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Some metric and topological properties of certain linear spaces of measurable functions

by

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Abstract. Locally convex spaces of measurable functions, which are projective limits of the normed function spaces of Zaanen and Luxemburg, are studied from the topological point of view. Some duality results are obtained, in which topologies of uniform convergence on families of solid sets of functions are important. In the case in which such spaces are metrizable, a completeness condition can be given, and it is then often possible to represent the metric dual explicitly as a function space.

1. Introduction. It is the purpose of this note to communicate some results concerning topological vector spaces whose elements are measurable functions. These results are in part an extension of some older work of Dieudonné [1], and are related to the papers of Welland [6], [7]. We make use also of the recent work of Luxemburg and Zaanen on normed Köthe spaces [4].

The function spaces which we consider are all projective limits, by linear mappings of a very natural type, of a family of normed Köthe spaces (which need not be complete).

The term "locally convex space" will be used to mean "locally convex topological vector space". An "absolutely convex" set in a vector space will be a set which is both convex and balanced, and a "neighborhood" in a topological vector space will always be, unless stated otherwise, a neighborhood of the origin. Φ will stand for the scalar field which we now assume to be held fixed; Φ may be either the real or the complex number system. We assume from now on that μ is a non-trivial, non-negative, countably additive, (totally) σ — finite measure on the non-void set X; we assume that the Carathéodory extension procedure has been applied to μ, so that μ cannot be further extended by this proce-

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dure. "Measurable" will always mean μ-measurable; "a.e." will always mean μ-almost everywhere. Let Λ be the family of measurable subsets of X. M will stand for the set of all measurable functions on X of the form $f = a+bi$ where a and b take their values in the extended real number system (and $b = 0$ if Φ is the real field); see the remarks concerning such functions in [4], Note I. M^+ is the set of all (a.e.) non-negative elements of M. Without further comment, we will identify elements of M which are a.e. equal, as is usual in the theory of Köthe function spaces. If $f, g \in M$ and have real (possibly infinite) values, $f \leqslant g$ will mean $f(x) \leqslant g(x)$ a.e. $f = 0$ means $f(x) = 0$ a.e. If $Y \subseteq X$ then χ_Y is the characteristic function of Y, and $\int d\mu$ means $\int_X d\mu$. Without further mention, whenever a vector space E occurs whose elements are members of M, we will assume that the algebraic operations of E are the usual ones of pointwise addition and multiplication by a scalar. A subset K of M will be called solid if $|f_1| \leqslant |f_2|$, $f_2 \in K$ implies $f_1 \in K$. Note that K is solid if and only if (1) $0 \leqslant u \leqslant v \in K$ implies $u \in K$, and (2) $f \in K$ if and only if $|f| \in K$.

2. Saturation and spaces in duality. Let E be a solid vector space over Φ of elements of M; assume that for all $f \in E$, $|f(x)| < \infty$ a.e. We use the Köthe notation $E^\times$ to denote the set of all $g \in M$ such that $\int fg d\mu$ exists for all $f \in E$. We will extend slightly the notion of saturation ([4], Note IV) to obtain a simple condition that E and $E^\times$ be a pair of vector spaces in duality under the bilinear form $\int fg d\mu$.

LEMMA 1. *Suppose* $\emptyset \neq \Gamma \subseteq \Lambda$ *where* Γ *has the following properties:*

(1) *If* $A \in \Gamma$ *then* $\mu(A) > 0$ *and every subset of* A *with positive measure belongs to* Γ.

(2) Γ *is closed under the taking of countable unions.*

Then there exists a maximal element X_∞ *of* Γ; *that is, no subset of* $X - X_\infty$ *belongs to* Γ. *The set* X_∞ *is* μ*-uniquely determined; that is, if* $X'_\infty \in \Gamma$ *and* $X - X'_\infty$ *has no subsets belonging to* Γ, *then* X'_∞ μ*-almost equals* X_∞.

The proof of this lemma, which makes use of the σ-finiteness of μ, is omitted since it follows that of [4], Note IV, Theorem 8.3 (i).

We extend the terminology of [4], Note IV, with

DEFINITION 1. Any set A with positive measure such that every $f \in E$ vanishes a.e. on A will be called *E-purely infinite*. If there are no E-purely infinite sets then E will be called *saturated.*

COROLLARY. *If* E *is not saturated, then there exists a maximal* E*-purely infinite set* X_∞; *i.e.,* $X - X_\infty$ *does not have any* E*-purely infinite subsets.* X_∞ *is* μ*-uniquely determined.*

It is no loss of generality, therefore, to assume that E is saturated, for we can always consider the set $X - X_\infty$ in place of X (since every $f \in E$ has the property that $f = f\chi_{X-X_\infty}$). Throughout most of the remainder of this note we will be dealing with saturated spaces.

LEMMA 2. *If* E *is saturated then every element of* $E^\times$ *is finite a.e.*

Proof. If $g \in E^\times$ and $|g(x)| = \infty$ on A where $\mu(A) > 0$, then every $f \in E$ vanishes a.e. on A. For if this is not so for some $f \in E$, let $f(x) \neq 0$ on $A_0 \subseteq A$ with $\mu(A_0) > 0$. Then $\int_{A_0} |fg| d\mu = \infty$ and hence $\int |fg| d\mu = \infty$, which contradicts the fact that $g \in E^\times$. Therefore E is not saturated, contradiction.

COROLLARY. *If* E *is saturated then* $E^\times$ *is a solid vector space over* Φ.

Proof. Since the elements of $E^\times$ are (a.e.) finite, $E^\times$ is closed under the algebraic operations. The rest is obvious.

Suppose $\{X_n\}$ is an increasing sequence of measurable subsets of X whose union is X. Following [4], we will call a measurable set A bounded relative to $\{X_n\}$ if $A \subseteq X_k$ for some k, and a function $f \in M$ locally summable relative to $\{X_n\}$ if $\int_{X_n} |f| d\mu < \infty$ for all n.

LEMMA 3. *If* E *is saturated, then there is a sequence* $X_1 \subseteq X_2 \subseteq \ldots$ *of finitely measurable subsets of* X *such that* $X = \bigcup X_n$, *and for every* A *bounded with respect to* $\{X_n\}$, *we have* $\chi_A \in E$.

Proof. Since μ is σ-finite choose a sequence $\{Y_n\}$ such that $Y_n \uparrow X$ and $\mu(Y_n) < \infty$. Let Π be the collection of measurable sets B such that $\chi_B \in E$. If now, A has positive measure and is bounded with respect to $\{Y_n\}$, then since A cannot be E-purely infinite we can find some $f \in E$ such that $f(x) \neq 0$ on $A_0 \subseteq A$, $\mu(A_0) > 0$. We may shrink A_0 a bit if necessary so as to have $|f(x)| > \delta > 0$ (δ constant) on all of A_0. Then $\delta\chi_{A_0} \leqslant |f| \in E$ so $\chi_{A_0} \in E$ and $A_0 \in \Pi$; what we have shown is that A has a subset of positive measure belonging to Π. By [4], Note IV, Lemma 8.6, we can find a sequence $\{X_n\}$ with the required property.

COROLLARY. *If* E *is saturated, then for every* $g \in E^\times$ *with* $g \neq 0$, *there is some* $f \in E$ *with* $\int fg d\mu \neq 0$.

Proof. If A is the set on which $g(x) \neq 0$ then $0 < \mu(A \cap X_m) < \infty$ for some m. But $A \cap X_m$ is bounded with respect to $\{X_n\}$ so $\chi_{A \cap X_m} \in E$. Since $g\chi_{A \cap X_m} \neq 0$ there is some measurable $B \subseteq X$ such that $\int_B g\chi_{A\cap X_m} d\mu \neq 0$. Let $f = \chi_{B \cap A \cap X_n}$; then $f \in E$ and $\int fg d\mu \neq 0$.

THEOREM 1. *Suppose* E *is saturated. If every* $f \in E$ *is locally summable relative to some fixed sequence, then* $(E, E^\times)$ *is a dual system under the bilinear form* $\langle f, g \rangle = \int fg d\mu$.

Proof. Suppose $0 \neq f \in E$. Let A be the set of all x for which $f(x) \neq 0$; then $\mu(A) > 0$; there is an X_m belonging to the given sequence $\{X_n\}$, such that $\mu(A\cap X_m) > 0$. Since every element of E is summable over X_m, we have $\chi_{X_m}\in E^\times$. Since $f\chi_{A\cap X_m}\neq 0$ there is some measurable set B such that

$$\int f\chi_{B\cap A\cap X_m}d\mu = \int_B f\chi_{A\cap X_m}d\mu \neq 0.$$

But $E^\times$ is solid, so $g = \chi_{B\cap A\cap X_m}\in E^\times$; we have shown that $< f, g > \neq 0$. The rest follows from the preceding corollary.

3. Solid topologies. We continue our study of the space E of Section 2; throughout this section we will assume that E is saturated, and that there is a fixed sequence $\{Y_n\}$ relative to which every $f\in E$ is locally summable. Hence $(E, E^\times)$ is a dual pair under the bilinear form $< f, g >$ of Section 2. We are going to introduce a class of topologies on E of uniform convergence on families of solid subsets of $E^\times$. Terminology will always be referred to the dual pair $(E, E^\times)$; for example, "weak" will pertain to one of the weak topologies $\sigma(E, E^\times)$ or $\sigma(E^\times, E)$, and a barrel will always be a barrel in one (and hence all) of the topologies of the dual pair, unless otherwise specified.

Assume that $\mathscr{A}^\times$ is a non-empty set of weakly bounded, solid subsets of $E^\times$ having the following well-known properties:

(B1) If $A^\times\in\mathscr{A}^\times$, $B^\times\in\mathscr{A}^\times$ then there exists $C^\times\in\mathscr{A}^\times$ such that $A^\times\cup B^\times\subseteq C^\times$.

(B2) $\mathscr{A}^\times$ is closed under multiplication by a scalar.

(B3) $\bigcup_{A^\times\in\mathscr{A}^\times} A^\times$ spans $E^\times$.

For each $A^\times\in\mathscr{A}^\times$, let

$$p_{A^\times}(f) = \sup_{g\in A^\times}|\langle f, g\rangle|, \quad f\in E.$$

Then the set $\{p_{A^\times}\colon A^\times\in\mathscr{A}^\times\}$ of seminorms determines the topology on E of $\mathscr{A}^\times$-convergence, i.e., of uniform convergence on the sets of $\mathscr{A}^\times$.

DEFINITION 1. Any topology of $\mathscr{A}^\times$-convergence, where $\mathscr{A}^\times$ is as above, will be called a *solid topology*.

Thus, a solid topology is merely a topology on E of $\mathscr{A}^\times$-convergence in the usual sense, where the sets belonging to $\mathscr{A}^\times$ are solid subsets of $E^\times$, and the underlying duality is that of E, $E^\times$ determined by $\langle f, g\rangle = \int fg d\mu$.

LEMMA 1. (A) *Suppose ζ is the solid topology of $\mathscr{A}^\times$-convergence.*

(1) *Given $A^\times\in\mathscr{A}^\times$, we have*

$$p_{A^\times}(f) = \sup_{g\in A^\times}\int|fg|d\mu, \quad f\in E.$$

(2) *If $A^\times\in\mathscr{A}^\times$, the seminorm $p_{A^\times}$ is solid; that is, if f_1, f_2 are measurable functions such that $|f_1|\leqslant|f_2|$, $f_2\in E$, then $f_1\in E$ and $p_{A^\times}(f_1)\leqslant p_{A^\times}(f_2)$.*

(3) *The polars $A^{\times 0}$ in E of the sets $A^\times\in\mathscr{A}^\times$ form a neighborhood base (at 0) for ζ consisting of absolutely convex solid sets.*

(B) *If A is a solid subset of E (or $E^\times$) then its polar in $E^\times$ (or E) is solid.*

Proof. (A) (1) All we have to show is that

$$\sup_{g\in A^\times}\int|fg|d\mu \leqslant p_{A^\times}(f).$$

Given $g\in A^\times$, set $g_1 = |g|/\mathrm{sgn}f$, where we define $\mathrm{sgn}f = 1$ wherever $|f| = 0$ or ∞. Since $A^\times$ is solid, we have $g_1\in A^\times$, and $|\int fg_1 d\mu| = \int|fg|d\mu$; thus $\int|fg|d\mu\leqslant p_{A^\times}(f)$.

(2) follows from (1).

(3) All we have to prove is that each $A^{\times 0}$ is solid; the rest is well-known. Assume $|f_1|\leqslant|f_2|$ and $f_2\in A^{\times 0}$. Then $f_1\in E$ since E is solid; by (2) and the fact that $f_2\in A^{\times 0}$ we have $p_{A^\times}(f_1)\leqslant p_{A^\times}(f_2)\leqslant 1$, which says that $f_1\in A^{\times 0}$.

(B) is now obvious.

Our aim at present is to obtain several characterizations of a solid topology. We will use freely the terminology and results of Luxemburg and Zaanen in [4], especially Note I. We will deal with a family of function seminorms ϱ_a. See [4], Note I, Definition 3.1 and the following results, which we will assume known. If f is a measurable function and $\varrho_a(f) < \infty$, we will denote by $[f]_a$ that element (equivalence class) of the normed linear space L_{ϱ_a} which contains the function f.

LEMMA 2. *Suppose E under the topology ξ is a Hausdorff locally convex space with a neighborhood base $\{S_a\colon a\in\Gamma\}$ at 0 consisting of convex, solid sets. Then there exists a family $\{\varrho_a\colon a\in\Gamma\}$ of function seminorms such that (1) if $f\in E$ then $\varrho_a(f) < \infty$ for all $a\in\Gamma$; (2) if we define the linear mapping v_a of E into L_{ϱ_a} by $v_a(f) = [f]_a$, then E under ξ is the projective limit by the mappings v_a of the normed function spaces L_{ϱ_a}.*

Proof. For each $a\in\Gamma$, S_a, being solid, is balanced; being a neighborhood, it absorbs every element of E. Let p_a be the gauge or Minkowski functional in E of the set S_a, and define $\varrho_a(f) = p_a(f)$ if $f\in E$, $\varrho_a(f) = \infty$ otherwise. Then ϱ_a is a function seminorm, and $\{f\colon \varrho_a(f) < 1\}\subseteq S_a \subseteq\{f\colon \varrho_a(f)\leqslant 1\}$. We have now proven assertion (1). Next,

$$\bigcap_{a\in\Gamma} v_a^{-1}(0) = \{0\}.$$

For, if $f\in E$, $f\neq 0$, then $f\notin S_a$ for some a, since ξ is a Hausdorff topology; therefore $\varrho_a(f)\geqslant 1$ and so $v_a(f) = [f]_a$ is not the zero element of L_{ϱ_a}. Thus we may define ξ_L to be the projective limit topology on E of the normed spaces L_{ϱ_a} by the linear mappings v_a. We now have to show that $\xi = \xi_L$.

First, every v_α is ξ-continuous. For, suppose the net $f_\nu \to f$ in E relative to ξ. Given $\varepsilon > 0$, choose ν_0 such that if $\nu \geqslant \nu_0$ then $\varepsilon^{-1}(f_\nu - f) \in S_\alpha$. Hence, for $\nu \geqslant \nu_0$,

$$\varrho_\alpha\big(v_\alpha(f_\nu) - v_\alpha(f)\big) = \varrho_\alpha(f_\nu - f) \leqslant \varepsilon;$$

thus $v_\alpha(f_\nu) \to v_\alpha(f)$, and v_α is ξ-continuous. Since ξ_L is the weakest topology on E under which every v_α is continuous it follows that $\xi_L \subseteq \xi$. We will prove the converse by showing that every S_α is a ξ_L-neighborhood. The sets

$$V_\varepsilon = \{[f]_\alpha \in L_{\varrho_\alpha}\colon \varrho_\alpha(f) \leqslant \varepsilon\}, \quad \varepsilon > 0,$$

form a base of (absolutely convex) neighborhoods in L_{ϱ_α}, and $v_\alpha^{-1}(V_{1/2}) \subseteq S_\alpha$. Therefore $v_\alpha^{-1}(V_{1/2})$ is a ξ_L-neighborhood and so is S_α.

The normed spaces L_{ϱ_α} of Lemma 2 are not necessarily Banach spaces, but it may be of interest to note that we can obtain a condition, using known properties of normed function spaces L_ϱ, that E under the topology ξ be the projective limit of Banach function spaces L_{λ_α}, so that each function seminorm λ_α has the Riesz–Fisher property ([4], Note I, Definition 4.3). We will use the notation $|S_\alpha|$ for $\{|f|\colon f \in S_\alpha\}$. Suppose each $|S_\alpha|$ has the property that, whenever $\{u_n\}$ is a sequence of non-negative measurable functions such that $u_n(x) \uparrow u(x)$ a.e. as $n \to \infty$, and $\varepsilon u_n \in |S_\alpha|$ for every scalar $\varepsilon > 0$ and every n, then $\varepsilon u \in |S_\alpha|$ for every scalar $\varepsilon > 0$. (This property was called "null-monotone closure" in [2], Section 3.) Suppose, moreover, that for all $\alpha \in \Gamma$, the gauge p_α of S_α has the following property: If $\{u_n\}$ is a sequence such that $0 \leqslant u_n \in E$ for all n, and $u_n(x) \downarrow 0$ a.e. as $n \to \infty$, and $p_\alpha(u_m - u_n) \to 0$ as $m, n \to \infty$, then $p_\alpha(u_n) \to 0$ as $n \to \infty$. Then there exists a family $\{\lambda_\alpha\colon \alpha \in \Gamma\}$ of function seminorms such that each L_{λ_α} is a Banach function space and E is the projective limit by linear mappings w_α of the spaces L_{λ_α}. Indeed, L_{λ_α} is the metric space completion of L_{ϱ_α}. To prove this, we note that every ϱ_α has the Fatou null property ([4], Note II, Definition 5.1 and Section 4 below) by what we are assuming about $|S_\alpha|$, and every ϱ_α has the property of Theorem 6.8 (iii) of [4], Note III. From this it follows that there exists, for each $\alpha \in \Gamma$, a function seminorm $\lambda_\alpha \leqslant \varrho_\alpha$ such that L_{λ_α} is the metric space completion of L_{ϱ_α}. The reader is referred to [4], Note III, Section 6, for a full discussion of this matter. If $\varrho_\alpha(f) < \infty$ then $\lambda_\alpha(f) < \infty$, so let i_α be the natural imbedding of L_{ϱ_α} in L_{λ_α} given by [4], Note III, Theorem 6.8; i_α is an isometry and an algebraic isomorphism. Let $w_\alpha = i_\alpha \circ v_\alpha$. Then, again,

$$\bigcap_{\alpha \in \Gamma} w_\alpha^{-1}(0) = \{0\}.$$

We may thus write down ξ_* for the projective limit topology on E of the spaces L_{λ_α} by the mappings w_α. Since every w_α is ξ-continuous, we have $\xi_* \subseteq \xi$. On the other hand, suppose the net $f_\nu \to f$ in E relative to ξ_*; then $w_\alpha(f_\nu) \to w_\alpha(f)$ for all α, and hence since each i_α is bicontinuous, we have $v_\alpha(f_\nu) \to v_\alpha(f)$. Thus every v_α is ξ_*-continuous and therefore $\xi = \xi_L \subseteq \xi_*$. Hence E under the given topology ξ is the projective limit of the Banach function spaces L_{λ_α} by the linear mappings w_α.

The next lemma is the converse to Lemma 2.

Lemma 3. *Suppose that $\{\varrho_\alpha\colon \alpha \in \Gamma\}$ is any collection of function seminorms (with domain M), and $\{v_\alpha\colon \alpha \in \Gamma\}$ is a collection of linear mappings $v_\alpha\colon E \to L_{\varrho_\alpha}$ such that each v_α has the property that if S is a solid subset of L_{ϱ_α} then $v_\alpha^{-1}(S)$ is a solid subset of E. Suppose moreover, that*

$$\bigcap_{\alpha \in \Gamma} v_\alpha^{-1}(0) = \{0\}.$$

If ξ is the projective limit topology on E of the spaces L_{ϱ_α} by the mappings v_α, then E under ξ has a neighborhood base at 0 consisting of solid convex sets.

Proof. For all $\alpha \in \Gamma$, let $\mathscr{V}_\alpha$ be the collection of all subsets of L_{ϱ_α} of the form $\{[f]_\alpha\colon \varrho_\alpha(f) \leqslant \varepsilon\}$, $\varepsilon > 0$. The collection of all sets of the form

$$V = \bigcap_{i=1}^{n} v_{\alpha_i}^{-1}(V_i), \qquad V_i \in \mathscr{V}_{\alpha_i}, \ \{\alpha_1, \ldots, \alpha_n\} \subseteq \Gamma$$

is a neighborhood base at 0 for the topology ξ, consisting of convex sets. Since every V_i is a solid subset of $L_{\varrho_{\alpha_i}}$ it follows by hypothesis that the set V, above, is also solid.

Lemma 4. *If E under the topology ξ is a Hausdorff locally convex space and there is a neighborhood base at the origin for ξ consisting of solid convex sets, then there is a neighborhood base at the origin for ξ consisting of solid ξ-barrels.*

Proof. Any solid, convex set is absolutely convex, so let $\mathscr{V}$ be a neighborhood base at 0 for ξ consisting of absolutely convex, solid sets; for each $V \in \mathscr{V}$ let p_V be the gauge or Minkowski functional of V. The family of all sets of the form $\{f \in E\colon p_V(f) \leqslant \varepsilon\}$ where $V \in \mathscr{V}$, $\varepsilon > 0$, is then the required neighborhood base.

We are now ready for the main theorem of this section. Recall that we are assuming that E is saturated and that there is a fixed sequence relative to which every function in E is locally summable, and that in consequence, E and $E^\times$ are in duality under the bilinear form $\langle f, g\rangle = \int fg\,d\mu$.

Theorem 1. *Let ξ be a locally convex topology on E which is stronger than the weak topology $\sigma(E, E^\times)$. Then the following conditions are equivalent:*

(ST 1) *ξ is a solid topology.*

(ST 2) *ξ has a neighborhood base at 0 consisting of solid $\sigma(E, E^\times)$-barrels.*

If ξ has one and hence both of these properties, then ξ has the following:

(ST 3) *ξ has a neighborhood base at 0 consisting of solid sets.*

(ST 4) *There exists a collection $\{\varrho_a\colon a\in\Gamma\}$ of function seminorms (with domain M) such that E under ξ is the projective limit by the linear mappings $v_a\colon E\to L_{\varrho_a}$ of the normed Köthe spaces L_{ϱ_a}, where $v_a(f) = [f]_a$, and for which $\bigcap_{a\in\Gamma} v_a^{-1}(0) = \{0\}$. If, moreover, the topology ξ is a topology of the dual system $(E, E^\times)$, then* (ST 1) — (ST 4) *are all equivalent.*

Proof. Assume (ST 1). Then ξ is a topology of $\mathscr{A}^\times$-convergence, where $\mathscr{A}^\times$ is a family of weakly bounded, solid subsets of $E^\times$ having the above properties (B1), (B2), and (B3). By Lemma 1, the family $\mathscr{U}$ of sets $U^{\times 0}$, where $U^\times\in\mathscr{A}^\times$, is a ξ-neighborhood basis at 0, consisting of absolutely convex, solid sets; every element of $\mathscr{U}$, being the polar of a set in $E^\times$, is automatically $\sigma(E, E^\times)$-closed, and is therefore a $\sigma(E, E^\times)$-barrel. This proves (ST 2). Now assume (ST 2). Let $\mathscr{U}$ be a ξ-neighborhood base at 0 consisting of solid, $\sigma(E, E^\times)$-barrels; add to $\mathscr{U}$ also all the sets δU for $\delta > 0$, $U\in\mathscr{U}$, and the set E itself. $\mathscr{U}$ is now closed under multiplication by positive scalars. Let $\mathscr{A}^\times$ be the collection of all polars in $E^\times$ of the sets in $\mathscr{U}$. Then $\mathscr{A}^\times$ has (B1)–(B3) and every set in $\mathscr{A}^\times$ is weakly bounded. For, (B1) and (B2) are trivial; we will prove (B3) by showing that $E^\times = \bigcup_{U\in\mathscr{U}} U^0$. Given $g\in E^\times$, let $N = \{f\in E\colon |\langle f, g\rangle| \leqslant 1\}$. Since $\sigma(E, E^\times)\subseteq\xi$, the linear functional $f\to\langle f, g\rangle$ is ξ-continuous on E. Therefore N is a ξ-neighborhood, and there is some $U\in\mathscr{U}$ with $U\subseteq N$; then $g\in U^0$. This proves (B3). Now if $U\in\mathscr{U}$ then U absorbs the points of E; hence so does U^{00}; hence U^0 is a weakly bounded subset of $E^\times$. By Lemma 1, the sets in $\mathscr{A}^\times$ are solid. Hence, if we let ξ_0 be the topology of $\mathscr{A}^\times$-convergence, then ξ_0 is a solid topology and we are to prove that $\xi = \xi_0$, thus showing that (ST 2) implies (ST 1). Suppose the net $f_\nu\to f$ in E relative to ξ. Take any $U\in\mathscr{U}$; then it is straightforward to prove that

$$\langle f_\nu, g\rangle \to \langle f, g\rangle \quad \text{uniformly for} \quad g\in U^0.$$

Therefore $f_\nu\to f$ relative to ξ_0, and $\xi_0\subseteq\xi$. Again let $U\in\mathscr{U}$ be arbitrary. If A is any subset of E and η is any topology on E, $\mathscr{I}(A)$ will stand for the absolutely convex hull of A and $\bar{A}^\eta$ will stand for the η-closure of A. In this notation, we have

$$U^{00} = \overline{\mathscr{I}(U)}^{\sigma(E,E^\times)} = \overline{U}^{\sigma(E,E^\times)} = U$$

by a basic result of duality theory (see, e.g., [5], Ch. II, Theorem 4) and the fact that U is absolutely convex and weakly closed. Therefore U is the polar of a set in $\mathscr{A}^\times$ and so is a ξ_0-neighborhood. Therefore $\xi\subseteq\xi_0$. We have established (ST 1) and have thus shown that (ST 1) and (ST 2) are equivalent.

That (ST 2) implies (ST 3) and (ST 4) follows by Lemma 2.

Now suppose the topology ξ is consistent with the duality $(E, E^\times)$. Assume (ST 4). By Lemma 3, E under the topology ξ has a neighborhood basis at 0 consisting of convex solid sets, and hence, by Lemma 4, of solid ξ-barrels. But ξ is a topology of the dual pair and hence every ξ-barrel is a $\sigma(E, E^\times)$-barrel. Therefore (ST 2) is established. That (ST 1) implies (ST 3) follows by Lemma 1. Assume (ST 3); we will prove (ST 1). We may assume, by adding to $\mathscr{U}$ appropriate sets, that ξ has a neighborhood base $\mathscr{U}$ (at 0) consisting of solid sets, that $\mathscr{U}$ is closed under multiplication by a positive scalar, and that $E\in\mathscr{U}$. Let $\mathscr{A}^\times = \{U^0\colon U\in\mathscr{U}\}$. Since every $U^{00}(U\in\mathscr{U})$ is absorbing, the sets in $\mathscr{A}^\times$ are weakly bounded; the properties (B1)–(B3) for $\mathscr{A}^\times$ are easy to establish in the usual way. Since ξ is the topology of uniform convergence on the ξ-equicontinuous subsets of $E^\times$, ξ is the topology of $\mathscr{A}^\times$-convergence. The sets in $\mathscr{A}^\times$ are solid (by Lemma 1), so ξ is a solid topology. We have shown that (ST 3) implies (ST 1), and the theorem is proved.

4. Metrizable function spaces and completeness. In this section we will continue to study function spaces with the kind of projective limit structure determined in the last section, but in most cases the family $\{\varrho_a\colon a\in\Gamma\}$ of function seminorms will be countable, and the spaces under consideration will in consequence be metrizable. We will develop a criterion for the completeness of such spaces. It will not be assumed in the present section that our functions are a.e. finite; indeed, we will obtain a generalization of the spaces L_ϱ considered in [4], Note I for the case where ϱ is a function seminorm, as well as of certain well-known metrizable, non-normable function spaces.

Let $\{\varrho_a\colon a\in\Gamma\}$ be any non-empty collection of function seminorms (over M). We will write $f\equiv g$ $(f, g\in M)$ if and only if $\varrho_a(f-g) = 0$ for all $a\in\Gamma$. Let $M_F = \{f\in M\colon \varrho_a(f) < \infty$ for all $a\in\Gamma\}$. Then M_F is solid, and it follows from [4], Note I, Section 3, that $\equiv$ partitions M_F into equivalence classes $[\cdot]$. The first lemma generalizes and follows at once from [4], Note I, Lemma 3.4, and is needed for the same reason as is that lemma.

LEMMA 1. *If $f\in M_F$ then there exists $g\in M_F$ such that $f\equiv g$ and g is finite everywhere.*

Proof. Let B be the set on which $|f(x)| = \infty$. Then $\varrho_a(f-f\chi_{X-B}) = 0$ for all $a\in\Gamma$, and $g = f\chi_{X-B}$ is finite everywhere.

Let F be the set of all equivalence classes $[f]$, $f\in M_F$. Define, for $f, g\in M_F$ and $\lambda\in\Phi$,

$$[f]+[g] = [f+g], \qquad \lambda[f] = [\lambda f].$$

It follows from [4], Note I, Lemma 3.2 that if $f \equiv f'$, $g \equiv g'$, $\lambda \in \Phi$, then

$$f+g \equiv f'+g', \quad \lambda g \equiv \lambda g'$$

so these definitions are unambiguous; it follows from Lemma 1 that F with these operations is a vector space over Φ.

Let $p_a([f]) = \varrho_a(f)$ for $f \in M_F$, $a \in \Gamma$. This definition is independent of the representative element of $[f]$. (In the usual vector space sense) p_a is a seminorm on F, and:

LEMMA 2. *Under the topology determined by the family $\{p_a: a \in \Gamma\}$ of seminorms, F is a Hausdorff locally convex space. If Γ is countable, F with this topology is metrizable.*

Proof. To show that F is Hausdorff, suppose $[f] \neq 0$. Then $f \not\equiv 0$ so $p_a(f) > 0$ for some $a \in \Gamma$.

For reference, we record two properties which an arbitrary function seminorm ϱ may or may not possess; these properties have been exhaustively studied in [4], and we will make important use of them in the next theorem.

DEFINITION 1 (Riesz–Fischer property). We say that ϱ has the *Riesz–Fischer property* if for any sequence $\{u_n\}$ of non-negative functions in M, $\sum_1^\infty \varrho(u_n) < \infty$ implies the existence of functions u_n' $(n = 1, 2, \ldots)$ with $\varrho(u_n' - u_n) = 0$ such that $\varrho(\sum_1^\infty u_n') < \infty$.

The normed space L_ϱ is complete if and only if ϱ has the Riesz–Fischer property ([4], Note I, Theorem 4.8).

DEFINITION 2 (Fatou null property). ϱ has the *Fatou null property* if, whenever $0 \leqslant u_n \uparrow u$ pointwise $(u_n \in M)$ and $\varrho(u_n) = 0$ for all n, we have $\varrho(u) = 0$.

We return now to the space F of Lemma 2.

THEOREM 1. *Suppose that Γ is countable and that each $\varrho_a (a \in \Gamma)$ has the Fatou null and Riesz–Fischer properties. Then, under the topology determined by the family $\{p_a: a \in \Gamma\}$ of seminorms, F is a Fréchet space. Moreover, there exists a measurable subset X_0 of X such that $[\chi_{X_0}] = 0$, and having the following property: If the sequence $\{[f_n]\} \subseteq F$ converges to $[g] \in F$ in this topology, then every subsequence of $\{f_n\}$ has a subsequence which converges pointwise a.e. to g on $X - X_0$; and therefore $\{f_n\}$ converges to g in measure on every finitely measurable subset of $X - X_0$.*

Proof. That the second measure-theoretic property follows from the first is well-known. We will give a proof of the theorem which demonstrates simultaneously the completeness of F and the existence of X_0 having the first measure-theoretic property.

Write $\Gamma = \{1, 2, \ldots\}$ and suppose that $\{[f_n]\}$ is a Cauchy sequence

in F. Then, for each k, $\varrho_k(f_m - f_n) \to 0$ as $m, n \to \infty$. We will denote the element (equivalence class) of L_{ϱ_k} containing f by $[f]_k$. With this notation, $\{[f_n]_1\}$ is a Cauchy sequence in the normed space L_{ϱ_1} which is complete (ϱ_1 has the Riesz–Fischer property). Let the limit in L_{ϱ_1} of this sequence be denoted by $[g_1]_1$. Since ϱ_1 has the Fatou null property, there is a maximal ϱ_1–null subset X_1 of X. (ϱ_1-null means $\varrho_1(\chi_{X_1}) = 0$; this set X_1 is μ-unique; see [4], Note II, Theorem 5.10.) The mapping which carries the element $[f]_1 \in L_{\varrho_1}$ onto the restriction of f to $X - X_1$ is an isometric isomorphism of L_{ϱ_1} onto the space $L_{\varrho_1}(X - X_1)$ of all such restrictions, this space being given the norm ϱ_1; on this space, ϱ_1 is a function norm having the Riesz–Fischer property. Therefore $\{f_n\}$ has a subsequence $\{f_{1n}\}$ such that $f_{1n}(x) \to g_1(x)$ pointwise a.e. on $X - X_1$ (part (a) of the proof of [4], Theorem 4.8). Now $\varrho_2(f_{1m} - f_{1n}) \to 0$ as $m, n \to \infty$, so we can choose $[g_2]_2 \in L_{\varrho_2}$ such that $\varrho_2(f_{1n} - g_2) \to 0$ as $n \to \infty$. Let X_2 be the maximal ϱ_2-null subset of X, and pass to a subsequence $\{f_{2n}\}$ of $\{f_{1n}\}$ such that $f_{2n}(x) \to g_2(x)$ point-wise a.e. on $X - X_2$. Continue inductively in this manner. Let $X_0 = \bigcap_1^\infty X_i$. For each k we have from the properties of the function seminorm ϱ_k that

$$\varrho_k(\chi_{X_0}) \leqslant \varrho_k(\chi_{X_k}) = 0, \quad \text{so} \quad \varrho_k(\chi_{X_0}) = 0.$$

Since F is Hausdorff, $[\chi_{X_0}] = 0$. Now consider the sequence $\{f_{nn}\}_{n=1}^\infty$. This is a subsequence of $\{f_{1n}\}$; in the same way, $\{f_{nn}\}_{n=k}^\infty$ is a subsequence of $\{f_{in}\}_{n=K}^\infty$ for $i = 2, 3, \ldots, K$. Therefore,

$$\lim_{n\to\infty} f_{nn}(x) = g_k(x) \quad \text{a.e. on } X - X_k.$$

It follows that

$$g_{k_1}(x) = g_{k_2}(x) \quad \text{a.e. on } (X - X_{k_1}) \cap (X - X_{k_2}).$$

Hence there is no ambiguity in defining

$$g(x) = \begin{cases} g_k(x) = \lim_{n\to\infty} f_{nn}(x), & x \in X - X_k, \\ 0, & x \in X_0. \end{cases}$$

We will prove that $g \in M_F$ and that $[f_n] \to [g]$ in the metric topology of F. We have for each k,

$$\begin{aligned} \varrho_k(g) &\leqslant \varrho_k(|g|\chi_{X-X_k}) + \varrho_k(|g|\chi_{X_k}) \\ &= \varrho_k(|g|\chi_{X-X_k}) = \varrho_k(|g_k|\chi_{X-X_k}) \leqslant \varrho_k(|g_k|) < \infty \end{aligned}$$

where we have used the fact that ϱ_k has the Fatou null property and X_k is a ϱ_k-null set in order to observe that the second term on the right in the first inequality vanishes. Since k was arbitrary, we see that $g \in M_F$.

We will complete the proof by showing that $\varrho_k(f_n - g) \to 0$ as $n \to \infty$. We will prove at first that $\varrho_k(f_{1n} - g) \to 0$ as $n \to \infty$. Since X_k is a ϱ_k-null set and ϱ_k has the Fatou null property, we note that, as before, $\varrho_k((f_{km}-g)\chi_{X_k}) = 0$. Therefore

$$\begin{aligned}\varrho_k(f_{1n}-g) &\leqslant \varrho_k(f_{1n}-f_{km})+\varrho_k(f_{km}-g)\\ &\leqslant \varrho_k(f_{1n}-f_{km})+\varrho_k((f_{km}-g)\chi_{X-X_k})+\varrho_k((f_{km}-g)\chi_{X_k})\\ &= \varrho_k(f_{1n}-f_{km})+\varrho_k((f_{km}-g_k)\chi_{X-X_k}).\end{aligned}$$

Since $\{f_{1n}\}$ is a Cauchy sequence in L_{ϱ_k},

$$\lim_{m,n\to\infty} \varrho_k(f_{1n}-f_{km}) = 0.$$

Moreover,

$$\varrho_k((f_{km}-g_k)\chi_{X-X_k}) \leqslant \varrho_k(f_{km}-g_k)$$

which tends to zero as $m\to\infty$ since $f_{km}\to g_k$ in the normed space L_{ϱ_k}. Therefore $\varrho_k(f_{1n}-g)\to 0$ as $n\to\infty$, as desired. But

$$\varrho_k(f_n-g) \leqslant \varrho_k(f_n-f_{1m})+\varrho_k(f_{1m}-g).$$

Since $\{f_n\}$ is Cauchy in the normed space L_{ϱ_k}, the first term on the right tends to zero as $m,n\to\infty$. Therefore $\varrho_k(f_n-g)\to 0$ as $n\to\infty$. (See [4], Note I, Section 2 (a) for a justification of some of the above applications of the triangle inequality.) It follows that $[f_n]\to[g]$ in the metric topology of F. The existence, for $\{f_n\}$, and (by the same method) for any subsequence of $\{f_n\}$, of a subsequence having the required property, has also been demonstrated.

We remark that it often turns out that, in the space F, $f\equiv g$ if and only if $f(x) = g(x)$ a.e. — see the following examples. In this case, the statement in Theorem 1 that $[\chi_{X_0}] = 0$ becomes $\mu(X_0) = 0$; that is, every subsequence of $\{f_n\}$ now has a subsequence which converges pointwise a.e. to g on X.

In order to look at some examples, suppose ϱ is a single non-trivial function norm having the Riesz–Fischer property. (To say that ϱ is non-trivial means that there is some $u\geqslant 0$ such that $0<\varrho(u)<\infty$.) Let Λ_0 be a non-empty, countable subset of Λ. For all $A\in\Lambda_0$, define ϱ_A by $\varrho_A(u) = \varrho(u\chi_A)$ ($u\geqslant 0$ measurable). Then each ϱ_A is a function seminorm, having the Fatou null property (since ϱ is a norm) and the Riesz–Fischer property; to prove this last statement, assume that $\sum_1^\infty \varrho_A(u_n)<\infty$; then by [4], Note I, Theorem 4.2 applied to ϱ, we have $\varrho_A(\sum_1^\infty u_n)<\infty$. We now let $M_F(\Lambda_0) = \{f\in M\colon \varrho_A(f)<\infty \text{ for all } A\in\Lambda_0\}$ and let $F(\Lambda_0)$ be the corresponding metrizable locally convex space of equivalence classes $[\cdot]$, as given by Lemma 2. It follows from Theorem 1 that $F(\Lambda_0)$ is a Fréchet space. If $\{\varrho_A\colon A\in\Lambda_0\}$ is total in the sense that, for every $f\in F(\Lambda_0)$ which does not vanish a.e., there is some $A\in\Lambda_0$ such that $\varrho_A(f)$ $=\varrho(f\chi_A)>0$, then $f\equiv g$ in $F(\Lambda_0)$ if and only if $f=g$, and therefore the set X_0 given by Theorem 1 is μ-null.

Perhaps the most interesting case to consider is that in which the above function norm ϱ is saturated: that is, for every set C of positive measure there is a set D also of positive measure such that $D\subseteq C$ and $\varrho(\chi_D)<\infty$. ([4], Note IV, Definition 8.4.) In other words, the space L_ϱ is saturated (in our terminology), and there exists a sequence $\{X_n\}$ of finitely measurable subsets of X such that $X_1\subseteq X_2\subseteq\ldots$ and $X=\bigcup X_n$, and for every set A bounded with respect to $\{X_n\}$, we have $\varrho(\chi_A)<\infty$. Here let $\Lambda_0 = \{X_n\}$. Then $F(\Lambda_0)$ is the collection of all functions f which are "locally finite" relative to $\{X_n\}$, i.e., of all f such that $\varrho(f\chi_A)<\infty$ for every A which is bounded with respect to $\{X_n\}$. Since ϱ is a function norm, we see in this case that $f\equiv g$ in $F(\Lambda_0)$ if and only if $f(x)=g(x)$ a.e. Moreover, the family of seminorms $\varrho_A(f)=\varrho(f\chi_A)$, A a bounded set with respect to $\{X_n\}$, also determines the topology of $F(\Lambda_0)$. In the case in which ϱ is the L^1-norm: $\varrho(f)=\int|f|\,d\mu$, $F(\Lambda_0)$ is the well-known space of functions integrable on every set bounded with respect to $\{X_n\}$, under the topology of convergence in the mean on every such set.

5. The duals of the space F. We continue our study of the space F of Section 4, Theorem 1; we are assuming that Γ is countable; say $\Gamma=\{1,2,\ldots\}$, and that each ϱ_n ($n\in\Gamma$) has the Fatou null and Riesz–Fischer properties. We assume further that $f\equiv g$ in F if and only if $f(x)=g(x)$ a.e., so that the element $[f]\in F$ contains only functions which equal f a.e.; we now write $f\in F$ instead of $[f]\in F$. From Section 4, Lemma 1, it is clear that every $f\in F$ is finite a.e. Moreover, F is solid.

Now suppose that F is saturated. Then, from Section 2 we know that $F^\times$ is a solid vector space, and exery element of $F^\times$ is finite a.e. If g is an a.e. finite measurable function such that $\varrho_n'(g)<\infty$ for some $n\in\Gamma$, where ϱ_n' stands for the associate seminorm of ϱ_n as defined in [4], Note IV, Section 9, then for all $f\in F$ we have by [4], Note IV, Theorem 9.3, that fg is summable. In other words, $g\in F^\times$. We define F' to be the span in $F^\times$ of the set of all a.e. finite $g\in M$ such that there exists $n\in\Gamma$ with $\varrho_n'(g)<\infty$. Let F^* be the metric dual of F. In this section we shall study the relations among the various dual spaces $F', F^\times, F^*$. Given $g\in F^\times$, we now let $\varphi_g(f)=\int fg\,d\mu$ ($f\in F$); then φ_g is a linear functional on F. We are ready to prove:

THEOREM 1. *Suppose F is saturated. Then the mapping $g\to\varphi_g$ of F' into F^* is linear and one-one, hence embeds F' as a subspace of F^*.*

Proof. If $g\in F'$ then φ_g is (metrically) continuous on F, hence belongs to F^*. To prove this, we can write $g=\sum_1^N \alpha_i g_i$ where $|g_i(x)|<\infty$ a.e., $\varrho_{n_i}'(g_i)<\infty$ ($i=1,\ldots,N$). Suppose the sequence $f_k\to 0$ in the metric topology of F. Then, for $i = 1, \ldots, N$, we have $\varrho_{n_i}(f_k) \to 0$ as $k \to \infty$. The mapping $f \to \int f g_i d\mu$ is a continuous linear functional on $L_{\varrho_{n_i}}$ ([4], Note IV, Theorem 10.1), so $\int f_k g_i d\mu \to 0$ as $k \to \infty$, from which it follows that $\int f_k g d\mu \to 0$ as $k \to \infty$. Thus $\varphi_g \in F^*$. The mapping $g \to \varphi_g$ is obviously linear, and we prove it is one-one as follows: Suppose $\varphi_g = \varphi_h$ for $g, h \in F'$. This means that $\int f(g-h)\,d\mu = 0$ for all $f \in F$. Since F is saturated, choose a sequence $\{X_n\}$ of finitely measurable sets with $X_n \uparrow X$ such that $\chi_A \in F$ for every A bounded with respect to this sequence (Section 2, Lemma 3). We will show $g(x) = h(x)$ a.e. on every X_n. If B is a measurable set, then $\chi_{B \cap X_n} \in F$, so

$$\int_B \chi_{X_n} \cdot (g-h)\,d\mu = \int \chi_{B \cap X_n} \cdot (g-h)\,d\mu = 0.$$

Since B is arbitrary, this proves that $\chi_{X_n} \cdot (g-h)$ vanishes a.e., as desired.

The next definition is taken from [4], Note II, Definition 5.1, and from [3], Ch. 1, Section 2.

DEFINITION 1. The function seminorm ϱ has the *Fatou property* if $0 \leqslant u_n$, $u_n \uparrow u$ a.e., implies $\varrho(u_n) \uparrow \varrho(u)$. ϱ is *absolutely continuous* (AC) if $\{u_n\} \subseteq L_\varrho$, $u_n \downarrow 0$ a.e., implies $\varrho(u_n) \downarrow 0$.

The Fatou property implies both the Fatou null and Riesz–Fischer properties.

The next lemma is not new:

LEMMA 1. *Let ϱ be a saturated function seminorm which has the Fatou property and is AC. Then the metric dual L_ϱ^* of L_ϱ is identified by the usual canonical isometric isomorphism $\varphi_g \leftrightarrow g$, where*

$$\varphi_g(f) = \int f g d\mu, \quad f \in L_\varrho,$$

with the associate space L_ϱ' of L_ϱ ([4], Note IV, Section 9). *Every function in L_ϱ' is a.e. finite.*

Proof. By [4], Note V, Theorem 12.1, every $g \in L_\varrho'$ vanishes a.e. on the maximal ϱ-null set X_0. Let $Y = X - X_0$. Then $\varrho(f) = \varrho(f\chi_Y)$ for every $f \in M$, and if we delete X_0 from X and consider the restriction of each function to Y, then ϱ becomes a function norm on the set of all μ-measurable functions on Y ([4], Note II, Theorem 5.10). Let $L_\varrho(Y)$ be the Banach function space consisting of the set of all such restrictions f such that $\varrho(f) < \infty$. (L_ϱ and $L_\varrho(Y)$ are isometrically isomorphic under the mapping $[f] \to (f$ restricted to $Y)$, where $[f]$ denotes an arbitrary equivalence class in L_ϱ.) ϱ and ϱ' are now both saturated function norms and every $g \in L_\varrho'$ is finite a.e. Since ϱ is still AC and has the Fatou property, $L_\varrho(Y)^*$ and $L_\varrho(Y)'$ are identified as in the statement of the theorem. See [3], Chapter 1, Section 2. It is easy to see that this identification carries over to L_ϱ^* and L_ϱ'.

Recall that in [4], Note V, if ϱ is a function seminorm and $\{X_k\}$ is a sequence of measurable sets, then $\{X_k\}$ is said to be ϱ-exhaustive if $X_k \uparrow X$ and $\varrho(\chi_{X_k}) < \infty$ for all k. We will say that ϱ is saturated if the space L_ϱ is saturated. This is equivalent to the following property: For any set E of positive measure there is a set $F \subseteq E$ of positive measure such that $\varrho(\chi_F) < \infty$ ([4], Note IV, Section 8).

We return now to the space F of Theorem 1.

THEOREM 2. *Suppose the function seminorms $\varrho_1, \varrho_2, \ldots$ which determine F have the Fatou property and are AC; suppose moreover that there is a fixed sequence which is ϱ_n-exhaustive for all n. Then F' is isometrically isomorphic to F^* under the mapping $g \to \varphi_g$; i.e., this mapping identifies F' with all of F^*.*

Proof. F, and each L_{ϱ_n}, are saturated; we prove this for F. If not, let A be a positively measurable set such that every $f \in F$ vanishes a.e. on A. Choose k such that $\mu(A \cap X_k) > 0$, where $\{X_k\}$ is the sequence which is ϱ_n-exhaustive for all n. Then $\chi_{A \cap X_k} \leqslant \chi_{X_k} \in F$, so $\chi_{A \cap X_k} \in F$ but does not vanish a.e. on A, contradiction. In view, then, of Theorem 1, it suffices, for the proof of Theorem 2, to show that the mapping $g \to \varphi_g$ of $F' \to F^*$ is onto. If $f \in M$, let $[f]_n$ be the element (equivalence class) of L_{ϱ_n} containing f; let the linear mapping v_n of $F \to L_{\varrho_n}$ be given by $v_n(f) = [f]_n$. Then F (with the metric topology) is the projective limit of the spaces L_{ϱ_n} by the mappings v_n. Proof: The fact that $\bigcap_1^\infty v_n^{-1}(0) = \{0\}$ follows from the fact that $f \equiv 0$ if and only if $f = 0$ (a.e.). Let ξ, ξ_L be the metric and projective limit topologies on F, respectively; we are showing that $\xi = \xi_L$. Each mapping v_n is continuous if F is given ξ and L_{ϱ_n} is given its norm topology; therefore ξ_L is weaker than ξ. Conversely: The seminorms ϱ_n determine ξ so a base of ξ-neighborhoods is formed by the sets of the form

$$U = \{f \in F\colon \varrho_{n_i}(f) \leqslant \varepsilon,\ i = 1, \ldots, N\},$$

where $\varepsilon > 0$. We have finished if we can show that U is a ξ_L-neighborhood; we may assume here that $N = 1$. The sets

$$V_{n,\lambda} = \{[f]_n \in L_{\varrho_n}\colon \varrho_n(f) \leqslant \lambda\}, \quad \lambda > 0,$$

form a base of absolutely convex neighborhoods in L_{ϱ_n}. Hence $U = v_{n_1}^{-1}(V_{n_1,\varepsilon})$ is a ξ_L-neighborhood.

By the usual representation of the dual of a projective limit, an arbitrary element $\varphi \in F^*$ can be expressed as follows: There are integers $n_1, \ldots, n_K > 0$ and $\varphi_i \in L_{\varrho_{n_i}}^*$ $(i = 1, \ldots, K)$ such that

$$\varphi = \sum_{i=1}^K \varphi_i \circ v_{n_i}.$$

By Lemma 1, choose $g_i \in L'_{\varrho_{n_i}}$, finite a.e., such that

$$\varphi_i([f]_{n_i}) = \int f g_i \, d\mu,$$

for all "elements" f of $L_{\varrho_{n_i}}$. For every $f \in F$ we now have

$$\varphi(f) = \sum_{i=1}^{K} \varphi_i(v_{n_i}(f)) = \sum_{i=1}^{K} \varphi_i([f]_{n_i}) = \sum_{i=1}^{K} \int f g_i \, d\mu = \int f \Big(\sum_{i=1}^{K} g_i\Big) d\mu.$$

Hence $\varphi = \varphi_g$ where $g = \sum_{1}^{K} g_i \in F'$.

We conclude with a condition, different from that of Section 2, which is sufficient that F and $F^\times$ be in duality under the usual bilinear form $\langle f, g\rangle = \int fg\,d\mu$.

THEOREM 3. *If there exists a sequence which is ϱ_n-exhaustive for all n, then F is saturated and (F, F'), $(F, F^\times)$ are dual systems under $\langle f, g\rangle$.*

Proof. As in the last proof, F and each L_{ϱ_n} are saturated. Since every ϱ_n has the Fatou null property, L'_{ϱ_n} is a total subspace of the metric dual $L^*_{\varrho_n}$ of L_{ϱ_n} ([4], Note V, Theorem 15.2). By Section 2, Lemma 3, Corollary, we know that for all $g \in F^\times$ (and hence for all $g \in F'$) with $g \neq 0$, there is some $f \in F$ with $\langle f, g\rangle \neq 0$. But if $0 \neq f \in F$ then $\varrho_n(f) \neq 0$ for some n, so that there exists $g \in L'_{\varrho_n}$ such that $\langle f, g\rangle \neq 0$. Moreover, ϱ'_n is a function norm ([4], Note IV, Theorem 9.7) so g is finite a.e. and hence belongs to F' and to $F^\times$.

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A Cantor–Lebesgue theorem for double trigonometric series

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Abstract. Let $\xi = (x, y)$ be points of the plane, $\nu = (m, n)$ – lattice points, and $\langle \nu \cdot \xi\rangle = mx + ny$. It is shown that given any set E of positive measure situated in the square $0 \leqslant x < 1$, $0 \leqslant y < 1$, there is a constant $A = A_E$ such that for any trigonometric polynomial $T(\xi)$ of the form $\sum_{|\nu|=R} c_\nu e^{2\pi i(\nu\cdot\xi)}$ we have

$$\sum |c_\nu|^2 \leqslant A \int_E |T(\xi)|^2 d\xi.$$

In particular, if an infinite series $\sum c_\nu e^{2\pi i(\nu,\xi)}$ converges circularly in a set of positive measure, then $\sum_{|\nu|=R} |c_\nu|^2 \to 0$ as $R \to \infty$.

1. Let $\xi = (x, y) \in R^2$ and let $\nu = (m, n)$ denote lattice points in R^2. Consider a double trigonometric series

$$\sum_{\nu} c_\nu e^{2\pi i\langle \nu\cdot\xi\rangle}, \tag{T}$$

where $\langle \nu\cdot\xi\rangle = mx + ny$, and its circular partial sums

$$T_R(\xi) = \sum_{|\nu|\leqslant R} c_\nu e^{2\pi i\langle \nu\cdot\xi\rangle}.$$

We shall also write

$$A_R(\xi) = \sum_{|\nu|=R} c_\nu e^{2\pi i\langle \nu\cdot\xi\rangle}.$$

Recently, R. L. Cooke proved the following result (see [1]).

THEOREM 1. *If $A_R(\xi) \to 0$ almost everywhere as $R \to \infty$ (and, in particular, if T converges almost everywhere), then $c_\nu \to 0$ as $|\nu| \to \infty$. More generally, we then have*

$$\sum_{|\nu|=R} |c_\nu|^2 \to 0 \qquad (R \to \infty). \tag{1.1}$$

In this note we prove a somewhat more general result.

THEOREM 2. *If $A_R(\xi) \to 0$ at each point ξ of a set of positive measure, we have* (1.1).