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Projective and inductive limits of Banach spaces

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Abstract. We consider the projective and inductive limits of sequences of maps of Banach spaces. New results are given on when these are, respectively, reduced and Hausdorff. Applications are made to the question of the density of a countable intersection of dense subspaces and to the problem of constructing a Fréchet space with a given sequence of canonical maps. Also, an application to the theory of λ-nuclear spaces is indicated.

Let $T_n\colon F_{n+1} \to F_n$, $n = 1, 2, \ldots$ be a sequence of continuous linear maps of locally convex spaces. The *projective limit* of the sequence (T_n) is the locally convex space $F = F[\tau]$ where F is the vector space of all sequences $x = (x_n) \in \prod F_n$ with the property that each $x_n = T_n x_{n+1}$ and τ is the induced product topology. We have the *canonical projection maps* $P_n\colon F \to F_n$ defined by $P_n x = x_n$. Obviously $P_n = T_n P_{n+1}$ for all n. We say that the projective limit is *reduced* if each $P_n(F)$ is dense in F_n.

Let $S_n\colon E_n \to E_{n+1}$, $n = 1, 2, \ldots$ be a sequence of continuous linear maps of locally convex spaces. Let $Q_n\colon E_n \to \oplus E_n$ be the usual injection map which sends $x \in E_n$ into a sequence whose nth coordinate is x and all other coordinates are 0. Let H be the vector subspace of $\oplus E_n$ generated by the elements $Q_k x - Q_{n-1} S_{n-1} \circ \ldots \circ S_k x$, $k < n$, $x \in E_k$. Then the *inductive limit* of the sequence (S_n) is the topological vector space $E[\tau] = \oplus E_n / H$. It is Hausdorff (and locally convex) if and only if H is closed.

Let $T_n\colon F_{n+1} \to F_n$, $R_n\colon G_{n+1} \to G_n$, $n = 1, 2, \ldots$ be two sequences of continuous linear maps of Banach spaces. We say that (T_n), (R_n) are *equivalent* if there exist isometries $U_n\colon F_n \to G_n$ such that $U_n^{-1} R_n U_{n+1} = T_n$, $n = 1, 2, \ldots$

Let F be a Fréchet space and let (p_n) be a fundamental sequence of seminorms defining the topology. Consider the seminormed space (F, p_n) and let F_{p_n} be the normed space formed by taking the vector space $F/\ker p_n$ with the norm induced by p_n. Let $\hat{F}_{p_n}$ be the completion of F_{p_n}. There is no loss of generality if we assume that $p_n(x) \leqslant p_{n+1}(x)$

for all n and $x \in F$. In this case the identity map on F induces a continuous linear map $F_{p_{n+1}} \to F_{p_n}$ and its extension to the completions, $\hat{F}_{p_{n+1}} \to \hat{F}_{p_n}$ is called the nth *canonical map* of the sequence (p_n).

If E is a Banach space we shall sometimes consider more than one topology on E. In this case, for simplicity of notation, we adopt the convention that if no topology is mentioned then the norm topology is understood. Thus, E', E'' will stand for the first and second Banach space duals of E.

LEMMA. *Let $T_n\colon F_{n+1} \to F_n$, $n = 1, 2, \ldots$ be a sequence of linear continuous maps of Banach spaces with dense range. Then for each k, the set $\{b \in F_k\colon \exists (x_n)_n$ with $x_n = T_n x_{n+1}$ and $x_k = b\}$ is dense in F_k.*

Proof. Choose k, $\varepsilon > 0$ and $a \in F_k$. There is no loss of generality if we assume that $\|T_n\| \leqslant 1$ for all n. We construct two sequences (x_j^{k+j-1}), (x_j^{k+j}), $j = 1, 2, \ldots$ as follows. Take $x_1^k = T_k(x_1^{k+1})$ with $\|x_1^k - a\| \leqslant \varepsilon/2$. Then for each $j > 1$ take $x_j^{k+j-1} = T_k(x_j^{k+j})$ with $\|x_j^{k+j-1} - x_{j-1}^{k+j-1}\| \leqslant \varepsilon/2^j$. It follows that for each $n \geqslant k$, the sequence $(x_j^n)_j$ is Cauchy in F_n and hence convergent to $x^n \in F_n$. By continuity we have $x^n = T_n x^{n+1}$ for all $n \geqslant k$ and $\|x^k - a\| \leqslant \|x^k - x_1^k\| + \|x_1^k - a\| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$. We define $x^n = T_n \circ \ldots \circ T_{k-1} x^k$ for $k < n$ and it follows that (x^n) is the desired sequence. ∎

Using this lemma, we now obtain two results; one on the density of an intersection of dense subspaces and the other characterizing reduced projective limits of sequences of Banach spaces.

PROPOSITION 1. *Let X be a Banach space and $(X_n)_n$ a decreasing sequence of subspaces ∋ for each n ∃ a linear continuous map $A_n\colon F_n \to X$ of Banach spaces with $A_n(F_n) = X_n$ and $A_n^{-1}(X_{n+1})$ dense in F_n. Then $\bigcap_n X_n$ is dense in the closure of X_1.*

Proof. First suppose that A_n is not 1-1. Then we can replace it by $\bar{A}_n\colon F_n/\ker A_n \to X$. Clearly $\bar{A}_n(F_n/\ker A_n) = A_n(F_n) = X_n$ and if $\Pi_n\colon F_n \to F_n/\ker A_n$ is the quotient map, then $\bar{A}_n^{-1}(X_{n+1}) = \Pi_n A_n^{-1}(X_{n+1})$ which is dense because Π_n is onto. Thus we may assume that A_n is 1-1.

Now we define $T_n\colon F_{n+1} \to F_n$ by $T_n(y) = A_n^{-1} A_{n+1}(y)$. This map is defined because $X_{n+1} \subset X_n$; it is obviously linear and its continuity follows from the closed graph theorem. Moreover, $T_n(F_{n+1}) = A_n^{-1} A_{n+1}(F_{n+1}) = A_n^{-1}(X_{n+1})$ so T_n has dense range. If we define $T_0\colon F_1 \to \bar{X}_1$ by $T_0(x) = A_1(x)$, then T_0 also has dense range. Applying the lemma with $k = 0$ we conclude that $\bigcap_n T_0 \ldots T_n(F_{n+1})$ is dense in $\bar{X}_1$. But this intersection is clearly equal to $\bigcap_n A_{n+1}(F_{n+1}) = \bigcap_n X_n$. ∎

Remark 1. Proposition 1 is a slight generalization of an unpublished result of W. Wojtyński who assumed that X was a Hilbert space, and

each A_n was a fixed map, $A\colon X \to X$ where A was 1-1 and had dense range. The converse of our result is clearly false for we can take each X_n to be a fixed dense subspace which is not the image of any Banach space.

On the other hand some hypotheses are necessary for it is true that *in every infinite dimensional separable Banach space there is a decreasing sequence of dense subspaces whose intersection is* $\{0\}$. Indeed we need only take a dense sequence (x_n) and perturb it slightly so that it remains dense and becomes linearly independent. Then if X_n is the linear span of x_n, $x_{n+1}, \ldots$, the sequence (X_n) has the desired properties.

PROPOSITION 2. *The projective limit of a sequence of linear continuous maps of Banach spaces is reduced if and only if each map has dense range.*

Proof. We have the maps $T_n\colon F_{n+1} \to F_n$ and projective limit F with canonical projections $P_n\colon F \to F_n$, $n = 1, 2, \ldots$

Clearly if each P_n has dense range, then the relation $P_n = T_n P_{n+1}$ implies that $T_n(F_{n+1}) \supset T_n P_{n+1}(F) = P_n(F)$ so T_n has dense range.

Conversely, if each T_n has dense range, the lemma immediately implies that each P_n has dense range. ∎

Next we apply Proposition 2 to obtain a general method for constructing a Fréchet space so that it has a fundamental sequence of seminorms with preassigned canonical maps.

THEOREM. *Let $T_n\colon F_{n+1} \to F_n$, $n = 1, 2, \ldots$ be a sequence of linear continuous maps of Banach spaces which are 1-1 and have dense range. Then ∃ a unique (up to isomorphism) Fréchet space F which has a fundamental sequence of seminorms for which the sequence of canonical maps is equivalent to $(T_n)_n$. Moreover F admits a continuous norm.*

Proof. Given (T_n), let F be the projective limit. A fundamental sequence of seminorms for F is given by the sequence $(p_n)_n$, where

$$p_n(x) = \|P_n(x)\|, \quad x \in F$$

and $P_n\colon F \to F_n$ is the canonical map. Since T_n is 1-1 and $p_n(x) = \|T_n P_{n+1}(x)\|$ it follows, for a given k, that if $p_k(x) = 0$, then $P_n(x) = 0$ for all n so $x = 0$. Thus each p_n is a norm. In particular F admits a continuous norm. Also, F_{p_n} is just the normed space (F, p_n) and the canonical map $\hat{F}_{p_{n+1}} \to \hat{F}_{p_n}$ is the extension of the identity on F. Moreover, by definition, the map $P_n\colon F_{p_n} \to F_n$ is norm preserving so its extension $U_n\colon \hat{F}_p \to F_n$ is also norm preserving. By Proposition 2, P_n has dense range so U_n is an isometry of $\hat{F}_{p_n}$ onto F_n. Finally, the map $U_n^{-1} T_n U_{n+1}\colon \hat{F}_{p_{n+1}} \to \hat{F}_{p_n}$ equals the identity on $F_{p_{n+1}}$ so it is the canonical map and F has the desired properties.

Since any complete locally convex space is isomorphic to the projective limit of its canonical maps, the uniqueness of F follows from the easily checked fact that any two projective limits of two equivalent sequences of maps are isomorphic. ■

Remark 2. It would be interesting to know if this theorem has a converse. In [2], Proposition 1, a converse is proved for the case in which F is a Fréchet space whose topology is defined by a sequence of norms which come from inner products (the assumption of nuclearity in that result is obviously irrelevant), but we are unable to prove a more general result.

Remark 3. The theorem has an application which actually provided the motivation for these considerations. If $1 \leqslant p < q < \infty$ and $\xi \in l_q - l_p$, let λ be the Banach space of all sequences $x = (x_j)$ which can be represented $x = s + \xi t$, $s \in l_p$, $t \in c_0$ (coordinatewise arithmetic) with norm $\inf(\|s\| + \|t\|)$ where the inf is taken over all representations. Then $l_p \subset \lambda \subset l_q$ and it can be shown that with an appropriate choice of ξ, $\exists$ a continuous linear map $T\colon \lambda \to \lambda$ which factors $\lambda \to l_\infty \xrightarrow{\Delta} \lambda$ with Δ diagonal, but no power of T is absolutely summing. Along with the above theorem, this leads to an example of a λ-nuclear Fréchet space which is not nuclear. Details will appear subsequently in a joint paper with M. S. Ramanujan.

Finally we dualize Proposition 2 to obtain a new sufficient condition for an inductive limit to be Hausdorff. The dualization is obtained via the following well known result ([3], p. 139).

Duality theorem. *If $T_n\colon F_{n+1} \to F_n$ is a sequence of linear continuous maps of locally convex spaces whose projective limit is reduced, then the inductive limit of the maps $T_n'\colon F_n'[T_k(F_n)] \to F_{n+1}'[T_k(F_{n+1})]$ is Hausdorff. (Here T_k refers to the Mackey topology.)*

Proposition 3. *Let $S_n\colon E_n \to E_{n+1}$, $n = 1, 2, \ldots$ be a sequence of linear continuous maps of Banach spaces ∍ for each n, the second adjoint, $S_n''\colon E_n'' \to E_{n+1}''$ is 1-1. Then the inductive limit of (S_n) is Hausdorff.*

Proof. Let $F[\tau]$ be the projective limit of the maps of Banach spaces, $S_n'\colon E_{n+1}' \to E_n'$ with projections $P_n\colon F \to E_n'$. Now S_n'' is 1-1 if and only if $S_n'(E_{n+1}')$ is dense in E_n' so by Proposition 2 each $P_n(F)$ is dense in E_n'.

Now from the fact that S_n' is the adjoint of S_n it follows that each map $S_n'\colon E_{n+1}'[\tau_s(E_{n+1})] \to E_n'[\tau_s(E_n)]$ is continuous (here τ_s refers to the weak topology) and hence ([3], p. 158) each of the maps $S_n'\colon E_{n+1}'[\tau_k(E_{n+1})] \to E_n'[\tau_k(E_n)]$ is continuous.

Let $F[\tau_1]$ be the projective limit of this last sequence of maps. Notice that the vector space F and the canonical projections $P_n\colon F \to E_n'$ are the same as before. Since the Mackey topology $\tau_k(E_n)$ on E_n' is weaker than the norm topology, it follows that each $P_n(F)$ is dense in $E_n'[\tau_k(E_n)]$ — that is, the projective limit $F[\tau_1]$ is reduced.

Finally we apply the duality theorem to this last projective limit and conclude that the inductive limit of the maps $S_n\colon E_n[\tau_k'(E_n')] \to E_{n+1}[\tau_k(E_n')]$ is Hausdorff. Since the Mackey topology, $\tau_k(E_n')$, on E_n is exactly the norm topology, we are finished. ■

Remark 4. The best previous theorem on this question is that of De Wilde [1] who shows that if each S_n is 1-1 and maps the closed unit ball of E_n onto a $\tau_s(E_{n+1}')$-compact set, then the inductive limit is Hausdorff. This hypothesis and the hypothesis of Proposition 3 do not compare in either direction as is easily seen by considering, respectively, the two maps $S\colon c_0 \to c_0$, $S\colon l_1 \to l_1$ where each map is defined by the relation,

$$S((x_n)) = \left(\frac{1}{n^2} x_n\right).$$

It would be interesting to have simple necessary and sufficient conditions for the inductive limit of continuous linear maps of Banach spaces to be Hausdorff — even in the case when each map is 1-1.

References

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