumption $0 < \varepsilon < 1$ is not essential.

Remark. Making use of a corollary in [3] one may prove in the same way as above a more general theorem which asserts in terms of the measure M_n' and operators B_n^{ε} the necessary and sufficient conditions for sequence (1) to be shift-convergent.

Lemma 2.1. *If the distribution* $p * {}^{-}p$ *is normal in* H, *then the distribution* p *is normal in* H.

Proof. By assumption we have $|\hat{p}(f)|^2 = e^{-(Sf,f)}$, where S is an S-operator.

Let $f \in H$. Consider the distribution $(p * {}^{-}p)^f = p^f * {}^{-}p^f$ on a straight line; the characteristic function of this distribution is $|\hat{p}(t\cdot f)|^2 = e^{-t^2(Sf,f)}$.

By Cramer's theorem (see [1])

$$\hat{p}(t\cdot f) = \exp[-\tfrac{1}{2}t^2(Sf,f)+itm(f)]. \tag{15}$$

Since $m(f_1+f_2)$ may be interpreted as the expected value of the random variable (f_1+f_2, ξ), where ξ is a random variable with the values from H and with the distribution p, the functional $m(f)$ is additive.

Let $\|f_n\| \to 0$ as $n \to \infty$; then by Theorem 1.10 in [10] the sequence p^{f_n} converges weakly to p^0. The distributions p^{f_n} are normal on a straight line and thus $m(f_n) \to 0$ as $n \to \infty$. Thus the functional $m(f)$ is linear, i.e. there exists an $x_0 \in H$ such that $m(f) = (x_0, f)$. Putting in (15) $t = 1$, we obtain the assertion.

LEMMA 2.2. *If the sequence $q_n * {}^{-}q_n$ converges weakly to a normal distribution in H, then the sequence q_n is shift-convergent to a normal distribution in H.*

Proof. Let a sequence of distributions ${}^{\circ}q_n$ converge weakly to the distribution q and let $\hat{q}(f) = e^{-(Sf,f)}$, where S is an S-operator.

By Theorem 1.1 the sequence q_n is shift-compact. Thus there exists a sequence $x_n \in H$ such that each subsequence of the sequence $q_n * \delta_{x_n}$ includes a subsequence converging to some distribution. By Lemma 2.1 the limit distributions are normal, have the same dispersion operator S and differ in the factor δ_x. Thus the sequence x_n may be so modified that every subsequence of the sequence $q_n * \delta_{x_n}$ includes a subsequence converging to the same normal distribution, e.g. to the distribution with the characteristic functional $e^{-\frac{1}{2}(Sf,f)}$, and thus the sequence q_n is shift-convergent to a normal distribution in H.

Introduce the notation

$$\mathring{M}_n = \sum_{k=1}^{k_n} {}^{\circ}\mu_{n,k}, \tag{16}$$

$$(\mathring{B}_n^{\varepsilon} g, h) = \int_{\|x\|\leqslant\varepsilon} (x, g)(x, h)\mathring{M}_n(dx). \tag{17}$$

COROLLARY 2.2. *The sequence* (1) *is shift-convergent to a normal distribution if and only if*

1° $\lim\limits_{n\to\infty} \int_{\|x\|\geqslant\varepsilon} \mathring{M}_n(dx) = 0$,

2° $\lim\limits_{N\to\infty} \sup\limits_{n} \sum_{i=N}^{\infty} (\mathring{B}_n^{\varepsilon} e_i, e_i) = 0$,

3° $\lim\limits_{n\to\infty} (\mathring{B}_n^{\varepsilon} f, f) = (Bf, f)$ *for every* $f \in H$

for arbitrary $\varepsilon > 0$.

Proof. By Lemma 2.2 one may consider the weak convergence of the sequence ${}^{\circ}\mu_n = \prod_{k=1}^{k_n}{}^{*}\, {}^{\circ}\mu_{n,k}$. It follows from Corollary 2.1 that the conditions 1°, 2°, 3° are equivalent to the shift-convergence of the sequence of distributions ${}^{\circ}\mu_n$ to a normal distribution in H. The distributions ${}^{\circ}\mu_n$ are symmetrised, and from Theorems 4.4 and 4.5 in [9] we easily obtain the weak convergence of the sequence ${}^{\circ}\mu_n$.

3. The set of distributions $p \in \mathfrak{M}$ for which there exists a sequence of positive numbers $\{a_n\}$ such that the sequence of distributions $T_{a_n} p^{n*}$ is shift-convergent to a distribution $q \in \mathscr{G}$ is called the *domain of attraction of the set of distributions* $\mathscr{G}$.

Denote by

$\Pi(B)$ — the domain of attraction of normal distribution determined by S-operator B;

$\Pi^{(nd)}$ — the domain of attraction of normal, non-degenerate distributions in H, i.e. $(Sf,f) \not\equiv 0$;

$\Pi^{(r)}$ — the domain of attraction of normal distributions in H and regular, i.e. $(Sf,f) > 0$ for $f \neq 0$;

$\Pi_1^{(nd)}$ — the domain of attraction of the family of one-dimensional normal non-degenerate distributions;

$\Pi^{(nd)}, \Pi_1^{(nd)}$ — the normal domain of attraction, i.e. $a_n = c/\sqrt{n}$, where $c > 0$.

COROLLARY 3.1. *A distribution p belongs to $\Pi(B)$ if and only if there exists a sequence of positive numbers $\{a_n\}$ such that the sequence of distributions $T_{a_n}{}^{\circ}p^{n*}$ converges weakly to the normal distribution determined by the S-operator B.*

LEMMA 3.1. *If the distribution p belongs to $\Pi^{(nd)}$ with a sequence $\{a_n\}$, then*

$$\lim_{n\to\infty} a_n = 0, \tag{1}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1 \quad \text{(see Lemma 2 in [4]).} \tag{2}$$

Proof. By assumption the sequence $T_{a_n}p^{n*}$ is shift-convergent to the non-degenerate normal distribution q. It is easy to find that there exists an element $f \in H$ such that the distribution q^f is non-degenerate on a straight line. Thus the sequence of non-degenerate one-dimensional distributions $T_{a_n}(p^f)^{n*}$ is shift-convergent to a non-degenerate distribution q^f. By a lemma from § 29 and Theorem 4 from § 14 of [2] we have the assertion.

COROLLARY 3.2. *The distribution p belongs to $\Pi(B)$ if and only if there exists a sequence of positive numbers $\{a_n\}$ such that*

1° $\lim_{n\to\infty} n \int_{\|x\|\geq\varepsilon} T_{a_n}{}^\circ p(dx) = 0,$

2° $\lim_{N\to\infty} \sup_n \sum_{i=N}^{\infty} n \int_{\|x\|\leq\varepsilon} (x, e_i)^2 T_{a_n}{}^\circ p(dx) = 0,$

where $\{e_i\}$ *is a basis in* H.

3° $\lim_{n\to\infty} n \int_{\|x\|\leq\varepsilon} (x, f)^2 T_{a_n}{}^\circ p(dx) = (Bf, f)$ *for every* $f \in H$,

for an arbitrary $\varepsilon > 0$.

To prove this it suffices to make use of Corollary 2.2 and the fact that the distributions $T_{a_n} p$ are uniformly asymptotically negligible.

THEOREM 3.1. *A non-degenerate distribution* p *belongs to* $\pi^{(nd)}$ *if and only if*

$$\int_H \|x\|^2 {}^\circ p(dx) < +\infty. \tag{3}$$

Suppose that 1°, 2°, 3° *hold with* $a_n = 1/\sqrt{n}$ *and* $(Bf, f) \not\equiv 0$. *Then for an arbitrary* $\varepsilon > 0$

$$\lim_{n\to\infty} n \int_{\|x\|\geq\varepsilon\sqrt{n}} {}^\circ p(dx) = 0, \tag{4}$$

$$\lim_{n\to\infty} \int_{\|x\|\leq\varepsilon\sqrt{n}} \|x\|^2 \, {}^\circ p(dx) = \sum_{i=1}^{\infty} (Be_i, e_i) = a^2 > 0. \tag{5}$$

Let ${}^\circ\xi$ be a random variable in H with the distribution ${}^\circ p$. From (4) and (5) it follows that the distribution of random variable $\|{}^\circ\xi\|$ belongs to $\pi_1^{(nd)}$ (Theorem 2, § 26 in [2]), and by Theorem 6, § 34 in [2], we have (3).

Suppose now that (3) holds. Condition 1° for $a_n = 1/\sqrt{n}$ is obtained from the inequality:

$$0 \leq \varepsilon^2 n \int_{\|x\|\geq\varepsilon\sqrt{n}} {}^\circ p(dx) \leq \int_{\|x\|\geq\varepsilon\sqrt{n}} \|x\|^2 \, {}^\circ p(dx).$$

Condition 2° follows from the inequality

$$\int_{\|x\|\leq\varepsilon\sqrt{n}} (x, e_i)^2 \, {}^\circ p(dx) \leq \int_H (x, e_i)^2 \, {}^\circ p(dx),$$

while condition 3° results from the monotonity and boundedness of the sequence $\int_{\|x\|\leq\varepsilon\sqrt{n}} (x, f)^2 \, {}^\circ p(dx)$ and from conditions 1° and 2°.

COROLLARY 3.3. *A non-degenerate distribution* p *belongs to* $\pi^{(nd)}$ *if and only if*

$$\int_H \|x\|^2 p(dx) < +\infty. \tag{6}$$

Proof. Condition (6) is equivalent to condition (3) (see the proof of Theorem 2.6. (vi) in [5]).

COROLLARY 3.4. *If* $p \in \Pi^{(r)}$ *and* $\int_H \|x\|^2 p(dx) = +\infty$, *then*

$$\int_H (x, f)^2 p(dx) = +\infty \quad \textit{for every } f \in H \textit{ and } f \neq \theta.$$

Proof. Suppose there exist an $\tilde{f} \in H$ and an $\tilde{f} \neq \theta$ such that

$$0 < \int_H (\tilde{f}, x)^2 \, {}^\circ p(dx) < +\infty;$$

then $p^{\tilde{f}} \in \pi_1^{(nd)}$ (see Theorem 6, § 34 in [2]). From the assumption, $p^{\tilde{f}} \in \Pi_1^{(nd)}$ with some sequence $\{a_n\}$. By Theorem 2, § 10 in [2] we have

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{\sqrt{n}}} = c > 0.$$

Thus the sequences of distributions $T_{a_n} p^{n*}$ and $T_{c/\sqrt{n}} p^{n*}$ are shift-convergent to the same normal distribution (Theorem 1.10 in [10]), and by Corollary 3.3 we have (6), which contradicts the assumption.

Let $p \in \mathfrak{M}$, define the distribution p_1 on a straight line by the formula

$$p_1(B) = p\{x \in H : \|x\| \in B\} \tag{7}$$

for every Borelian set B on a straight line.

THEOREM 3.2. *Let* p *belong to* $\Pi^{(nd)}$; *then* p_1 *belongs to* $\Pi_1^{(nd)}$.

Proof. If condition (6) is satisfied, we have $p_1 \in \Pi_1^{(nd)}$ (Theorem 6, § 34 in [2]). The opposite case remains to be considered.

By assumption and by Theorem 2.1 there exists a sequence of positive numbers $\{a_n\}$ such that

$1'$. $\lim_{n\to\infty} n \int_{\|x\|\geq\varepsilon/a_n} p(dx) = 0,$

$2'$. $\lim_{N\to\infty} \sup_n \sum_{i=N}^{\infty} n \cdot a_n^2 \Big\{ \int_{\|x\|\leq\varepsilon/a_n} (x, e_i)^2 p(dx) - \Big[\int_{\|x\|\leq\varepsilon/a_n} (x, e_i) p(dx) \Big]^2 \Big\} = 0,$

$3'$. $\lim_{n\to\infty} n \cdot a_n^2 \Big\{ \int_{\|x\|\leq\varepsilon/a_n} (x, f)^2 p(dx) - \Big[\int_{\|x\|\leq\varepsilon/a_n} (x, f) p(dx) \Big]^2 \Big\} = (Bf, f)$

for an arbitrary $\varepsilon > 0$.

From conditions $2'$ and $3'$ we obtain

(8)

$$\lim_{n\to\infty} n a_n^2 \Big\{ \int_{\|x\|\leq\varepsilon/a_n} \|x\|^2 p(dx) - \sum_{i=1}^{\infty} \Big[\int_{\|x\|\leq\varepsilon/a_n} (x, e_i) p(dx) \Big]^2 \Big\} = \sum_{i=1}^{\infty} (Be_i, e_i) = a^2 > 0.$$

Write $(g_n^\varepsilon, f) = \int_{\|x\|\leq\varepsilon/a_n} (x, f) \, p(dx)$; thus we find

$$\sum_{i=1}^{\infty} \Big[\int_{\|x\|\leq\varepsilon/a_n} (x, e_i) p(dx) \Big]^2 = \|g_n^\varepsilon\| \leq \Big[\int_{\|x\|\leq\varepsilon/a_n} \|x\| p(dx) \Big]^2. \tag{9}$$

Since $\int\limits_H \|x\|^2 p(dx) = +\infty$, we have

(10) $$\lim_{n\to\infty} \frac{\Big[\int\limits_{\|x\|\leqslant \varepsilon/a_n} \|x\| p(dx)\Big]^2}{\int\limits_{\|x\|\leqslant \varepsilon/a_n} \|x\|^2 p(dx)} = 0$$ (see the proof of Theorem 1, § 34 in [2]).

Making use of (10), we get the equivalence of the following conditions:

(11) $$\lim_{n\to\infty} na_n^2 \Big\{\int\limits_{\|x\|\leqslant \varepsilon/a_n} \|x\|^2 p(dx) - \Big[\int\limits_{\|x\|\leqslant \varepsilon/a_n} \|x\| p(dx)\Big]^2\Big\} = a^2 > 0,$$

(12) $$\lim_{n\to\infty} na_n^2 \int\limits_{\|x\|\leqslant \varepsilon/a_n} \|x\|^2 p(dx) = a^2 > 0.$$

From (9) it follows simultaneously that (12) is equivalent to (8); thus we have proved that condition (11) is satisfied. By theorem 2, § 26 in [2] and by conditions 1′ and (11) we get the assertion.

COROLLARY 3.5. *If a distribution p belongs to $\Pi^{(nd)}$, then*

$$\int\limits_H \|x\|^\delta p(dx) < +\infty \quad \textit{for } 0 \leqslant \delta < 2.$$

This corollary follows from Theorem 3.2 and from Theorem 5, § 34 in [2].

Let $\{e_i\}$ be a basis in H.

THEOREM 3.3. *A distribution p belongs to $\Pi^{(r)}$ if and only if*

(a) *the distribution p^{e_i} belongs to $\Pi_1^{(nd)}$ for every $i = 1, 2, \ldots$ with a universal sequence of positive numbers $\{a_n\}$ (independent of i) and $\sum_{i=1}^{\infty} \sigma_i^2 < +\infty$, where σ_i^2 are the variances of corresponding limit distributions.*

(b) *The sequence of distributions $T_{a_n} P^{n*}$ is shift-compact.*

Proof. The necessity of these conditions is obvious.

To prove the sufficience we prove the weak convergence of the sequence $T_{a_n} \circ p^{n*}$; by (b) it suffices to prove the convergence of the sequence of characteristic functionals $\widehat{T_{a_n} \circ p^{n*}}(f)$ to the functional $e^{-\frac{1}{2}(Bf,f)}$, where B is S-operator.

Define the S-operator B as follows:

(13) $$(Bg, h) = \sum_{i=1}^{\infty} (g, e_i)(h, e_i)\sigma_i^2.$$

By assumption

(14) $$\lim_{n\to\infty} |\hat{p}(a_n \cdot t \cdot e_i)|^{2n} = e^{-\frac{1}{2}(Be_i, e_i)t^2} \text{ for every } t \text{ and for } i = 1, 2, \ldots$$

Let ξ_n stand for a random variable with the values in H and with the distribution $p_n = T_{a_n} \circ p^{n*}$. Let f be an arbitrary element of H,

$f_i = (f, e_i)$ and let N be an arbitrary natural number. Consider a sequence of random vectors in R^N

$$X_n = [(\xi_n, f_1 e_1), (\xi_n, f_2 e_2), \ldots, (\xi_n, f_N e_N)].$$

Let η be a random variable in H with the normal distribution determined by the S-operator B. Consider the random vector in R^N

$$X = [(\eta, f_1 e_1), (\eta, f_2 e_2), \ldots, (\eta, f_N e_N)].$$

The characteristic function of the random vector X_n is

(15) $$\varphi_n(t_1, \ldots, t_N) = \hat{p}_n\Big(\sum_{j=1}^{N} t_j \cdot f_j \cdot e_j\Big)$$

and the characteristic function of the random variable X is

(16) $$\psi(t_1, \ldots, t_N) = \exp\Big[-\tfrac{1}{2}\sum_{j=1}^{N} t_j^2 \cdot f_j (Be_j, e_j)\Big].$$

The sequence of distributions of random vectors X_n is compact, by (14) the boundary distributions of the sequence of vectors X_n converge weakly to the corresponding boundary distributions of the random vector X. From (16) it follows that the distribution of the vector X is uniquely determined by its boundary distributions. Thus every weakly convergent subsequence of the sequence of distributions of the vectors X_n converges to the distribution of the vector X. Thus we have

(17) $$\lim_{n\to\infty} \varphi_n(t_1, \ldots, t_N) = \psi(t_1, \ldots, t_N)$$

and hence

(18) $$\lim_{n\to\infty} \hat{p}_n(f_N) = \exp[-\tfrac{1}{2}(Bf_N, f_N)], \quad \text{where } f_N = \sum_{j=1}^{N} f_j e_j.$$

Simultaneously we have

(19) $$\lim_{N\to\infty} \hat{p}_n(f_N) = \hat{p}_n(f).$$

Employing Theorem 1.3 and the fact that the sequence of distributions p_n is compact, one may easily show that the convergence in (19) is uniform and thus

$$\lim_{n\to\infty} \hat{p}_n(f) = e^{-\frac{1}{2}(Bf,f)} \quad \text{for every } f \in H.$$

Assign to a distribution $p \in \mathfrak{M}$ the family of S-operators S_X defined by the bilinear form

(20) $$(S_X g, h) = \int\limits_{\|x\|\leqslant X} (x, g)(x, h)^\circ p(dx).$$

THEOREM 3.4. *A non-degenerate distribution p belongs to $\Pi^{(nd)}$ if and only if there exists an element $g^* \in H$ satisfying the conditions*

(a) $$\lim_{X\to\infty} \frac{X^2 \int\limits_{\|x\|\geqslant X} {}^{\circ}p(dx)}{(S_X g^*, g^*)} = 0,$$

(b) $$\lim_{N\to\infty} \sup_X \sum_{i=N}^{\infty} \frac{(S_X e_i, e_i)}{(S_X g^*, g^*)} = 0, \quad \textit{where } \{e_i\} \textit{ is a basis in } H,$$

(c) $$\lim_{X\to\infty} \frac{(S_X f, f)}{(S_X g^*, g^*)} = (Df, f) \quad \textit{for every } f \in H.$$

Proof. *Sufficiency.* If condition (6) is satisfied, then $p \in \Pi^{(nd)}$. Thus let $\int\limits_H \|x\|^2 p(dx) = +\infty$. Define the sequence

$$C_n(\delta) = \inf\{X: n \int\limits_{\|x\|\geqslant X} {}^{\circ}p(dx) < \delta\}.$$

For every $\delta > 0$, $\lim\limits_{n\to\infty} C_n(\delta) = +\infty$ because $\int\limits_H \|x\|^2 p(dx) = +\infty$. Making use of (a) in an analogous way to that followed in the proof of Theorem 1, § 34 in [2], we find

$$\lim_{n\to\infty} \frac{n}{C_n^2(\delta)} \int\limits_{\|x\|\leqslant C_n(\delta)} (x, g^*)^2 \, {}^{\circ}p(dx) = +\infty \quad \text{for every } \delta > 0;$$

thus there exists a sequence $\delta_n \to 0$ as $n \to \infty$ such that

(21) $$\lim_{n\to\infty} \frac{n}{C_n^{*2}} \int\limits_{\|x\|\leqslant C_n^*} (x, g^*)^2 \, {}^{\circ}p(dx) = +\infty, \quad \text{where } C_n^* = C_n(\delta_n)$$

and

(22) $$\lim_{n\to\infty} n \int\limits_{\|x\|\geqslant C_n^*} {}^{\circ}p(dx) = 0.$$

Define the sequence

(23) $$a_n^2 = \Big[n \int\limits_{\|x\|\leqslant C_n^*} (x, g^*)\, {}^{\circ}p(dx)\Big]^{-1}.$$

From (21), (22) and (23) we obtain

(24) $$\lim_{n\to\infty} n \int\limits_{\|x\|\geqslant \varepsilon/a_n} {}^{\circ}p(dx) = 0 \quad \text{for an arbitrary } \varepsilon > 0.$$

For an arbitrary $f \in H$, an arbitrary $\varepsilon > 0$ and sufficiently large n we have

$$n a_n^2 \int\limits_{\|x\|\leqslant \varepsilon/a_n} (x, f)^2 \, {}^{\circ}p(dx) = n \cdot a_n^2 \int\limits_{\|x\|\leqslant C_n^*} (x, f) \, {}^{\circ}p(dx) + n \cdot a_n^2 \int\limits_{C_n^* < \|x\| \leqslant \varepsilon/a_n} (x, f)^2 \, {}^{\circ}p(dx).$$

By assumption (c) and (23) we find

(25) $$\lim_{n\to\infty} n \cdot a_n^2 \int\limits_{\|x\|\leqslant \varepsilon/a_n} (x, f)^2 \, {}^{\circ}p(dx) = (Df, f)$$

for every $f \in H$ and arbitrary $\varepsilon > 0$,

(26) $$(Dg^*, g^*) = 1.$$

It follows from assumption (b) that

$$\lim_{N\to\infty} \sup_n \sum_{i=N}^{\infty} \frac{n \cdot a_n^2 \int\limits_{\|x\|\leqslant \varepsilon/a_n} (x, e_i)^2 \, {}^{\circ}p(dx)}{n \cdot a_n^2 \int\limits_{\|x\|\leqslant \varepsilon/a_n} (x, g^*)^2 \, {}^{\circ}p(dx)} = 0 \quad \text{for arbitrary } \varepsilon > 0.$$

Hence and from (25) and (26) we get

(27) $$\lim_{N\to\infty} \sup_n \sum_{i=N}^{\infty} n \cdot a_n^2 \int\limits_{\|x\|\leqslant \varepsilon/a_n} (x, e_i)^2 \, {}^{\circ}p(dx) = 0 \quad \text{for arbitrary } \varepsilon > 0.$$

Basing on Corollary 3.2 and also on (24), (25), (26), (27), we find that $p \in \Pi^{(nd)}$.

Assume now that there exists a sequence of positive numbers $\{a_n\}$ such that conditions (24), (25), (27) are satisfied and the S-operator D is such that $(Dg^*, g^*) \neq 0$ for some $g^* \in H$ (the limit distribution is non-degenerate). We do not reduce the generality of our argument if we assume that (26) holds. It follows from (1) that for X sufficiently large there exists such an n that

$$\frac{1}{a_n} \leqslant X \leqslant \frac{1}{a_{n+1}}.$$

Employing (2) and the above argument, we can easily find that conditions (24), (25), (26), (27) imply conditions (a), (b), (c) of the theorem for an element $g^* \in H$.

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On the trace of some operators

by

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Abstract. Let X and Y be Banach spaces and $1 \leqslant p < \infty$. We consider the space $\mathrm{TR}_p(Y, X)$ of all bounded linear operators A from Y to X, for which the functional $N \to \mathrm{trace}(AN)$ is continuous for the p-nuclear norm on the space of all finite-dimensional operators from X to Y. Each such A defines in a unique way a cont. linear functional Tr_A on $N_p(X, Y)$ — the space of all p-nuclear operators from X to Y. It is shown, that every p'-integral operator $(1/p+1/p' = 1)$ from Y to X belongs to $\mathrm{Tr}_p(Y, X)$, and that every element of this space is p'-absolutely summing. This result is used to prove that if $n \geqslant 3$ is such that $p \leqslant p'(n-1)$ and $A_1, A_2, \ldots, A_n$ are p-integral operators on X, then $\mathrm{Tr}_{A_1}(A_2, A_3, \ldots, A_n) = \mathrm{Tr}_{A_{i_1}}(A_{i_2}, \ldots, A_{i_n})$ for each cyclic permutation $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$. This trace formula was conjectured by R. Sikorski. It should be mentioned that X is not assumed to have the approximation property.

Introduction and background. Let X be a Banach space. If $N\colon X \to X$ is a finite dimensional operator, i.e. there are finite sets $\{x_1, x_2, \ldots, x_k\} \subseteq X$, $\{x_1^*, x_2^*, \ldots, x_k^*\} \subseteq X^*$ such that

$$Nx = \sum_{n=1}^{k} x_n^*(x)x_n \quad \text{for all } x \in X,$$

then it is well known that N has a uniquely determined trace $\mathrm{Tr}(N)$, defined by

$$\mathrm{Tr}(N) = \sum_{n=1}^{k} x_n^*(x_n),$$

i.e. $\mathrm{Tr}(N)$ does not depend on the actual finite dimensional representation of N.

If N is a nuclear operator on X with nuclear representation $N = \sum_{n=1}^{\infty} x_n^* \otimes x_n$, we shall say that N has a uniquely determined trace $\mathrm{Tr}(N)$, if the sum $\sum_{n=1}^{\infty} x_n^*(x_n)$ depends only on N and not on the actual representation, and in that case we put

$$\mathrm{Tr}(N) = \sum_{n=1}^{\infty} x_n^*(x_n).$$