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Projective and inductive limits of Banach spaces

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Abstract. We consider the projective and inductive limits of sequences of maps of Banach spaces. New results are given on when these are, respectively, reduced and Hausdorff. Applications are made to the question of the density of a countable intersection of dense subspaces and to the problem of constructing a Fréchet space with a given sequence of canonical maps. Also, an application to the theory of i-nuclear spaces is indicated.

Let \( T_n : F_n \rightarrow F_{n+1}, n = 1, 2, \ldots \) be a sequence of continuous linear maps of locally convex spaces. The projective limit of the sequence \((T_n)\) is the locally convex space \( F = F[\tau] \) where \( F \) is the vector space of all sequences \( x = (x_n) \in \prod F_n \) with the property that each \( x_n = T_n x_{n+1} \) and \( \tau \) is the induced product topology. We have the canonical projection maps \( P_n : F \rightarrow F_n \) defined by \( P_n x = x_n \). Obviously \( P_n = T_n P_{n+1} \) for all \( n \).

We say that the projective limit is reduced if each \( P_n(F) \) is dense in \( F_n \).

Let \( S_n : E_n \rightarrow E_{n+1}, n = 1, 2, \ldots \) be a sequence of continuous linear maps of locally convex spaces. Let \( Q_n : E_n \rightarrow \oplus E_n \) be the usual injection map which sends \( x \in E_n \) into a sequence whose \( n \)th coordinate is \( x \) and all other coordinates are 0. Let \( H \) be the vector subspace of \( \oplus E_n \) generated by the elements \( Q_n x - Q_{n-1} S_{n-1} \cdots S_2 x, k < n, x \in E_k \). Then the inductive limit of the sequence \((S_n)\) is the topological vector space \( E[\tau] = \oplus E_n / H \). It is Hausdorff (and locally convex) if and only if \( H \) is closed.

Let \( T_n : F_{n+1} \rightarrow F_n, R_n : G_{n+1} \rightarrow G_n, n = 1, 2, \ldots \) be two sequences of continuous linear maps of Banach spaces. We say that \((T_n)\), \((R_n)\) are equivalent if there exist isometries \( U_n : F_n \rightarrow G_n \) such that \( U_n^{-1} R_n U_{n+1} = T_n, n = 1, 2, \ldots \).

Let \( F \) be a Fréchet space and let \((p_n)\) be a fundamental sequence of seminorms defining the topology. Consider the seminormed space \((F, p_n)\) and let \( F_{p_n} \) be the normed space formed by taking the vector space \( F[\ker p_n] \) with the norm induced by \( p_n \). Let \( \tilde{F}_{p_n} \) be the completion of \( F_{p_n} \). There is no loss of generality if we assume that \( p_n(x) = p_{n+1}(x) \)
for all $n$ and $x \in F$. In this case the identity map on $F$ induces a continuous linear map $F_{n+1} \to F_n$ and its extension to the completions, $\hat{F}_{n+1} \to \hat{F}_n$, is called the $n$th canonical map of the sequence $(p_n)$. If $E$ is a Banach space we shall sometimes consider more than one topology on $E$. In this case, for simplicity of notation, we adopt the convention that if no topology is mentioned then the norm topology is understood. Thus, $E_0, E'$ will stand for the first and second Banach space duals of $E$.

Lemma. Let $T_n: F_n \to F_{n+1}$, $n = 1, 2, \ldots$ be a sequence of linear continuous maps of Banach spaces with dense range. Then for each $k$, the set $(T_k \circ \cdots \circ T_1)(E_k)$ with $E_k = T_0, T_1, \ldots, T_{k-1}$ is dense in $F_k$.

Proof. Choose $k, \varepsilon > 0$ and $x \in F_k$. There is no loss of generality if we assume that $\|T_n\| \leq 1$ for all $n$. We construct two sequences $(x_j^n)$, $(a_j^n)$, $j = 1, 2, \ldots$ as follows. Take $x_1^n = T_1(a_1^n)$ with $\|a_1^n - x_1^n\| \leq \varepsilon/2$. Then for each $j > 1$ take $x_j^{n+1} = T_j(a_j^n)$ with $\|x_j^n - x_j^{n+1}\| \leq \varepsilon/2$. It follows that for each $n \geq k$, the sequence $(a_j^n)$ is Cauchy in $E_n$ and hence convergent to $x_k^\infty E_k$. By continuity we have $x_k = T_k x_k^\infty$ for all $n$, and $\|x^n - a^n\| = \|x^n - x^n + x^n - a^n\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. We define $x_k^\infty = T_k x_k^\infty \cdots T_1 x_1^n$ for $k < n$ and it follows that $(x_k^\infty)$ is the desired sequence.

Using this lemma, we now obtain two results; one on the density of an intersection of dense subspaces and the other characterizing reduced projective limits of sequences of Banach spaces.

Proposition 1. Let $X$ be a Banach space and $(X_i)_i$ a decreasing sequence of subspaces of $X$; then a linear continuous map $A_i: F_i \to X$ of Banach spaces with $A_i(F_i) = X_i$ and $A_i((X_{i+1}))$ dense in $F_i$. Then $\bigcap X_i$ is dense in the closure of $X_1$.

Proof. First suppose that $A_i$ is not 1-1. Then we can replace it by $A_i: F_i/\ker A_i \to X$. Clearly $A_i(F_i/\ker A_i) = A_i(F_i)$ and if $H_i: F_i \to F_i/\ker A_i$ is the quotient map, then $A_i((X_{i+1})) = H_i A_i((X_{i+1}))$ which is dense because $H_i$ is onto. Thus we may assume that $A_i$ is 1-1.

Now we define $T_i: F_i \to X_i$ by $T_i(y) = A_i^{-1} A_i (y)$. This map is defined because $A_i$ is a continuous linear and its continuity follows from the closed graph theorem. Moreover, $A_i((X_{i+1})) = A_i^{-1} A_i((X_{i+1})) = A_i((X_{i+1}))$ to $X_i$ has dense range. If we define $T_i: F_i \to X_i$ by $T_i(y) = A_i(y)$, then $T_i$ also has dense range. Applying the lemma with $k = 0$ we conclude that $\bigcap T_i F_i = F_i$ is dense in $X_i$.

But this intersection is clearly equal to $\bigcap A_i((X_{i+1})) = \bigcap X_i$.

Remark. Proposition 1 is a slight generalization of an unpublished result of W. Wojtyński who assumed that $X$ was a Hilbert space, and each $A_i$ was a fixed map, $A_i: X \to X$ where $A_i$ was 1-1 and had dense range. The converse of our result is clearly false for we can take each $X_i$ to be a finite dimensional subspace which is not the image of any Banach space.

On the other hand some hypotheses are necessary for it is true that in every infinite dimensional separable Banach space there is a decreasing sequence of dense subspaces whose intersection is 0. Indeed we need only take a dense sequence $(x_n)$ and perturb it slightly so that it remains dense and becomes linearly independent. Then if $x_n$ is the linear span of $(x_n, x_{n+1}, \ldots)$, the sequence $(x_n)$ has the desired properties.

Proposition 2. The projective limit of a sequence of linear continuous maps of Banach spaces is reduced if and only if each map has dense range.

Proof. We have the maps $T_n: F_{n+1} \to F_n$ and projective limit $F$ with canonical projections $P_n: F \to F_n$, $n = 1, 2, \ldots$ Clearly each $P_n$ has dense range, then the relation $P_n = T_n P_{n+1}$ implies that $T_n(F_{n+1}) = P_n F_{n+1}$ for $F_n$ to have dense range. Conversely, if each $T_n$ has dense range, the lemma immediately implies that each $P_n$ has dense range.

Next we apply Proposition 2 to obtain a general method for constructing a Fréchet space so that it has a fundamental sequence of seminorms with preassigned canonical maps.

Theorem. Let $T_i: F_{i+1} \to F_i$, $i = 1, 2, \ldots$ be a sequence of linear continuous maps of Banach spaces which are 1-1 and have dense range. Then there is a unique (up to isomorphism) Fréchet space $F$ which has a fundamental sequence of seminorms for which the sequence $T_i(F_{i+1})$ is equivalent to $(T_i)_i$. Moreover $F$ admits a continuous norm.

Proof. Given $(T_i)$, let $F$ be the projective limit. A fundamental sequence of seminorms for $F$ is given by the sequence $(p_n)_n$, where $p_n(a) = \|T_n(a)\|, x \in F$ and $F_0 = F_n$ is the canonical map. Since $T_i$ is 1-1 and $p_n(a) = \|T_n a\|$ follows, for a given $k$, if $p_k(a) = 0$, then $p_n(a) = 0$ for all $n$ so $x = 0$. Thus each $p_n$ is a norm. In particular $F$ admits a continuous norm. Also, $F_0$ is just the normed space $(F, p_0)$ and the canonical map $F_0 \to F_n$ is the extension of the identity on $F_n$. Moreover, by definition, the norm $F_0, F_n = F_n$ is norm preserving its extension $U_n: F_0 \to F_n$ is also norm preserving. By Proposition 2, $F_0$ has dense range so $U_n$ is an isometry of $F_0$ onto $F_n$, Finally, the map $U_n U_{n+1}: F_{n+1} \to F_n$ equals the identity on $F_{n+1}$ so it is the canonical map and $F$ has the desired properties.

Since any complete locally convex space is isomorphic to the projective limit of its canonical maps, the uniqueness of $F$ follows from the
easily checked fact that any two projective limits of two equivalent sequences of maps are isomorphic. ■

Remark 2. It would be interesting to know if this theorem has a converse. In [2], Proposition 1, a converse is proved for the case in which $F$ is a Fréchet space whose topology is defined by a sequence of norms which come from inner products (the assumption of nuclearity in that result is obviously irrelevant), but we are unable to prove a more general result.

Remark 3. The theorem has an application which actually provided the motivation for these considerations. If $1 \leq p < q < \infty$ and $\xi \tau_k - \lambda$, let $\lambda$ be the Banach space of all sequences $a = (a_n)$ which can be represented $a = \xi + \theta / \tau_k$, $\theta \in \mathbb{C}$ (coordinatewise arithmetic) with norm inf $||\xi + \theta||$ where the inf is taken over all representations. Then $\lambda = \mathbb{C}$ and it can be shown that with an appropriate choice of $\xi$, $\lambda$ a continuous linear map $T: \lambda \rightarrow \lambda$ which factors $\lambda \rightarrow \lambda \rightarrow \lambda$ with $\lambda$ diagonal, but no power of $T$ is absolutely summing. Almost with the above theorem, this leads to an example of a $\lambda$-nuclear Fréchet space which is not nuclear. Details will appear subsequently in a joint paper with M. S. Ramanujan.

Finally we dualize Proposition 2 to obtain a new sufficient condition for an inductive limit to be Hausdorff. The dualization is obtained via the following well known result ([3], p. 139).

DUALITY THEOREM. If $T_n: E_{n+1} \rightarrow E_n$ is a sequence of linear continuous maps of locally convex spaces whose projective limit is reduced, then the inductive limit of the maps $T_n: E_n \rightarrow E_{n+1}$ is isomorphic to the inductive limit of the sequences $E_n$, where $E_{n+1}$ is the norm closure of $E_n$ in $E_{n+1}$.

PROPOSITION 3. Let $S_n: E_n \rightarrow E_{n+1}$, $n = 1, 2, \ldots$ be a sequence of linear continuous maps of Banach spaces $\tau$ for each $n$, the second adjoint, $S_n^\tau: E_n^\tau \rightarrow E_{n+1}^\tau$ is 1-1. Then the inductive limit of the sequences $S_n^\tau$ is 1-1 if and only if $S_n^\tau(E_{n+1})$ is dense in $E_n^\tau$ so by Proposition 2 each $P_n(F)$ is dense in $E_n^\tau$.

Now from the fact that $S_n$ is the adjoint of $S_n$, it follows that each map $S_n^\tau: E_{n+1}^\tau \rightarrow E_n^\tau$ is continuous when $\tau$ refers to the weak topology and hence ([3], p. 158) each of the maps $S_n^\tau: E_{n+1}^\tau \rightarrow E_n^\tau$ is continuous.

Let $F[\tau]$ be the projective limit of this last sequence of maps. Notice that the vector space $F$ and the canonical projections $P_n: F \rightarrow E_n$ are the same as before. Since the Mackey topology $\tau(E_n)$ on $E_n$ is weaker than the norm topology, it follows that each $P_n(F)$ is dense in $E_n^\tau$ — that is, the projective limit $F[\tau]$ is reduced.

Finally we apply the duality theorem to this last projective limit and conclude that the inductive limit of the maps $S_n: E_n[\tau(E_n)] \rightarrow E_{n+1}[\tau(E_{n+1})]$ is Hausdorff. Since the Mackey topology, $\tau(E_n)$, on $E_n$ is exactly the norm topology, we are finished. ■

Remark 4. The best previous theorem on this question is that of De Wilde [1] who shows that if each $S_n$ is 1-1 and maps the closed unit ball of $E_n$ onto a $\tau(E_{n+1})$-compact set, then the inductive limit is Hausdorff. This hypothesis and the hypothesis of Proposition 3 do not compare in either direction as it is easily seen by considering, respectively, the two maps $S_1: E_1 \rightarrow E_1$, $S_2: E_2 \rightarrow E_2$ where each map is defined by the relation,

$$S_1(E_n) = \left\{ \frac{1}{n+1} e_n \right\}.$$  

It would be interesting to have simple necessary and sufficient conditions for the inductive limit of continuous linear maps of Banach spaces to be Hausdorff — even in the case when each map is 1-1.

References


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