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Action of topological semigroups, invariant means, and fixed points

by

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Abstract. It is the main purpose of this paper to establish relations between fixed point properties on compact convex subsets of a locally convex space and the existence of an invariant mean for certain actions of a topological semigroup on an arbitrary topological space. Our results generalise some recent fixed point theorems of M. M. Day and T. Mitchell.

1. Introduction. Consider the following fixed point properties for an action of a topological semigroup S on a Hausdorff topological space X such that the mapping $S\times X\to X$ is continuous in the second variable:

(P_1) Whenever S acts affinely on a compact convex subset Y of a l.c.s. (locally convex linear topological space) for which the mapping $S\times Y\to Y$ is *continuous in the second variable* and there exists a continuous mapping Π from X into Y such that $\Pi(s\cdot x) = s\cdot\Pi(x)$ for all $s\in S$ and $x\in X$, then Y has a fixed point for S.

(P_2) Whenever S acts affinely on a compact convex subset Y of a l.c.s. for which the mapping $S\times Y\to Y$ is *separately continuous* and there exists a continuous mapping Π from X into Y such that $\Pi(s\cdot x) = s\cdot\Pi(x)$ for all $s\in S$ and $x\in X$, then Y has a fixed point for S.

(P_3) Whenever S acts affinely on a compact convex subset Y of a l.c.s. for which the mapping $S\times Y\to Y$ is *jointly continuous* and there exist a continuous mapping Π from X into Y such that $\Pi(s\cdot x) = s\cdot\Pi(x)$ for all $s\in S$ and $x\in X$, then Y has a fixed point for S.

If A is a norm closed S-translation invariant subspace of $m(X)$ (the set of bounded real functions on X) containing constants, and

(P) Whenever $\mathscr{S} = \{\eta(s);\ s\in S\}$ is a homomorphic representation of S as continuous affine mappings from a compact convex subset Y of a l.c.s. into Y and there exist a linear transformation T from $\mathscr{A}(Y)$ (the set of affine continuous real functions on Y) into A such that $T(1) = 1$, $T(f)\geqslant 0$ for $f\geqslant 0$ and ${}_sT(h) = T({}_{\eta(s)}h)$ for all $s\in S$, $h\in\mathscr{A}(Y)$, then Y has a fixed point for $\mathscr{S}$.

We show in this paper that A has an S-invariant mean iff (P). Furthermore if $WLUC(S,X)$ $[LUC(S,X)]$ denotes the set of all $f\in C(X)$ (the set of bounded continuous real functions on X) such that the mapping from S into $C(X)$ defined by $s\to {}_sf$ for all $s\in S$ is continuous when $C(X)$ has the weak [sup norm] topology, then we prove, by applying Property (P), the following:

THEOREM.

(a) *$C(X)$ has a S-invariant mean iff* (P_1).

(b) *$WLUC(S,X)$ has a S-invariant mean iff* (P_2).

(c) *$LUC(S,X)$ has an S-invariant mean iff* (P_3).

When $S=X$, then part (a) of this theorem becomes Theorem 4 in Day [4], and parts (b) and (c) becomes Theorem 4 and 2 in Mitchell [22], respectively.

When A is in addition an algebra, and n is a positive integer, we have obtained a Property $(Q(n))$, an analogue of (P), which is equivalent to A having an S-invariant mean of type $\frac{1}{k}\sum_1^k\varphi_i$, $1\leqslant k\leqslant n$, and each φ_i is multiplicative.

The relation between invariant means on space of functions and fixed point properties was first introduced by Day [3]. We shall show in Section 5 the relation of Properties (P) and $(Q(n))$ with the fixed point properties in Argabright [1] and Mitchell [21].

2. Some notations. All topological spaces considered in this paper are assumed to be Hausdorff.

Z^+ denotes the set of positive integers.

For any set A, $|A|$ will denote the *cardinality* of A. If A is a subset of a linear space E, then $\mathrm{Co}\,A$ is the *convex hull* of A in E.

Let S be a semigroup of transformations from a set X into X, and $m(X)$ be the space of bounded real functions on X. For any subset $T\subseteq X$, 1_T will denote the characteristic function on T and $f|T\in m(T)$ is the restriction of f to T for any $f\in m(X)$. Furthermore, if $f\in m(X)$ and $s\in S$, then $\|f\|=\sup_{x\in X}|f(x)|$, ${}_sf(x)=f(sx)$ for all $x\in X$.

If A is a norm closed, S-translation invariant (i.e. ${}_sf\in A$ whenever $f\in A$ and $s\in S$) subspace of $m(X)$ containing constants, then for any $s\in S$, define $l_s\colon A\to A$ by $(l_sf)(x)={}_sf(x)$ for all $x\in X$ and $f\in A$, and $L_s\colon A^*\to A^*$ by $(L_s\varphi)(f)=\varphi({}_sf)$ for all $f\in A$ and $\varphi\in A^*$ (A^* is the conjugate space of A). An element $\varphi\in A^*$ is a *mean* if $\varphi(f)\geqslant 0$ for all $f\geqslant 0$ and $\varphi(1_X)=1$; φ is *S-invariant* if $L_s\varphi=\varphi$ for all $s\in S$. As well-known, the set of means on A is compact in the w^*-topology of A^* (i.e. $\sigma(A^*,A)$).

For $x\in X$, let $p_x\in m(X)^*$ be the *point measure* at x i.e. $p_x(f)=f(x)$ for all $f\in m(X)$; an element in $\mathrm{Co}\{p_x;\ x\in X\}$ is called a *finite mean* on $m(X)$. Furthermore, $\varphi\in A^*$ is a (point measure) finite mean iff φ is the restriction of some (point measure) finite mean of $m(X)$ to A. As well-known, the set of finite means is $\sigma(A^*,A)$-dense in the set of means on A.

If A is in addition an algebra, then $\varphi\in A^*$ is *multiplicative* if $\varphi(fg)=\varphi(f)\varphi(g)$ for all $f,g\in A$. Furthermore, the set of multiplicative means on A, $\Delta(A)$, is $\sigma(A^*,A)$-compact, and the set of point measures on A is $\sigma(A^*,A)$-dense in $\Delta(A)$. A mean μ on A is *the average of k multiplicative means* if $\mu=\frac{1}{k}\sum_1^k\varphi_i$, $\varphi_i\in\Delta(A)$.

We shall denote by:

$\mathscr{K}X$ = the set of means on $m(X)$.

βX = the set of multiplicative means on $m(X)$ (which is the Stone–Čech compactification of X).

If X is a topological space and Y is a compact convex subset of a l.c.s. (locally convex linear topological space), then $C(X)$ and $\mathscr{A}(Y)$ will denote the Banach algebra of all bounded continuous real functions on X and the Banach space of all real affine continuous functions on Y respectively.

Remark 2.1. The followings are known and will be useful for our purpose:

(a) Let X be a set, A be a norm closed subspace of $m(X)$ containing constants and $M(A)$ be the set of means on A. Then for each $\mu\in M(A)$, μ has an extension $\mu^{\sim}\in\mathscr{K}X$ to $m(X)$ ([2], p. 513, Lemma 1). Furthermore, the mapping $\psi\colon f\to f^{\wedge}$, where $f^{\wedge}(\mu)=\mu(f)$ for all $f\in A$, $\mu\in M(A)$ is an isometry mapping A *onto* $\mathscr{A}(M(A))$ (ψ is onto follows from an application of the separation theorem [6] p. 64 Corollary 13; see for example [12], Corollary 2.4).

When A is also an algebra and $\Delta(A)$ is the set of multiplicative means on A, then any $\varphi\in\Delta(A)$ has an extension $\varphi^{\sim}\in\beta X$ to $m(X)$. In fact if $\{p_{x_\alpha}\}$ is a net of point measure on $m(X)$ such that $\lim p_{x_\alpha}(f)=\varphi(f)$ for all $f\in A$, then any cluster point $\varphi^{\sim}$ in $\{p_{x_\alpha}\}$ is such an extension. Furthermore, the mapping $f\to f^{\wedge}$ where $f^{\wedge}(\varphi)=\varphi(f)$ for all $\varphi\in\Delta(A)$ and $f\in A$ is an isometry mapping A *onto* $C(\Delta(A))$ (see [6], p. 274).

(b) If X is a compact space, then any multiplicative mean φ on $C(X)$ is a point measure ([6], Lemma 25, p. 278).

(c) If Y is a compact convex subset of a l.c.s. E, then functions of the form $h|Y+c\cdot 1_Y$ where $h\in E^*$ and c is real are uniformly dense in $\mathscr{A}(Y)$ ([24], p. 31, Lemma 4.5). Consequently $\mathscr{A}(Y)$ separates points. Furthermore, any mean μ on $\mathscr{A}(Y)$ is a point measure ([1], Lemma 2. See also [5], Lemma 6.2).

A semigroup $(S,\cdot)$ can be regarded as a semigroup of transformations from S into S defined by $s(t)=s\cdot t$ for all $s,t\in S$. In this case, S-invariant means and S-translation invariant subspaces are also referred to as *left invariant mean* (LIM) and *left translation invariant subspaces*. A subspace A of $m(S)$ is right translation invariant if $r_a(A) \subseteq A$ for all $a \in S$, where $r_a(f)(s) = f_a(s) = f(sa)$ for all $s \in S$ and $f \in A$; A is *translation invariant* if A is both left and right translation invariant. A left translation invariant subspace A is *left introverted* if $\Pi_\varphi(A) \subseteq A$ for all $\varphi \in A^*$, where $\Pi_\varphi(f)(s) = \varphi({}_sf)$ for all $s \in S$. If A is also an algebra, then A is *left M-introverted* if $\Pi_\varphi(A) \subseteq A$ for all multiplicative mean φ on A. It is known and easy to see that left M-introverted (and left introverted) implies right translation invariant ([21], p. 121).

3. Semigroup of transformations and fixed points. In this section we shall prove two theorems which are basic to the rest of our work.

Let X, Y be sets and A, B be norm closed subspaces of $m(X)$, $m(Y)$ respectively containing constants. Denote by

$K[A, B] =$ the set of all linear transformations T from A to B such that $T(1_X) = 1_Y$ and $T(f) \geqslant 0$ if $f \geqslant 0$.

If A, B are algebras, then $T \in K[A, B]$ is multiplicative if $T(fg) = T(f)T(g)$ for all $f, g \in A$.

THEOREM 3.1. *Let S be a semigroup of transformations from a set X into X and A be a norm closed S-translation invariant subspace of $m(X)$ containing constants. Then A has an S-invariant mean iff*

(P) *whenever $\mathscr{S} = \{\eta(s);\ s \in S\}$ is a homomorphic representation of S as continuous affine mappings from a compact convex subset Y of a l.c.s. into Y and there exists $T \in K[\mathscr{A}(Y), A]$ such that ${}_sT(h) = T({}_{\eta(s)}h)$ for all $s \in S$, $h \in \mathscr{A}(Y)$, then Y has a fixed point for $\mathscr{S}$.*

Proof. Let φ be an S-invariant mean on A and $y \in Y$ such that the restriction of p_y to $\mathscr{A}(Y)$ coincide with the mean $T^*(\varphi)$ on $\mathscr{A}(Y)$ (Remark 2.1 (c)). If $s \in S$, then

$$h(\eta(s)\cdot y) = \varphi(T({}_{\eta(s)}h)) = \varphi({}_sT(h)) = \varphi(T(h)) = h(y)$$

for all $h \in \mathscr{A}(Y)$. Since $\mathscr{A}(Y)$ separates points (Remark 2.1 (c)), y is a fixed point for $\mathscr{S}$.

Conversely, if (P) is satisfied, consider the homomorphic representation $\mathscr{S} = \{L_s;\ s \in S\}$ of S as continuous affine mappings from the $\sigma(A^*, A)$-compact convex subset Y of A^* into Y, where Y is the set of means on A. Define T: $\mathscr{A}(Y) \to m(X)$ by $T(h)(x) = h(p_x)$ for all $h \in \mathscr{A}(Y)$ and $x \in X$. It is easy to see that T is linear, $T(1_Y) = 1_X$, $T(h) \geqslant 0$ if $h \geqslant 0$ and $T({}_{\eta(s)}h) = {}_sT(h)$ for all $h \in \mathscr{A}(Y)$ and $s \in S$, where $\eta(s) = L_s$. Furthermore, if $h \in \mathscr{A}(Y)$ there exists $f \in A$ such that $h(\varphi) = \varphi(f)$ for all $\varphi \in Y$ (Remark 2.1 (a)). Consequently $T(h)(x) = h(p_x) = p_x(f) = f(x)$ i.e. $T(h) = f \in A$. It follows that $T \in K[\mathscr{A}(Y), A]$ and Y has a fixed point φ for $\mathscr{S}$; φ is in S-invariant mean on A.

Remark. If A is a norm closed *left* translation invariant subspace of $m(S)$ containing constants, where S is a semigroup, then Theorem 2.1 furnishes us with a fixed point property which is equivalent to A being left amenable i.e. A has a LIM. (See [1], Theorem 1, 2 and also Section 5A of this paper). The idea in using the fact that a mean on $\mathscr{A}(Y)$, where Y is a compact convex subset of a l.c.s., can be represented by a point measure on $\mathscr{A}(Y)$ in the proof of Theorem 3.1 is from Argabright [1] (Lemma 2, and Theorem 1).

THEOREM 3.2. *Let S be a semigroup of transformations from a set X into X, A be a norm closed S-translation invariant subalgebra of $m(X)$ containing constants and $n \in Z^+$. Then A has an S-invariant mean which is the average of k-multiplicative means for some $k \in Z^+$, $k \leqslant n$ iff*

(Q(n)) *whenever $\mathscr{S} = \{\eta(s);\ s \in S\}$ is a homomorphic representation of S as continuous mapping from a compact space Y into Y and there exists a multiplicative $T \in K[C(Y), A]$ such that ${}_sT(h) = T({}_{\eta(s)}h)$ for all $s \in S$ and $h \in C(Y)$, then there exists a non-empty finite subset $F \subseteq Y$, $|F| \leqslant n$ such that $\eta(s)F = F$ for all $s \in S$.*

Proof. Let $\varphi = \frac{1}{k}\sum_1^k \varphi_i$ be an S-invariant mean on A, where each φ_i is a multiplicative mean on A. For each $1 \leqslant i \leqslant k$, pick $y_i \in Y$ such that the restriction of the point measure p_{y_i} to $C(Y)$ coincides with the multiplicative mean $T^*(\varphi_i)$ on $C(Y)$ (Remark 2.1 (b)). If $s \in S$, then

$$\frac{1}{k}\sum_1^k h(\eta(s)\cdot y_i) = \varphi(T({}_{\eta(s)}h)) = \varphi({}_sT(h)) = \varphi(T(h)) = \frac{1}{k}\sum_1^k h(y_i)$$

for all $h \in C(Y)$. Since $C(Y)$ separates closed sets, it follows that $\eta(s)F = F$ for all $s \in S$ where F is the set of distinct elements from $\{y_1, \ldots, y_k\}$.

Conversely, if (Q(n)) is satisfied, consider the homomorphic representation $\mathscr{S} = \{L_s;\ s \in S\}$ of S as continuous mappings from the $\sigma(A^*, A)$-compact subset Y of A^* into Y where Y is the set of multiplicative means on A. Define a multiplicative linear transformation T: $C(Y) \to m(X)$ by $T(h)(x) = h(p_x)$ for all $h \in C(Y)$ and $x \in X$. It is easy to see that $T(1_Y) = 1_X$, $T(h) \leqslant 0$ if $h \geqslant 0$ and ${}_sT(h) = T({}_{\eta(s)}(h)$ for all $h \in C(Y)$ and $s \in S$, where $\eta(s) = L_s$. Furthermore, if $h \in C(Y)$, there exists $f \in A$ such that $h(\varphi) = \varphi(f)$ for all $\varphi \in Y$ (Remark 2.1 (a)). Consequently $T(h) = f \in A$ and $T \in K[C(Y), A]$. Hence there exists a non-empty subset $F = \{\varphi_1, \ldots, \varphi_k\} \subseteq Y$, $k \leqslant n$ such that $\eta(s)F = F$ for all $s \in S$. If $\varphi = \frac{1}{k}\sum_1^k \varphi_i$, then φ is an S-invariant mean on A.

REMARK 3.3. (a) When $n = 1$, then the element in F of $(Q(1))$ becomes a fixed point for $\mathscr{S}$.

(b) When S is a semigroup and A is a norm closed *left* translation invariant subalgebra of $m(S)$ containing constants, Theorem 3.2 ($n = 1$) yields a fixed point property on compacta which is equivalent to A being extremely left amenable i.e. A has a multiplicative LIM (see [21], Theorems 1, 2 and also Section 5B of this paper). The idea in using the fact that a multiplicative mean on $C(Y)$, where Y is a compact space, can be represented as a point measure on $C(Y)$, is from Mitchell [21] (Theorem 1).

(c) (See also Theorem 5.1). If S, X, A, η are as in Theorem 3.2 and A has an S-invariant mean of the type $\varphi = \frac{1}{k}\sum_{1}^{k}\varphi_i$, where each φ_i is a multiplicative mean on A and $k \leqslant n$, we may assume $\varphi = \sum_{j=1}^{m}\lambda_j\varphi_i$ where $H_0 = \{\varphi_1, \ldots, \varphi_m\}$ is the set of distinct elements from $\{\varphi_1, \ldots, \varphi_k\}$, $\lambda_j > 0$. Then for each $a \in S$, $\sum_{j=1}^{m}\lambda_j f(L_a\varphi_j) = \sum_{j=1}^{m}\lambda_j\, f(\varphi_j)$ for all $f \in C(\Delta(A))$ ($\Delta(A)$ is the set of multiplicative means on A). Since $\Delta(A)$ separates closed sets, it follows that $L_a H_0 = H_0$ for all $a \in S$. Define on S the two-sided stable equivalence relation E (i.e. $a(\mathrm{E})b$ implies $ac(\mathrm{E})bc$ and $ca(\mathrm{E})cb$ for all $a, b, c, \in S$ [19], p. 39): $a(\mathrm{E})b$ iff $L_a\varphi = L_b\varphi$ for all $\varphi \in H_0$. If S/H_0 is the factor semigroup of S defined by E, then S/H_0 is a finite group. Furthermore, a non-empty finite subset $F_0 \subseteq Y$ in $(Q(n))$ can be chosen such that $\eta(s)F_0 = F_0$ for all $s \in S$ and

$$(*) \qquad |S/H_0| = |F_0|\,|N(y)|$$

for all $y \in F_0$, where $N(y)$ is the subgroup of S/H_0 defined by $N(y) = \{\bar{a} \in S/H_0;\ \eta(a)y = y\}$, and $\bar{a}$ denotes the homomorphic image of $a \in S$ in S/H_0. To see this, let F be the non-empty finite subset of Y such that $\eta(s)F = F$ for all $s \in S$ chosen as in the proof of Theorem 3.2 (and we shall from now on use the notation in there). For any $a, b \in S$, if $L_a\varphi = L_b\varphi$ for all $\varphi \in H_0$, then $h(\eta(a)\cdot y_i) = T^*(\varphi_i)({}_{\eta(a)}h) = T^*(\varphi_i)({}_{\eta(b)}h) = h(\eta(b)\cdot y_i)$ for all $h \in C(Y)$ and $1 \leqslant i \leqslant k$. Since $C(Y)$ separates points, it follows that $ay = by$ for all $y \in F$. Consequently, S/H_0 may be regarded as a finite group of transformations from F onto F defined by $\bar{a}(y) = \eta(a)(y)$ for all $y \in F$ and $a \in S$. Let $y_0 \in F$ be fixed, and $F_0 = \{\bar{a}y_0,\ \bar{a} \in S/H_0\}$. Then $\eta(s)F_0 = F_0$ for all $s \in S$ and $|S/H_0| = |F_0|\,|N|$, where $N = \{\bar{a} \in S/H_0;\ \bar{a}y_0 = y_0\}$ ([15], p. 22). Since $F_0 = \{\bar{a}y;\ \bar{a} \in S/H_0\}$ for any $y \in F_0$, $(*)$ follows.

EXAMPLES. Let S be a semigroup of transformations from a set X into X.

(1) If $\varphi \in \mathscr{K}X$ and

$$A_\varphi = \{f \in m(X);\ \varphi({}_sf) = \varphi(f) \text{ for all } s \in S\}.$$

Then A_φ is norm closed, S-translation invariant linear subspace of $m(X)$ containing constants. Furthermore, the restriction of φ to A is an S-invariant mean. Consequently A_φ satisfies property (P) in Theorem 3.1. Conversely, if A is a norm closed S-translation invariant subspace of $m(X)$ containing constants admitting an S-invariant mean φ, then for any extension $\varphi^{\sim}$ to φ to $m(X)$ (Remark 2.1 (a)), $A \subseteq A_{\varphi^{\sim}}$.

(2) If $\varphi \in \beta X$, then A_φ in (1) is even an algebra and the restriction of φ to A_φ is a multiplicative S-invariant mean. Consequently, A_φ satisfies $(Q(1))$ in Theorem 3.2. Conversely, if A is a norm closed S-translation invariant subalgebra of $m(X)$ containing constants and has a multiplicative S-invariant mean φ, then φ has an extension $\varphi^{\sim} \in \beta X$ to $m(X)$ (Remark 2.1 (a)) and $A \subseteq A_{\varphi^{\sim}}$. (See also [17], Theorem 1).

(3) Let $G = \{g_1, \ldots, g_n\}$ be a finite group of order n, $T = S \times G$ be the product semigroup with coordinatewise multiplication and $Y = X \times G$ be the product set. Consider T as a semigroup of transformations from Y into Y defined by $t\colon (x, g) \to (sx, g_ig)$ for each $t \in T$, $t = (s, g_i)$ and all $(x, g) \in Y$. Let A be a fixed norm closed S-translation invariant subalgebra of $m(X)$ containing constants, and A has a multiplicative S-invariant mean φ. Define

$$A_G = \{h \in m(Y);\ \Pi_g(h) \in A \quad \text{for all } g \in G\}$$

where $(\Pi_g h)(x) = h(x, g)$ for all $x \in X$, $g \in G$. Then it can be easily verified that A_G is a norm closed T-translation invariant subalgebra of $m(Y)$ containing constants. For each $i = 1, \ldots, n$, define $\psi_i \in \beta Y$ by $\psi_i(h) = \varphi(\Pi_{g_i}h)$ for all $h \in m(Y)$. Then the restriction of $\psi = \frac{1}{n}\sum_{1}^{n}\psi_i$ to A_G is a T-invariant mean since if $t = (s_0, g_0) \in T$ and $h \in A_G$, then

$$\psi({}_th) = \frac{1}{n}\sum_{i=1}^{n}\varphi({}_{s_0}\Pi_{g_0g_i}h) = \frac{1}{n}\sum_{i=1}^{n}\varphi(\Pi_{g_i}h) = \psi(h).$$

Consequently A_G satisfies $(Q(n))$ in Theorem 3.2.

4. Action of semitopological semigroups and fixed points. A mapping $\Pi\colon X \times Y \to Z$, where X, Y, Z are topological spaces, is *separately continuous* if ψ is continuous in each of the two variables when the other one is kept fixed; ψ is *jointly continuous* if ψ is continuous when $X \times Y$ has the product topology.

A *semitopological semigroup* is a semigroup S with a topology such that for each $a \in S$, the mapping from S into S defined by $s \to as$ for all $s \in S$ is continuous. A *topological semigroup* is a semitopological semigroup such that the mapping from $S \times S$ into S defined by $(s, t) \to st$ for all $s, t \in S$ is separately continuous.

An *action* is an order pair (S, X), where S is a semitopological semigroup and X is a topological space, with a mapping $\psi: S\times X\to X$, denoted by $(s,x)\to s\cdot x$ such that

(1) $s_1\cdot(s_2\cdot x) = (s_1 s_2)\cdot x$ for all $s_1, s_2\in S$, $x\in X$,

(2) the mapping $X\to X$, $x\to s\cdot x$, $x\in X$ is continuous for each fixed $s\in S$.

Furthermore, (S, X) is *separately continuous* (*jointly continuous*) if ψ is a separately continuous (jointly continuous). When X is a convex subset of a l.c.s., then (S, X) is *affine* if for each $s\in S$, the mapping from X into X defined by $x\to s\cdot x$ for all $x\in X$ is *affine*. If (T, Y) is another action, then (T, Y) is a *continuously homomorphic* to (S, X) if T is a continuous homomorphic image of S with homomorphism η, and there exists a continuous mapping $\Pi: X\to Y$ such that $\Pi(s\cdot x) = \eta(s)\cdot\Pi(x)$ for all $s\in S$ and $x\in X$.

EXAMPLE. Let T be a discrete semigroup and S be the set of means on $m(T)$. Then S with the Arens product $\odot$ defined by $\mu\odot\varphi(f) = \mu(\Pi_\varphi f)$ where $(\Pi_\varphi f)(t) = \varphi({}_t f)$ for all $\mu, \varphi\in S$, $f\in m(T)$ and $t\in T$ is a semigroup. Furthermore, the dual semigroup $(S, *)$, where $\mu*\varphi = \varphi\odot\mu$ for all $\mu,\varphi\in S$, with the w^*-topology on S is a semitopological semigroup but not necessarily a topological semigroup (see Day [2] pp. 526–531). Some examples of actions are given in Lemma 4.3.

Let (S, X) be an action. We shall denote by $WLUC(S, X)$ [$LUC(S, X)$] to be the set of all those $f\in C(X)$ for which the mapping from S into $C(X)$ defined by $s\to {}_s f$ for all $s\in S$ is continuous when $C(X)$ has the weak [sum norm] topology, or equivalently, $f\in WLUC(S, X)$ [$LUC(S, X)$] iff $f\in C(X)$ and whenever $s_\alpha\to s$, $s_\alpha, s\in S$, then $|\mu({}_{s_\alpha}f) - \mu({}_s f)|\to 0$ for all $\mu\in C(X)^*$ [$\sup_{x\in X}|f(s_\alpha\cdot x) - f(s\cdot x)|\to 0$]. Furthermore, $LMC(S, X)$ will denote the set of all those $f\in C(X)$ such that whenever $s_\alpha\to s$, $s_\alpha, s\in S$, then $\varphi({}_{s_\alpha}f)\to\varphi({}_s f)$ for all *multiplicative* means φ on $C(X)$.

Greenleaf [10] considers jointly continuous action (S, X) where S is a locally compact group and X a locally compact space for which $LUC(S, X)$ has an S-invariant mean.

Certainly any semitopological semigroup S can be considered as an action (S, S) defined by the mapping $(s,t)\to st$ for all $s, t\in S$. In this case, $LUC(S, S)$, $WLUC(S, S)$ and $LMC(S, S)$ will be denoted by $LUC(S)$, $WLUC(S)$ and $LMC(S)$ respectively.

Namioka [23] considers topological semigroups S for which $LUC(S)$ has a LIM and Mitchell [22] recently shows that certain fixed point properties are equivalent to the existence of a [multiplicative] LIM on $LUC(S)$, $WLUC(S)$ [$LUC(S)$, $LMC(S)$]. Topological semigroups S for which $LUC(S)$ has a LIM of type $\frac{1}{n}\sum_{i=1}^{n}\varphi_i$ where each φ_i is multiplicative has been studied by the author in [16] and [18].

REMARK 4.1. For any action (S, X)

(a) $C(X)\supseteq LMC(S, X)\supseteq WLUC(S, X)\supseteq LUC(S, X)$,

(b) It is easy to verify, or arguments similar to those given by Mitchell [22] (Lemmas 2, 3) for $LMC(S)$ and $WLUC(S)$, and Namioka [23] (p. 64, 68, 72) for $LUC(S)$, will show that each of the spaces listed in (a) is norm closed, S-translation invariant and contains constants. Furthermore, $LUC(S, X)$ and $LMC(S, X)$ are subalgebras of $C(X)$.

(c) If $f\in m(X)$, the condition: "whenever $s_\alpha\to s$, s_α, $s\in S$ then $\sup_{x\in X}|f(s_\alpha\cdot x) - f(s\cdot x)|\to 0$" does *not* imply that $f\in C(X)$ since any $f\in m(X)$ such that ${}_s f = c\cdot 1_X$ for some fixed c and all $s\in S$ will satisfy this condition (see [11], p. 299, footnote).

(d) If S is a semitopological semigroup, then $C(S)$ is left translation invariant but *not* necessarily right translation invariant. When S has a completely regular topology and $a\in S$ such that $r_a(C(S))\subseteq C(S)$, then the mapping from S into S, $s\to sa$ for all $s\in S$, is continuous. In fact if $s_\alpha\to s$, s_α, $s\in S$, then $p_{s_\alpha a}(f)\to p_{sa}(f)$ for all $f\in C(S)$. Consequently $s_\alpha a\to sa$ ([6], Theorem 22, p. 276). In particular, if S is a completely regular semitopological semigroup, then $C(S)$ is translation invariant iff S is a topological semigroup. However, the subspaces $LUC(S)$, $WLUC(S)$ are left introverted, and $LMC(S)$ is left M-introverted (which already implies that they are right translation invariant) for any semitopological semigroups S. Furthermore, it follows from [25] (Theorem 1) that $WLUC(S)$ is the unique maximal left-introverted subspace of $C(S)$; and it can be shown that $LMC(S)$ is the unique maximal left M-introverted subalgebra of $C(S)$ (this observation is due to Mitchell [22] when S is a topological semigroup).

LEMMA 4.2. *Let (S, X) be an action.*

(a) *If (S, X) is separately continuous and X is compact then $LMC(S, X) = C(X)$.*

(b) *If (S, X) is separately continuous and affine and X is a compact convex subset of a l.c.s., then $WLUC(S, X)\supseteq\mathscr{A}(X)$.*

(c) *If (S, X) is jointly continuous and $f\in C(X)$, then the mapping from S to $C(X)$ defined by $s\to {}_s f$ for all $s\in S$ is continuous when $C(X)$ has the topology of uniform convergence on compacta. In particular, if X is compact, then $LUC(S, X) = C(X)$.*

Proof. (a) For any multiplicative mean φ on $C(X)$, let $x\in X$ such that the restriction of p_x to $C(X)$ coincide with φ (Remark 2.1 (b)). Then for any $f\in C(X)$, if $s_\alpha\to s$, $s_\alpha, s\in S$, then $\varphi({}_{s_\alpha}f) = f(s_\alpha\cdot x)\to f(s\cdot x) = \varphi({}_s f)$.

(b) If φ is a mean on $C(X)$ and $x\in X$ be such that $\mu(h) = h(x)$ for all $h\in\mathscr{A}(X)$ (Remark 2.1 (c)). If $h\in\mathscr{A}(X)$ and $s_\alpha\to s$, s_α, $s\in S$, then ${}_s h$, ${}_{s_\alpha}h\in\mathscr{A}(X)$ for each α and $\mu({}_{s_\alpha}h) = h(s_\alpha\cdot x)\to h(s\cdot x) = \mu({}_s h)$.

(c) Let $f \in C(X)$, $s_\alpha \to s$, $s_\alpha, s \in S$ and K be a compact subset of X. For each α, pick $x_\alpha \in K$ such that the continuous real function $|{}_{s_\alpha}f - {}_sf|$ attains its maximum on K. If $\sup_{x \in K} |f(s_\alpha \cdot x) - f(s \cdot x)|$ does not converge to 0, then there exists $\varepsilon > 0$ and subnets $\{s_\gamma\}$, $\{x_\gamma\}$ of the nets $\{s_\alpha\}$ and $\{x_\alpha\}$ respectively such that $|f(s_\gamma \cdot x_\gamma) - f(s \cdot x_\gamma)| \geqslant \varepsilon$ for all γ. By compactness of K, we may assume that x_γ converges to some $x \in K$. Consequently we arrive a contradiction that

$$0 < \varepsilon \leqslant |f(s_\gamma \cdot x_\gamma) - f(s \cdot x_\gamma)| \leqslant |f(s_\gamma \cdot x_\gamma) - f(s \cdot x)| + |f(s \cdot x) - f(s \cdot x_\gamma)| \to 0$$

since $s_\gamma \cdot x_\gamma \to s \cdot x$ and $s \cdot x_\gamma \to s \cdot x$ by jointly continuity of (S, X).

Remark. Lemma 4.3 (c) has been shown by the author in [18] (Lemma 2.1) for the case when S is a topological semigroup, and the proof given there can be carried to our case. However we give a (different) proof for the sake of completeness.

LEMMA 4.3. *Let (S, X) be an action, A be a norm closed S-translation invariant subspace of $C(X)$ containing constants and Y be a $\sigma(A^*, A)$-compact subset of the set of means on A containing the set of point measures. Let $P: S \times Y \to Y$ be defined by $(s, \varphi) \to L_s\varphi$ for all $s \in S$ and $\varphi \in Y$. Then*

(a) *P defines an action (S, Y) which is continuously homomorphic to (S, X).*

(b) *P defines a separately continuous action (S, Y) iff for each $f \in A$ and $s_\alpha \to s$, $s_\alpha, s \in S$, then $\varphi({}_{s_\alpha}f) \epsilon \varphi({}_sf)$ for all $\varphi \in Y$.*

(c) *P defines a jointly continuous action (S, Y) iff $A \subseteq LUC(S, X)$.*

Proof. (a) Define a continuous mapping Π from X into Y by $\Pi(x) = p_x$ restricted to A, $x \in X$. Then $(s \cdot x) = L_s p_x$ for all $s \in S$ and $x \in X$.

(b) is trivial; to prove (*b*), if (S, Y) is jointly continuous, let $f \in A$ and define $\bar{f} \in C(Y)$ by $\bar{f}(\varphi) = \varphi(f)$ for all $\varphi \in Y$. Now if $s_\alpha \to s$, $s_\alpha, s \in S$, then

$$\|{}_{s_\alpha}f - {}_{s_0}f\| \leqslant \sup_{\varphi \in Y} |{}_{s_\alpha}\bar{f}(\varphi) - {}_s\bar{f}(\varphi)| \to 0$$

since $\bar{f} \in LUC(S, Y) = C(Y)$ by Lemma 4.2 (c). Conversely, if $A \subseteq LUC(S, X)$, let $f \in A$, $s_\alpha \to s$, $s_\alpha, s \in S$ and $\varphi_\beta \to \varphi$, $\varphi_\beta, \varphi \in Y$, then

$$|\varphi_\beta({}_{s_\alpha}f) - \varphi({}_sf)| \leqslant |\varphi_\beta({}_{s_\alpha}f) - \varphi_\beta({}_sf)| + |\varphi_\beta({}_sf) - \varphi({}_sf)|$$
$$\leqslant \|{}_{s_\alpha}f - {}_sf\| + |\varphi_\beta({}_sf) - \varphi({}_sf)| \to 0.$$

Consider the following properties for a given action (S, X)

(P_1) Whenever (T, Y) is affine and continuously homomorphic to (S, X) and Y is a compact convex subset of a l.c.s., then Y has a fixed point for T.

(P_2) Whenever (T, Y) is affine, *separately* continuous and continuously homomorphic to (S, X) and Y is a compact convex subset of a l.c.s., then Y has a fixed point for T.

(P_3) Whenever (T, Y) is affine, *jointly* continuous and continuously homomorphic to (S, X) and Y is a compact convex subset of a l.c.s., then Y has a fixed point for T.

THEOREM 4.4. *For any action (S, X)*

(a) *$C(X)$ has an S-invariant mean iff (P_1);*

(b) *$WLUC(S, X)$ has an S-invariant mean iff (P_2);*

(c) *$LUC(S, X)$ has an S-invariant mean iff (P_3).*

Proof. For each of (a), (b) and (c), let Π be a continuous mapping from X into Y such that $\Pi(s \cdot x) = \eta(s)\Pi(x)$ for all $s \in S$ and $x \in X$, where η is a continuous homomorphism from S onto T. Define $\tilde{\Pi}: C(Y) \to C(X)$ by $(\tilde{\Pi}h)(y) = h(\Pi(y))$ for $h \in C(Y), y \in Y$. Then $\tilde{\Pi}$ is linear, $\tilde{\Pi}(1_X) = 1_Y$, $\tilde{\Pi}(h) \geqslant 0$ for $h \geqslant 0$ and $\tilde{\Pi}({}_{\eta(s)}h) = {}_s\tilde{\Pi}(h)$ for all $s \in S$, $h \in C(Y)$.

(a) Since $\tilde{\Pi}(\mathscr{A}(Y)) \supseteq C(Y)$, it follows that Y has a fixed point for T whenever $C(X)$ has an S-invariant mean (Theorem 3.1). Conversely, if (P_1) holds, consider the action of S on Y the set of means on $C(X)$ defined by $(s, \varphi) \to L_s\varphi$ for all $s \in S$ and $\varphi \in Y$. Then (S, Y) is affine and is continuously homomorphic to (S, X) (Lemma 4.3 (a)). Since Y is a w^*-compact convex subset of $C(X)^*$, it follows that Y has a fixed point ψ for S; and ψ is an S-invariant mean on $C(X)$.

(b) Let $f \in WLUC(T, Y)$. If $\mu \in C(X)^*$ then $\tilde{\Pi}^*(\mu) \in C(Y)^*$ and hence

$$|\mu({}_{s_\alpha}\tilde{\Pi}f) - \mu({}_s\tilde{\Pi}f)| = |\tilde{\Pi}^*\mu({}_{\eta(s_\alpha)}f) - \tilde{\Pi}^*\mu({}_{\eta(s)}f)| \to 0$$

whenever $s_\alpha \to s$, $s_\alpha, s \in S$. Hence $\Pi(f) \in WLUC(S, X)$. Since $WLUC(T, Y) \supseteq \mathscr{A}(Y)$ (Lemma 4.2 (b)), it follows that $\tilde{\Pi}(\mathscr{A}(Y)) \subseteq WLUC(S, X)$. Consequently Y has a fixed point whenever $WLUC(S, X)$ has an S-invariant mean (Theorem 3.2). Conversely, if (S, X) satisfies (P_2), consider the action of S on Y the set of means on $WLUX(S, X)$ defined by $(s, \varphi) \to L_s\varphi$ for all $s \in S$ and $\varphi \in Y$. Then (S, Y) is affine, separately continuous and continuously homomorphic to (S, X) (Lemma 4.3 (a), (b)). Since Y is a w^*-compact convex subset of $WLUC(S, X)^*$, it follows that Y has a fixed point ψ for S; and ψ is then an S-invariant mean on $WLUC(S, X)$.

(c) If $f \in LUC(T, Y)$ and $s_\alpha \to s$, $s_\alpha, s \in S$, then $\|{}_{s_\alpha}(\tilde{\Pi}f) - {}_s(\tilde{\Pi}f)\| = \|\tilde{\Pi}({}_{\eta(s_\alpha)}f) - \tilde{\Pi}({}_{\eta(s)}f)\| \leqslant \|{}_{\eta(s_\alpha)}f - {}_{\eta(s)}f\| \to 0$. Hence $\tilde{\Pi}f \in LUC(S, X)$. Since $LUC(T, Y) = C(Y)$ (Lemma 4.2 (c)), it follows that $\tilde{\Pi}(\mathscr{A}(Y)) \subseteq LUC(S, X)$. Consequently Y has a fixed point whenever $WLUC(S, X)$ has an S-invariant mean (Theorem 3.2). Conversely if (S, X) satisfies (P_3), consider the action of S on Y the set of means on $LUC(S, X)$ defined by $(s, \varphi) \to L_s\varphi$ for all $s \in S$ and $\varphi \in Y$. Then (S, Y) is affine, jointly continuous and continuously homomorphic to (S, X) (Lemma 4.3 (a), (c)). Since Y is a w^*-compact convex subset of $LUC(S, X)^*$, it follows that Y has a fixed point ψ for S; and ψ is then an S-invariant mean on $LUC(S, X)$.

Remark. (a) When S is a topological semigroup and $\{\eta(s);\ s \in S\}$ a homomorphic representation of S as continuous mappings from a topological space Y into Y, then the following are equivalent:

(1) there exists a continuous mapping Π: $S \to Y$ such that $\Pi(s_1 s_2) = \eta(s_1)\Pi(s_2)$ for all $s_1, s_2 \in S$.

(2) there exists $y \in Y$ such that the mapping $s \to \eta(s)\cdot y$, $s \in S$, is continuous from S into Y.

Consequently Theorem 4.4 (a) becomes Theorem 4 in Day [4] when S is a topological semigroup and $S = X$. Note that if S is only a semitopological semigroup, then (1) does not imply (2) in general

(b) when S is a semitopological semigroup and (S, S) is the action defined by $(s, t) \to st$ for all $s, t \in S$, then every *separately* continuous action (T, Y), where T is a continuous homomorphic image of S with homomorphism η, is continuously homomorphic to (S, S) since the mapping Π: $S \to Y$, $\Pi(s) = \eta(s)\cdot y$ for some fixed $y \in Y$ and $s \in S$ is continuous and $\Pi(s_1\cdot s_2) = \eta(s_1)\Pi(s_2)$ for all $s_1, s_2 \in S$. Consequently Theorem 4.4 (b), (c) becomes Theorem 4, 2 respectively in Mitchell [22] when S is a topological semigroup and $S = X$.

For a given action (S, X) and $n \in Z^+$, if

$Q_1(n)$ whenever (T, Y) is continuously homomorphic to (S, X) and Y is compact, then there exists a non-empty finite subset $F \subseteq Y$ such that $|F| \leqslant n$ and $t\cdot F = F$ for all $t \in T$.

$(Q_2(n))$ whenever (T, T) is *separately* continuous and continuously homomorphic to (S, X) and Y is compact, then there exists a non-empty finite subset $F \subseteq Y$ such that $|F| \leqslant n$ and $t\cdot F = F$ for all $t \in T$.

$(Q_3(n))$ whenever (T, Y) is *jointly* continuous and continuously homomorphic to (S, X) and Y is compact, then there exists a non-empty finite subset $F \subseteq Y$ such that $|F| \leqslant n$ and $t\cdot F = F$ for all $t \in T$.

Then with simple modification of the proof of Theorem 4.4 and applying Theorem 4.2, one obtains:

THEOREM 4.5. *For any action (S, X) and $n \in Z^+$*

(a) *$C(X)$ has an S-invariant mean which is the average of k multiplicative means for some $k \in Z^+$, $k \leqslant n$ iff $(Q_1(n))$.*

(b) *$LMC(S, X)$ has an S-invariant mean which is the average of k multiplicative means for some $k \in Z^+$, $k \leqslant n$ iff $(Q_2(n))$.*

(c) *$LUC(S, X)$ has an S-invariant mean which is the average of k multiplicative means for some $k \in Z^+$, $k \leqslant n$ iff $(Q_3(n))$.*

Remark. (a) When S is a topological semigroup, $S = X$ and $n = 1$, Theorem 4.5 is due to Mitchell ([21], Theorem 1; [22], Theorems 1, 3). The author proves Theorem 4.5 (c) for the case when S is a topological semigroup and $S = X$ in [18], Theorem 2.2.

(b) When S is a locally compact group and Y is a locally compact space, R. Ellis [7] has shown that an action (S, Y) is separately continuous iff (S, Y) is jointly continuous. Consequently, for any action (S, X) where S is a locally compact group, then $(P_2) \Leftrightarrow (P_3)$ and $(Q_2(n)) \Leftrightarrow (Q_3(n))$ for all $n \in Z^+$. Furthermore, we can prove, using Ellis's result something even stronger (the following proposition is due to Mitchell [22], Theorem 7, for the case when $S = X$):

PROPOSITION. *If (S, X) is an action and S is a locally compact group, then $LUC(S, X) = WLUC(S, X) = LMC(S, X)$.*

Proof. In any case

$$(*) \qquad LUC(S, X) \subseteq WLUC(S, X) \subseteq LMC(S, X).$$

Let Y be the set of multiplicative means on $LMC(S, X)$, then the mapping $S \times Y \to Y$, $(s, \varphi) \to L_s\varphi$ for all $s \in S$ and $\varphi \in M$ is separately continuous (Lemma 4.3 (b)) and hence jointly continuous (Ellis [7]). By Lemma 4.3 (c), $LMC(S, X) \subseteq LUC(S, X)$ forcing equality in $(*)$.

Remark. Semitopological semigroups S for which

$$(**) \qquad LUC(S) \text{ has a LIM of type } \frac{1}{n}\sum_{1}^{n}\varphi_i,$$

where each φ_i is a multiplicative mean on $LUC(S)$, and yet $m(S)$ does not even admit a LIM are abundant (see for example [16]). However the only examples of topological groups S which satisfy $(**)$ we can find are the finite groups. In fact, it is known ([19] Theorem 3) that the only topological subsemigroups S of a locally compact group which satisfy $(**)$ are the finite groups. Furthermore, it is also known ([8] Theorem 1, [27] Theorem 3.3.6) that when S has the discrete topology and S is right cancellative, then S satisfies $(**)$ iff S is a finite group. (See also Mitchell [22] discussion in Theorem 6).

5. Special cases.

A. Fixed point properties on convex compacta and subspaces of $m(S)$. Let S be a semigroup, A be a norm closed *left* translation invariant subspace of $m(S)$ containing constants and K_A be the set of all means φ on A such that $\Pi_\varphi(A) \subseteq A$, where $(\Pi_\varphi f)(s) = \varphi({}_sf)$

for all $s \in S$ and $f \in A$. If $\mathscr{S} = \{\eta(s);\ s \in S\}$ is a homomorphic representation of S as mappings from Y into Y, for each $y \in Y$, define $(Tyh)(s) = h(\eta(s)y)$ for all $h \in m(Y)$ and $s \in S$. Consider on the pair S, A the following fixed point properties (see [1], p. 128).

(E_1) Whenever $\mathscr{S}$ is a homomorphic representation of S as affine continuous mappings from a compact convex subset Y of a l.c.s. into Y such that $Ty(\mathscr{A}(Y)) \subseteq A$ for *some* $y \in X$, then Y has a fixed point for $\mathscr{S}$.

(E_2) Whenever $\mathscr{S}$ is a homomorphic representation of S as affine continuous mappings from a compact convex subset Y of a l.c.s. into Y such that $\{y;\ Ty(\mathscr{A}(Y)) \subseteq A\}$ is *dense* in Y, then Y has a fixed point for $\mathscr{S}$.

(E_3) Whenever $\mathscr{S}$ is a homomorphic representation of S as affine continuous mappings from a compact convex subset Y of a l.c.s. into Y such that $Ty(\mathscr{A}(Y)) \subseteq A$ for *all* $y \in Y$, then Y has a fixed point for $\mathscr{S}$.

In general we have from Theorem 3.1 that:

$$A \text{ has a LIM} \Leftrightarrow (P) \Rightarrow (E_1) \Rightarrow (E_2) \Rightarrow (E_3).$$

When K_A is non-empty (which is the case when S has an identity), then

$$A \text{ has a LIM} \Leftrightarrow (P) \Leftrightarrow (E_1) \Rightarrow (E_2) \Rightarrow (E_3).$$

To prove $(E_1) \Rightarrow (P)$, let $\varphi \in K_A$ and $y \in Y$ such that p_y coincide with the mean $T^*(\varphi)$ on $\mathscr{A}(Y)$ (Remark 2.1 (c)), then $Ty(\mathscr{A}(Y)) \subseteq A$.

When A is right translation invariant, then K_A includes all finite means on A and it follows from Theorem 1 in [1] that

$$A \text{ has a LIM} \Leftrightarrow (P) \Leftrightarrow (E_1) \Leftrightarrow (E_2) \Rightarrow (E_3).$$

When A is left introverted, then K_A includes all means on A and it follows from Theorem 2 in [1] that

$$A \text{ has a LIM} \Leftrightarrow (P) \Leftrightarrow (E_1) \Leftrightarrow (E_2) \Leftrightarrow (E_3).$$

B. Fixed point properties on compacta and subalgebras of $m(S)$. Let S be a semigroup, A be a norm closed left translation invariant subalgebra of $m(S)$ containing constants, H_A be the set of all $\varphi \in \Delta(A)$ (the set of all multiplicative means on A) such that $\Pi_\varphi(A) \subseteq A$ and $n \in Z^+$. Consider on the pair S, A the following properties (see [21] p. 118):

$(F_1(n))$ Whenever $\{\eta(s);\ s \in S\}$ is a homomorphic representation of S as continuous mappings from a compact space Y into Y such that $Ty(C(Y)) \subseteq A$ for *some* $y \in Y$, then there exists a non-empty finite subset $F \subseteq Y$ such that $|F| \leqslant n$ and $\eta(s)F = F$ for all $s \in S$.

$(F_2(n))$ Whenever $\{\eta(s);\ s \in S\}$ is a homomorphic representation of S as continuous mappings from a compact space Y into Y such that $\{y;\ Ty(C(Y)) \subseteq A\}$ is *dense* in Y, then there exists a non-empty finite subset $F \subseteq Y$ such that $|F| \leqslant n$ and $\eta(s)F = F$ for all $s \in S$.

$(F_3(n))$ Whenever $\{\eta(s);\ s \in S\}$ is a homomorphic representation of S as continuous mappings from a compact space Y into Y such that $Ty(C(Y)) \subseteq A$ for *all* $y \in Y$, then there exists a non-empty finite subset $F \subseteq Y$ such that $|F| \leqslant n$ and $\eta(s)F = F$ for all $s \in S$.

Let $(U(n))$ stand for the statement that A has a LIM of the type $\frac{1}{k}\sum_1^k \varphi_i$ where each $\varphi_i \in \Delta(A)$ and $k \leqslant n$. Then in general it follows from Theorem 3.2 that:

$$(U(n)) \Leftrightarrow (Q(n)) \Rightarrow (F_1(n)) \Rightarrow (F_2(n)) \Rightarrow (F_3(n)).$$

When H_A is non-empty (which is the case when S has an identity), then

$$(U(n)) \Leftrightarrow (Q(n)) \Leftrightarrow (F_1(n)) \Rightarrow (F_2(n)) \Rightarrow (F_3(n)).$$

To prove $(F_1(n)) \Leftrightarrow (Q(n))$, let $\varphi \in H_A$ and $y \in Y$ such that the restriction of p_y to $C(Y)$ coincide with the multiplicative mean $T^*(\varphi)$ on $C(Y)$, then $Ty(C(Y)) \subseteq A$.

When A is right translation invariant, then H_A includes all point measure on A; a simple modification of the proof of Mitchell [21] (Theorem 1, (3) $\Rightarrow$ (1)) shows that $(F_2(n)) \Rightarrow (U(n))$. Consequently

$$(U(n)) \Leftrightarrow (Q(n)) \Leftrightarrow (F_1(n)) \Leftrightarrow (F_2(n)) \Rightarrow (F_3(n)).$$

When A is left M-introverted, then $H_A = \Delta(A)$ and a simple modification of the proof in [21] (Theorem 2 (4) $\Rightarrow$ (1)) shows that $(F_3(n)) \Leftrightarrow (U(n))$. Consequently

$$(U(n)) \Leftrightarrow (Q(n)) \Leftrightarrow (F_1(n)) \Leftrightarrow (F_2(n)) \Leftrightarrow (F_3(n)).$$

C. M-introverted subalgebras. For certain classes of subalgebras, Theorem 3.2 assumes a stronger form:

THEOREM 5.1. *Let S be a semitopological semigroup, A be a norm closed, left translation subalgebra of $m(S)$ containing constants and $n \in Z^+$. If A satisfies either* (1) *or* (2):

(1) *A is left M-introverted;*

(2) *$LUC(S) \subseteq A \subseteq LMC(S)$;*

then A has a LIM of type $\frac{1}{k}\sum_1^k \varphi_i$ for some $k \in Z^+$, $k \leqslant n$ and k divides n iff

$(Q'(n))$ whenever $\{\eta(s);\ s \in S\}$ is a homomorphic representation of S as continuous mappings from a compact space Y into Y and there exists

a multiplicative $T \in K[C(Y), A]$ *such that* ${}_sT(h) = T({}_{\eta(s)}h)$ *for all* $s \in S$ *and* $h \in C(Y)$, *then there exists a non-empty finite subset* $F \subseteq Y$ *such that* $|F| \leqslant n$, $|F|$ *divides* n *and* $\eta(s)F = F$ *for all* $s \in S$.

Proof. Sufficiency follows easily from the proof of Theorem 3.2. To prove necessity, let H_0 be the set of distinct elements from $\{\varphi_1, \ldots, \varphi_k\}$, then as in Remark 3.3 (c), the factor semigroup $G = S/H_0$ of S defined by the equivalence relation E: $a(\mathrm{E})b$ iff $L_a\varphi = L_b\varphi$ for all $\varphi \in H_0$ is a finite group. Let $\{S_1, \ldots, S_m\}$ denotes the coset decomposition of S by G. If we can show that

$$(**) \qquad 1_{S_i} \in A \quad \text{for each} \quad i = 1, \ldots, m$$

define a linear mapping $\psi \colon m(G) \to A$ by $\psi(h)(s) = h(\bar{s})$ for all $s \in S$, where $\bar{s}$ is the homomorphic image of s in G. Then $\psi(h) = \sum_1^m h(\bar{a}_i)1_{S_i}$, where $\{a_1, \ldots, a_m\}$ is a coset representation of G. One readily checks that $\psi \in K[m(G), A]$, $\psi({}_{\bar{s}}h) = {}_s(\psi(h))$ for all $s \in S$, $h \in m(G)$ and ψ is multiplication. Consequently each $\psi^*(\varphi_i)$ is a point measure p_{g_i}, $i = 1, \ldots, k$ on $m(G)$, $g_i \in G$, and $\frac{1}{k}\sum_1^k p_{g_i}$ is a LIM on $m(G)$. By uniqueness of invariant means on a finite group, $\frac{1}{k}\sum_1^k p_{g_i} = \frac{1}{m}\sum_{g \in G} p_g$. Hence $|G|$ divides k. By Remark 3.3 (c), there exists a non-empty subset $F \subseteq Y$, $|F| \leqslant n$, such that $\eta(s)F = F$ for all $s \in S$ and $|F|$ divides $|G|$. Consequently $|F|$ divides k.

It remains to show (**).

(1) If A is left M-introverted, let $\bar{S}_i$ denote the $\sigma(A^*, A)$-closure of $\{p_s; p_s$ restricted to A and $s \in S_i\}$ in $\Delta(A)$. If $i \neq y$ and $\mu_0 \in \bar{S}_i \cap \bar{S}_j$, let $\{s_a\} \subseteq S_i$ and $\{t_\beta\} \subseteq S_j$ be nets such that $\lim_a p_{s_a}(f) = \mu_0(f)$ and $\lim_\beta p_{t_\beta}(f) = \mu_0(f)$ for all $f \in A$. Let $\varphi_0 \in H_0$ be such that $L_s\varphi_0 = L_t\varphi_0$ for all $s \in S_i$, $t \in S_j$. Since $\{L_{s_a}\varphi_0\} \subseteq H_0$, $\{L_{t_\beta}\varphi_0\} \subseteq H_0$, and H_0 is finite, there exists $\gamma_1, \gamma_2 \in H_0$ and a_0, β_0 such that $L_{s_a}\varphi_0 = \gamma_1$ and $L_{t_\beta}\varphi_0 = \gamma_2$ for all $a \geqslant a_0$ and $\beta \geqslant \beta_0$. By our choice of φ_0, $\gamma_1 \neq \gamma_2$. On the other hand,

$$\gamma_1(f) = \lim_a \varphi_0({}_{s_a}f) = \lim_a \mu_0(\Pi_{\varphi_0}f) = \lim_\beta \mu_0({}_{t_\beta}f) = \gamma_2(f)$$

for all $f \in A$, where $(\Pi_{\varphi_0}f)(s) = \varphi_0({}_sf)$ and $s \in S$, which is impossible. Consequently, $\{\bar{S}_i;\ i = 1, \ldots, m\}$ is a collection of *disjoint* $\sigma(A^*, A)$ closed subset of $\Delta(A)$ with union $\bigcup_{i=1}^m \bar{S}_i = \Delta(A)$. Hence each $\bar{S}_i$ is also open and $1_{\bar{S}_i} \in C(\Delta(A))$. Let $f_i \in A$ such that $\varphi(f_i) = 1_{\bar{S}_i}(\varphi)$ for all $\varphi \in \Delta(A)$ (Remark 2.1 (a)). It follows that $1_{S_i} = f_i \in A$ for all $i = 1, \ldots, m$ and hence (**).

(2) If $LUC(S) \subseteq A \subseteq LMC(S)$, let $S_i \in \{S_1, \ldots, S_m\}$ be arbitrary but fixed and $a_i \in S$. If $s_a \to s$, $s_a, s \in S_i$, and $\varphi \in H_0$, let $\varphi^{\sim}$ be an extension of φ to a multiplicative mean on $C(S)$ (Remark 2.1 (a)). Then for each $f \in A \subseteq LMC(S)$,

$$(L_{a_i}\varphi)(f) = \varphi({}_{s_a}f) = \varphi^{\sim}({}_{s_a}f) \to \varphi^{\sim}({}_sf) = \varphi({}_sf) = L_s\varphi(f).$$

Consequently $L_s\varphi = L_{a_i}\varphi$ for all $\varphi \in H_0$ and $s \in S_i$. Hence $\{S_1, \ldots, S_m\}$ is a collection of disjoint closed subset of S with union $\bigcup_{i=1}^m S_i = S$. It follows that each S_i is also open and $1_{S_i} \in C(S)$. Let $s_\beta \to s$, $s_\beta, s \in S$; then $s \in S_{i_0}$ for some $S_{i_0} \in \{S_1, \ldots, S_m\}$ and there exists β_0 such that $s_\beta \in S_{i_0}$ for all $\beta \geqslant \beta_0$. Consequently $\{a \in S;\ s_\beta a \in S_i\} = \{a \in S;\ sa \in S_i\}$ for all $\beta \geqslant \beta_0$, which implies $\|{}_{s_\beta}1_{S_i} - {}_s1_{S_i}\| = 0$ for all $\beta \geqslant \beta_0$ and $1_{S_i} \in LUC(S) \subseteq A$. Hence (**) holds.

Remark. We do not know whether or not Theorem 5.1 will still remain true without impossing that A must satisfy either condition (1) or condition (2).

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Some metric and topological properties of certain linear spaces of measurable functions

by

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Abstract. Locally convex spaces of measurable functions, which are projective limits of the normed function spaces of Zaanen and Luxemburg, are studied from the topological point of view. Some duality results are obtained, in which topologies of uniform convergence on families of solid sets of functions are important. In the case in which such spaces are metrizable, a completeness condition can be given, and it is then often possible to represent the metric dual explicitly as a function space.

1. Introduction. It is the purpose of this note to communicate some results concerning topological vector spaces whose elements are measurable functions. These results are in part an extension of some older work of Dieudonné [1], and are related to the papers of Welland [6], [7]. We make use also of the recent work of Luxemburg and Zaanen on normed Köthe spaces [4].

The function spaces which we consider are all projective limits, by linear mappings of a very natural type, of a family of normed Köthe spaces (which need not be complete).

The term "locally convex space" will be used to mean "locally convex topological vector space". An "absolutely convex" set in a vector space will be a set which is both convex and balanced, and a "neighborhood" in a topological vector space will always be, unless stated otherwise, a neighborhood of the origin. Φ will stand for the scalar field which we now assume to be held fixed; Φ may be either the real or the complex number system. We assume from now on that μ is a non-trivial, non-negative, countably additive, (totally) σ — finite measure on the non-void set X; we assume that the Carathéodory extension procedure has been applied to μ, so that μ cannot be further extended by this proce-

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