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Fractional powers of closed operators

by

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Abstract. In case of closed operators A in a Banach space such that the resolvent $R(\lambda; A)$ exists for $\lambda > 0$ and satisfies $\|\lambda R(\lambda; A)\| < M_0 (\lambda > 0)$, the fractional power $(-A)^\alpha$ $(0 < \alpha < 1)$ is defined as the strong limit of the family of integral operators $[\Gamma(\alpha)\Gamma(1-\alpha)]^{-1}\int_0^N \lambda^{\alpha-1}[I-\lambda R(\lambda; A)]d\lambda$ for $N \to \infty$. This definition enables one to study the fundamental properties of powers such as $(-A)^{\alpha+\beta} = (-A)^\alpha(-A)^\beta$, $((-A)^\alpha)^\beta = (-A)^{\alpha\beta}$. The methods are natural and elementary and depend on several characteristic identities involving Stieltjes transforms.

Introduction. The most general class of operators in a Banach space for which fractional powers were constructed is essentially that which we will call the class $\mathscr{K}$. It consists of those operators A which are defined and closed on a subset of a Banach space X with range also in X, and whose resolvent $R(\lambda; A)$ exists for each $\lambda > 0$ and satisfies the following condition of uniform boundedness with respect to $\lambda > 0$:

$$\|\lambda R(\lambda; A)\| < M_0 \quad (\lambda > 0).$$

For a rather wide subclass of $\mathscr{K}$, namely the negatives of infinitesimal generators of uniformly bounded strongly continuous semigroups of operators, E. Hille [4], R. S. Phillips [12] and A. V. Balakrishnan [1] developed an operational calculus of fractional powers in the framework of semigroup theory. Many other authors also worked in this field, as for instance E. Nelson [10] who studied fractional powers as a particular case of a more general functional calculus, furthermore K. Yosida [17], P. L. Butzer, H. Berens and U. Westphal [3], U. Westphal ([14], [15]) and others. In all of these investigations concerning fractional powers of semigroup generators Laplace transform methods play a more or less important role. Fractional powers for other subsets of $\mathscr{K}$ were examined for instance by M. A. Krasnoselskii and P. E. Sobolevskii [9], T. Kato [7] and J. Watanabe [13].

The study of fractional powers of operators of class $\mathscr{K}$ in general was started about 1960 by A. V. Balakrishnan [2]. For $A \in \mathscr{K}$ and $0 < \alpha < 1$ he defined the fractional power $(-A)^\alpha_B$ (B stands for Balakrishnan) as smallest closed extension of the operator J^α, given on the domain of A by

$$(0.1)\qquad J^\alpha f = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty \lambda^{\alpha-1}[I-\lambda R(\lambda; A)]f\,d\lambda.$$

In 1966 H. Komatsu [8] published the first of a series of papers about fractional powers of operators. He intended to summarize and generalize the results already known in that field from a unified point of view. His definition of fractional powers, just as Balakrishnan's, is a "process of closed extension", i. e. the fractional power in question is first defined by an integral representation on a certain subset and is then shown to have an extension which is a closed linear operator. In contrast to Balakrishnan's work, Komatsu's investigations depend upon the theory of intermediate spaces which may be the reason that his calculus is somewhat intricate. For the equivalence of various definitions of fractional powers see e. g. V. Nollau [11].

In the present paper we want to define and study fractional powers of operators of class $\mathscr{K}$ from a different point of view. It may roughly be characterized by the slogan: "limit process and transformation method". For fractional powers in connection with semigroup operators this program was carried out by U. Westphal [15] and turned out to be an elementary way in dealing with the problem. In [15] the power $(-A)^\alpha$, $0<\alpha<1$, A being the infinitesimal generator of a uniformly bounded semigroup $\{T(t); t\geqslant 0\}$ of class $(\mathscr{C}_0)$ in a Banach space X, is defined by

$$(0.2)\qquad (-A)^\alpha f = \mathrm{s}\text{-}\lim_{\varepsilon\to 0+}\frac{1}{K_\alpha}\int_\varepsilon^\infty u^{-1-\alpha}[I-T(u)]f\,du,$$

K_α a constant, whenever this limit exists in the norm of the space X. As mentioned above, the transformation which is the most important tool of this calculus is the Laplace transform.

In analogy with (0.2) and in contrast to Balakrishnan's and Komatsu's "method of closed extension" the purpose of this paper is to define $(-A)^\alpha$, $0<\alpha<1$, $A\in\mathscr{K}$, by

$$(0.3)\qquad (-A)^\alpha f = \mathrm{s}\text{-}\lim_{N\to\infty}\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^N \lambda^{\alpha-1}[I-\lambda R(\lambda; A)]f\,d\lambda$$

whenever this limit exists (the integrand having a singularity at infinity). Instead of the Laplace transform the Stieltjes transform will turn out to be the appropriate one in our present investigations. Therefore the first section of this paper deals with the Stieltjes transform, in particular with a convolution theorem which does not seem to be treated in the literature. It is mainly used for proving some identities which are characteristic in our calculus, for instance the identity (2.11) of Section 2. This identity gives a certain correspondence between the "differential quotient" $\int_0^N \lambda^{\alpha-1}[I-\lambda R(\lambda; A)]f\,d\lambda$ and the integral $\int_0^\infty p_\alpha(u)R(u; A)f\,du$ which is an element of $\mathsf{D}((-A)^\alpha)$. By means of (2.11), the fractional power defined by (0.3) is shown to be closed. Section 3 is concerned with a type of inversion formulae for $(-A)^\alpha$, the study of which seems to be new in principal (compare [9]). Section 4 is devoted to the properties of additivity and multiplicativity of fractional powers.

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1. The Stieltjes transform. Let the function f be defined on the interval $0<x<\infty$ and let f belong to $\mathsf{L}(0,r)$ for every positive r. Its Stieltjes transform (cf. D. V. Widder [16]) is given by

$$(1.1)\qquad \mathfrak{S}[f](s) = \int_0^\infty \frac{f(x)}{s+x}\,dx$$

whenever this integral exists, s being any positive real number (for the purpose of this paper it will be sufficient to restrict oneself to these values of s). If (1.1) converges for a point $s=s_0$, $s_0>0$, then it converges for every $s>0$. The uniqueness theorem reads: If $\mathfrak{S}[f](s)=0$ for $s\geqslant s_0$, then $f(x)=0$ almost everywhere on $(0,\infty)$. Moreover the Stieltjes transform may be regarded as an iteration of the Laplace transform. Indeed, denoting the Laplace transform of the function f by

$$(1.2)\qquad \mathfrak{L}[f](t) = f^\wedge(t) = \int_0^\infty e^{-tx}f(x)\,dx,$$

one has: If $\mathfrak{S}[f](s)$ converges for $s>0$, then $\mathfrak{L}[f](t)$ and $\mathfrak{L}[\mathfrak{L}(f)](s) = \mathfrak{L}[f^\wedge](s)$ exist for $t>0$ and $s>0$, respectively, the converse being also true under an appropriate additional assumption, as for instance $\mathfrak{L}[f](t)$ shall converge also for $t=0$ or $\mathfrak{L}[f](t)$ shall converge absolutely for $t>0$. In these cases one then has

$$(1.3)\qquad \mathfrak{S}[f](s) = \mathfrak{L}[\mathfrak{L}(f)](s)\qquad (s>0).$$

In the theory of Laplace transforms the concept of "convolution" plays an important role. But the results obtained there don't have their exact analogs for the Stieltjes transform. In case of Laplace transform one has: If $f, g \in \mathsf{L}(0, r)$ for every $r > 0$ and $f^\wedge(t)$, $g^\wedge(t)$ converge absolutely for $t \geqslant t_0$, then the convolution of f and g defined by

$$[f*g](x) = \int_0^x f(x-u)g(u)\,du$$

belongs to $\mathsf{L}(0, r)$ for every $r > 0$ and its Laplace transform converges absolutely for $t \geqslant t_0$, satisfying

$$[f*g]^\wedge(t) = f^\wedge(t)g^\wedge(t). \tag{1.4}$$

In particular, if $f, g \in \mathsf{L}(0, \infty)$, then its convolution also belongs to $\mathsf{L}(0, \infty)$, fulfilling

$$\|f*g\|_{\mathsf{L}} \leqslant \|f\|_{\mathsf{L}} \cdot \|g\|_{\mathsf{L}}$$

and (1.4) holds for $t \geqslant 0$.

For the Stieltjes transform, however, one has the following result due to I. I. Hirschman and D. V. Widder [5]: If $\alpha_1(x)$, $\alpha_2(x)$ are monotonely increasing and

$$A(s) = \int_0^\infty \frac{1}{s+x}\,d\alpha_1(x), \qquad B(s) = \int_0^\infty \frac{1}{s+x}\,d\alpha_2(x)$$

converge for $s > 0$, then there exists a monotonely increasing function $\gamma(x)$ such that

$$A(s^{\lambda_1})B(s^{\lambda_2}) = \int_0^\infty \frac{1}{s+x}\,d\gamma(x) \qquad (s > 0),$$

where $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 \leqslant 1$. This last condition is essential. It means that the case $\lambda_1 = \lambda_2 = 1$, which would give the desired convolution theorem, is not valid.

But even if $\alpha_i(x)$ is absolutely continuous with $x^{-1}\alpha_i'(x) \in \mathsf{L}(0, \infty)$ $(i = 1, 2)$, there may not exist a function $\gamma(x)$ having the same properties as $\alpha_i(x)$ and satisfying

$$\mathfrak{S}[\alpha_1'](s)\mathfrak{S}[\alpha_2'](s) = \mathfrak{S}[\gamma'](s).$$

Indeed, consider the following example kindly communicated by D. V. Widder to P. L. Butzer: Let $\varphi(x) = (2-x)^{-3/4}$ for $1 \leqslant x < 2$ and $\varphi(x) = 0$ elsewhere; then $x^{-1}\varphi(x) \in \mathsf{L}(0, \infty)$ and

$$\mathfrak{S}[\varphi(x)](s) = (s+2)^{-3/4}\left[\log\frac{(s+2)^{1/4}+1}{(s+2)^{1/4}-1} + 2\arctan\frac{1}{(s+2)^{1/4}}\right] \qquad (s \geqslant 0).$$

Moreover, if

$$\mathfrak{S}[\varphi(x)](s)\cdot\mathfrak{S}[\varphi(x)](s) = \mathfrak{S}[\omega(x)](s),$$

then

$$\omega(x) = 2\varphi(x)\,\mathrm{PV}\int_0^\infty \frac{\varphi(y)}{y-x}\,dy$$

which can be calculated explicitly to

$$\omega(x) = \begin{cases} 0 & (0 \leqslant x < 1), \\ 2[\varphi(x)]^2\left[\log\dfrac{1+(2-x)^{1/4}}{1-(2-x)^{1/4}} + 2\arctan\dfrac{1}{(2-x)^{1/4}}\right] & (1 < x < 2), \\ 0 & (x \geqslant 2). \end{cases}$$

This implies that

$$\lim_{x\to 2-} \omega(x)\cdot\varphi^{-2}(x) = 2\pi \quad \text{or} \quad x^{-1}\omega(x) = O([\varphi(x)]^2) \quad (x \to 2-),$$

and therefore $x^{-1}\omega(x)$ cannot be of class $\mathsf{L}(0, \infty)$.

Nevertheless we have the following convolution theorem which is a generalization of a particular example treated by Balakrishnan [2].

THEOREM 1.1. *If two functions g_1 and g_2 have the following properties for every $s > 0$*

(1) $$(s+x)^{-1}g_i(x) \in \mathsf{L}(0, \infty) \qquad (i = 1, 2),$$

(2) $$(s+x)^{-1}g_i(x)\int_0^1 \frac{|u^{-1}g_j(xu^{-1}) - g_j(xu)|}{1-u}\,du \in \mathsf{L}(0, \infty) \qquad (i, j = 1, 2,\ i \neq j),$$

(3) $(s+xu)^{-1}[g_1(xu)g_2(x) + g_1(x)g_2(xu)] \in \mathsf{L}(0, \infty)$ *for almost all u in $(0, 1)$,*

then the function

$$g(x) = g_1(x)\int_0^1 \frac{u^{-1}g_2(xu^{-1}) - g_2(xu)}{1-u}\,du + g_2(x)\int_0^1 \frac{u^{-1}g_1(xu^{-1}) - g_1(xu)}{1-u}\,du$$

exists for almost all x in $(0, \infty)$ and $(s+x)^{-1}g(x) \in \mathsf{L}(0, \infty)$ for every $s > 0$. Furthermore

$$\mathfrak{S}[g](s) = \mathfrak{S}[g_1](s)\cdot\mathfrak{S}[g_2](s) \qquad (s > 0).$$

$g(x)$ is called the convolution of $g_1(x)$ and $g_2(x)$, denoted by

$$g(x) = (g_1 \overset{*}{*} g_2)(x).$$

Proof. For $s > 0$ we have

$$\mathfrak{S}[g_1](s)\cdot\mathfrak{S}[g_2](s) = \int_0^\infty\int_0^\infty g_1(x)g_2(y)\frac{1}{(s+x)(s+y)}\,dx\,dy,$$

where the double integral is absolutely convergent on account of assumption (1) and can be rewritten as follows, the domain of integration being decomposed into two parts, one for which $x \geqslant y$, the other for which $x < y$:

$$\mathfrak{S}[g_1](s)\cdot\mathfrak{S}[g_2](s) = \int_0^1 du \int_0^\infty \frac{x}{(s+xu)(s+x)}[g_1(xu)g_2(x)+g_1(x)g_2(xu)]\,dx$$

$$= \int_0^1 \frac{du}{1-u}\int_0^\infty \left[\frac{1}{s+xu}-\frac{1}{s+x}\right][g_1(xu)g_2(x)+g_1(x)g_2(xu)]\,dx \qquad (s>0).$$

By means of property (3) we obtain

$$\int_0^\infty \frac{1}{s+xu}[g_1(xu)g_2(x)+g_1(x)g_2(xu)]\,dx$$

$$= \frac{1}{u}\int_0^\infty \frac{1}{s+x}\left[g_1(x)g_2\left(\frac{x}{u}\right)+g_1\left(\frac{x}{u}\right)g_2(x)\right]dx,$$

which gives for $s>0$

$$\mathfrak{S}[g_1](s)\cdot\mathfrak{S}[g_2](s) = \int_0^1 \frac{du}{1-u}\int_0^\infty \frac{1}{s+x}\left\{g_1(x)\left[\frac{1}{u}g_2\left(\frac{x}{u}\right)-g_2(xu)\right]+\right.$$

$$\left. + g_2(x)\left[\frac{1}{u}g_1\left(\frac{x}{u}\right)-g_1(xu)\right]\right\}dx.$$

As one may interchange the order of integration because of assumption (2) the theorem is proved.

Remark. Trivially the statement of the preceeding theorem is also true if assumptions (1)–(3) are fulfilled for $s=0$; we then even have $x^{-1}g(x)\in \mathsf{L}(0,\infty)$. Furthermore, in case $s=0$ we may replace the lower integral limit, namely 0, in assumption (2) by any $\varepsilon(0<\varepsilon<1)$ because of the estimate

$$\int_0^\infty \left|\frac{1}{x}g_i(x)\right|dx\int_0^\varepsilon \frac{|u^{-1}g_j(xu^{-1})-g_j(xu)|}{1-u}\,du \leqslant \frac{1+\varepsilon}{1-\varepsilon}\left\|\frac{1}{x}g_i(x)\right\|_{\mathsf{L}}\left\|\frac{1}{x}g_j(x)\right\|_{\mathsf{L}}.$$

2. Fractional powers. Let X be a complex Banach space with norm $\|\cdot\|$, and A a closed linear operator with domain $\mathsf{D}(A)$ dense in X and range in X. Moreover, suppose that the resolvent $R(\lambda; A)$ of A exists for each $\lambda>0$ and satisfies

(2.1) $$\|\lambda R(\lambda; A)\| < M_0 \qquad (\lambda>0),$$

M_0 being a constant independent of λ. The class of these operators A shall be denoted by $\mathscr{K}$. Let us mention first that for $A\in\mathscr{K}$ property (2.1), i. e. the uniform boundedness of the family of operators $\{\lambda R(\lambda; A); \lambda>0\}$, can be sharpened to an assertion of convergence for $\lambda\to\infty$, namely, if $A\in\mathscr{K}$, then

(2.2) $$s\text{-}\lim_{\lambda\to\infty}\lambda R(\lambda; A)f = f \qquad (f\in\mathsf{X});$$

this is obvious for $f\in\mathsf{D}(A)$ since $\lambda R(\lambda; A)f-f = R(\lambda; A)Af$, and follows for each $f\in\mathsf{X}$ by the theorem of Banach–Steinhaus.

DEFINITION 2.1. *Let $A\in\mathscr{K}$ and a be a real positive number with $0<a<1$. We define the fractional power $(-A)^a$ of $(-A)$ by*

(2.3) $$(-A)^a f = s\text{-}\lim_{N\to\infty} C_a^{-1}\int_0^N \lambda^{a-1}[I-\lambda R(\lambda; A)]f\,d\lambda,$$

whenever this limit exists, the constant C_a being given by $C_a = \Gamma(a)\Gamma(1-a) = \int_0^\infty \lambda^{a-1}(\lambda+1)^{-1}d\lambda$. The domain $\mathsf{D}((-A)^a)$ of $(-A)^a$ is the set of those elements $f\in\mathsf{X}$ for which the limit (2.3) exists.

One readily sees that $(-A)^a$ is a linear operator, furthermore the domain $\mathsf{D}((-A)^a)$ cannot be empty. Indeed, for $f\in\mathsf{D}(A)$ the integral

$$C_a^{-1}\int_0^\infty \lambda^{a-1}[I-\lambda R(\lambda; A)]f\,d\lambda$$

exists in the sense of Bochner and by definition is equal to $(-A)^a f$.

In analogy to [15] we now wish to verify the formula

(2.4) $$(-A)^a\left[\int_0^\infty p_a(\lambda/N)R(\lambda; A)f\,d\lambda\right] = \int_0^N \lambda^{a-1}[I-\lambda R(\lambda; A)]f\,d\lambda \qquad (N>0)$$

for each $f\in\mathsf{X}$, where $p_a(\lambda)$ is a numerically-valued function independent of the operator A. To calculate the function $p_a(\lambda)$ we will choose a particular A, namely that which is defined on the real number system by multiplication by a fixed negative number; i. e. let $\mathsf{X} = \mathsf{R}_1$ and

(2.5) $$A_0 f = -sf \qquad (s>0,\ \text{fixed};\ f\in\mathsf{R}_1).$$

Then the resolvent of A_0 is given by

$$R(\lambda; A_0)f = (\lambda+s)^{-1}f \qquad (\lambda\neq -s).$$

Obviously, A_0 is of class $\mathscr{K}$ with $M_0 = 1$ and

$$(-A_0)^a f = s^a f, \qquad \mathsf{D}((-A_0)^a) = \mathsf{R}_1.$$

Replacing A by A_0, in case $N=1$ the integral on the left-hand side of equation (2.4) (if it is valid) turns out to be the Stieltjes transform of p_a (multiplied by f). Thus we want to prove

LEMMA 2.2. *If* $0<a<1$, *then*

$$\int_0^1 \lambda^{a-1}(\lambda+s)^{-1}d\lambda = s^{a-1}\mathfrak{S}[p_a](s) \quad (s>0), \tag{2.6}$$

where $\mathfrak{S}[p_a](s)$ *is the Stieltjes transform of the fractional integral*

$$p_a(x) = [I^{1-a}h_a](x) = \frac{1}{\Gamma(1-a)}\int_0^x (x-u)^{-a}h_a(u)\,du \quad (x>0) \tag{2.7}$$

of order $(1-a)$ *of the function*

$$h_a(x) = \begin{cases} \dfrac{1}{\Gamma(a)}(1-x)^{a-1} & (0\leqslant x<1),\\ 0 & (x\geqslant 1).\end{cases}$$

Furthermore, $x^{-1}p_a(x)$ *belongs to* $\mathsf{L}(0,\infty)$ *and*

$$\lim_{s\to 0+}\mathfrak{S}[p_a](s) = \int_0^\infty p_a(x)x^{-1}dx = C_a. \tag{2.8}$$

Proof. Since $h_a\in\mathsf{L}(0,\infty)$, $p_a(x)=[I^{1-a}h_a](x)$ exists almost everywhere on $(0,\infty)$ and its Laplace transform converges absolutely for every $t>0$, satisfying

$$p_a^{\wedge}(t) = t^{a-1}h_a^{\wedge}(t) = t^{-1}g(t) \quad (t>0).$$

Here $g(t)=[I^a e^{-(\cdot)}](t)$, i. e. the fractional integral of the function e^{-t} of order a, the Laplace transform of which is given by

$$g^{\wedge}(\tau) = \tau^{-a}(\tau+1)^{-1} \quad (\tau>0).$$

The function $g^{\wedge}(\tau)$ is Lebesgue integrable on $(0,\infty)$ with respect to τ, and thus we have for every $s\geqslant 0$

$$\int_s^\infty \tau^{-a}(\tau+1)^{-1}d\tau = \int_s^\infty g^{\wedge}(\tau)\,d\tau = \mathfrak{L}[t^{-1}g(t)](s) = \mathfrak{L}[p_a^{\wedge}(t)](s). \tag{2.9}$$

Substituting $s/\tau=\lambda$ and $1/\tau=\lambda$, respectively, in the first integral of (2.9), we have

$$\int_s^\infty \tau^{-a}(\tau+1)^{-1}d\tau = s^{1-a}\int_0^1 \lambda^{a-1}(\lambda+s)^{-1}d\lambda = \int_0^{s^{-1}} \lambda^{a-1}(\lambda+1)^{-1}d\lambda \quad (s>0)$$

which tends to C_a for $s\to 0+$. The last term in (2.9) turns out to be the Stieltjes transform of the function p_a. Thus $s^{1-a}\int_0^1\lambda^{a-1}(\lambda+s)^{-1}d\lambda=\mathfrak{S}[p_a](s)$, and (2.6) is proven. As (2.9) is also valid for $s=0$ and $p_a(x)$ is non-negative on $(0,\infty)$, it is obvious that $x^{-1}p_a(x)\in\mathsf{L}(0,\infty)$, and (2.8) is satisfied.

Another representation of the function $p_a(x)$ easily follows by the substitution $u\to y=(x-u)(1-u)^{-1}$ on the right-hand side of (2.7), giving

$$p_a(x) = \begin{cases} C_a^{-1}\displaystyle\int_0^x y^{-a}(1-y)^{-1}dy & (0<x<1),\\ C_a^{-1}\displaystyle\int_x^\infty y^{-a}(y-1)^{-1}dy & (x>1).\end{cases} \tag{2.10}$$

From (2.6) it is clear that for $A=A_0$, defined in (2.5), the formula (2.4) is fulfilled. We now want to establish it for any operator A of class $\mathscr{K}$. For this purpose we need the fundamental identity given in Lemma 2.4; this is preceded by Lemma 2.3, generalizing the convolution Theorem 1.1.

LEMMA 2.3. *Let* $A\in\mathscr{K}$ *and let* g_1, g_2 *be two functions in* $\mathsf{L}(0,r)$ *for every* $r>0$ *such that the following conditions are satisfied for an element* $f\in\mathsf{X}$:

(1) $g_1(x)\|R(x;A)\|\int_0^\infty |g_2(u)|\|R(u;A)f\|\,du\in\mathsf{L}(0,\infty)$,

(2) $g_i(x)\|R(x;A)f\|\int_0^1 \dfrac{|u^{-1}g_j(xu^{-1})-g_j(xu)|}{1-u}\,du\in\mathsf{L}(0,\infty)$ $(i,j=1,2,\ i\neq j)$

(3) $[g_1(xu)g_2(x)+g_1(x)g_2(xu)]\|R(xu;A)f\|\in\mathsf{L}(0,\infty)$ *for almost all* u *in* $(0,1)$.

Then $(g_1 {}_*^* g_2)(x)$ *exists almost everywhere on* $(0,\infty)$ *and* $(g_1 {}_*^* g_2)(x)\|R(x;A)f\|\in\mathsf{L}(0,\infty)$. *Furthermore,*

$$\int_0^\infty (g_1 {}_*^* g_2)(x)R(x;A)f\,dx = \int_0^\infty g_1(x)R(x;A)\,dx\int_0^\infty g_2(u)R(u;A)f\,du.$$

Proof. In view of the first resolvent equation

$$xR(x;A)R(xu;A) = (1-u)^{-1}[R(xu;A)-R(x;A)]$$

the proof of Theorem 1.1 can be carried over to this lemma by replacing $(x+s)^{-1}$ by $R(x;A)$.

LEMMA 2.4. *For each* $f\in\mathsf{X}$, $0<a<1$, M, N *being real positive numbers there holds the identity*

$$\int_0^M \lambda^{a-1}[I-\lambda R(\lambda;A)]\,d\lambda\int_0^\infty p_a(u/N)R(u;A)f\,du = \int_0^N \lambda^{a-1}[I-\lambda R(\lambda;A)]\,d\lambda\int_0^\infty p_a(u/M)R(u;A)f\,du. \tag{2.11}$$

Proof. First of all we mention that the integral

$$\int_0^\infty p_a(u)R(u;A)f\,du$$

exists in the sense of Bochner for each $f\in X$ because of (2.1) and $u^{-1}p_a(u)\in L(0,\infty)$. Furthermore, the iterated integrals in (2.11), say on the left-hand side, define a family $\{J_{M,N}\}$ of bounded linear operators from X into itself with norm $\|J_{M,N}\|\leqslant M_0(M_0+1)a^{-1}M^aC_a$. Thus we have to show that

$$J_{M,N}f=J_{N,M}f\qquad (f\in X).$$

Introducing the auxiliary function $b_M(\lambda)=\lambda^a(0\leqslant\lambda\leqslant M)$, $b_M(\lambda)=0$ $(\lambda>M)$ for every positive M, $J_{M,N}$ can be rewritten in the form

$$J_{M,N}f=\{a^{-1}M^aI-\int_0^\infty b_M(\lambda)R(\lambda;A)\,d\lambda\}\int_0^\infty p_a(u/N)R(u;A)f\,du.$$

As proven in detail by Hövel [6], a rather lengthy calculation shows that the functions $b_M(u)$ and $p_a(u)$ satisfy the conditions of Lemma 2.3, such that

$$J_{M,N}f=\int_0^\infty P_{M,N}(u)R(u;A)f\,du, \tag{2.12}$$

where $\{P_{M,N}\}$ is the family of numerically-valued functions given by

$$P_{M,N}(u)=a^{-1}M^ap_a\left(\frac{u}{N}\right)-\left[b_M(\cdot)\mathop{*}_{*}p_a\left(\frac{\cdot}{N}\right)\right](u)$$

and $u^{-1}P_{M,N}(u)\in L(0,\infty)$. Of course from (2.12) we see that for $A=A_0$ the operator $J_{M,N}$ means multiplication by the Stieltjes transform of the function $P_{M,N}$, and

$$\mathfrak{S}[P_{M,N}](s)=\int_0^M\lambda^{a-1}\left[1-\frac{\lambda}{s+\lambda}\right]d\lambda\cdot\mathfrak{S}[p_a](s/N)$$

$$=s^{2-a}\int_0^M\frac{\lambda^{a-1}}{s+\lambda}\,d\lambda\int_0^N\frac{u^{a-1}}{s+u}\,du\qquad(s>0).$$

The last term of this equation shows that $\mathfrak{S}[P_{M,N}](s)$ is symmetric in M and N, i. e.

$$\mathfrak{S}[P_{M,N}](s)=\mathfrak{S}[P_{N,M}](s)\qquad(s>0),$$

implying $P_{M,N}=P_{N,M}$ by the uniqueness of the Stieltjes transform. This proves the identity.

PROPOSITION 2.5. *Let* $0<a<1$. *For each element* $f\in X$ *the integral*

$$\int_0^\infty p_a(\lambda)R(\lambda;A)f\,d\lambda$$

belongs to $D((-A)^a)$, *and for* $N>0$

$$(-A)^a\Big[\int_0^\infty p_a(\lambda/N)R(\lambda;A)f\,d\lambda\Big]=\int_0^N\lambda^{a-1}[I-\lambda R(\lambda;A)]f\,d\lambda. \tag{2.13}$$

Moreover, if $f\in D((-A)^a)$, then

$$\int_0^\infty p_a(\lambda/N)R(\lambda;A)(-A)^af\,d\lambda=\int_0^N\lambda^{a-1}[I-\lambda R(\lambda;A)]f\,d\lambda. \tag{2.14}$$

Proof. We first show that

$$\mathrm{s}\text{-}\lim_{M\to\infty}\int_0^\infty p_a(\lambda/M)R(\lambda;A)f\,d\lambda=C_af\qquad(f\in X). \tag{2.15}$$

Indeed, in view of (2.8), we have

$$C_af-\int_0^\infty p_a(\lambda/M)R(\lambda;A)f\,d\lambda=\int_0^\infty\lambda^{-1}p_a(\lambda/M)[I-\lambda R(\lambda;A)]f\,d\lambda,$$

where the integral on the right-hand side can be split into two parts, one from 0 to r, the other from r to infinity. For fixed r, the first integral in norm can be estimated by $(M_0+1)\|f\|\int_0^{r/M}|\lambda^{-1}p_a(\lambda)|\,d\lambda$ which tends to zero as $M\to\infty$. The second integral being estimated by

$$\sup_{r\leqslant\lambda<\infty}\|f-\lambda R(\lambda;A)f\|C_a$$

also converges to zero because of (2.2). Now using Lemma 2.4, equation (2.13) follows immediately. Indeed, as $M\to\infty$ the right-hand side of (2.11) converges to $C_a\int_0^N\lambda^{a-1}[I-\lambda R(\lambda;A)]f\,d\lambda$ for each $f\in X$, implying the existence of the limit on the left of (2.11). Thus by definition (2.1), $\int_0^\infty p_a(u/N)R(u;A)f\,du$ belongs to $D((-A)^a)$ for each $N>0$ and (2.13) is valid. To complete the proof of the proposition note that in (2.11) the order of integration may be interchanged giving (2.14) as $M\to\infty$ in case that $f\in D((-A)^a)$.

Having established this proposition, we are now able to prove that the operator $(-A)^a$ is closed.

THEOREM 2.6. *For* $0<a<1$ $(-A)^a$ *is a closed linear operator and its domain* $D((-A)^a)$ *is dense in* X.

Proof. Following upon Definition 2.1 we showed that $\mathsf{D}(A) \subset \mathsf{D}((-A)^\alpha)$, and therefore $\mathsf{D}((-A)^\alpha)$ is dense in X. To prove that $(-A)^\alpha$ is closed let $\{f_n\}_{n=1}^\infty$ be a sequence of elements in $\mathsf{D}((-A)^\alpha)$ converging to $f_0 \in \mathsf{X}$ with $(-A)^\alpha f_n$ converging to $h_0 \in \mathsf{X}$. Then by (2.14)

$$\int_0^N \lambda^{\alpha-1}[I-\lambda R(\lambda;A)]f_n d\lambda = \int_0^\infty p_\alpha(\lambda/N)R(\lambda;A)(-A)^\alpha f_n d\lambda,$$

and passing to the limit in norm as $n \to \infty$, we obtain

$$\int_0^N \lambda^{\alpha-1}[I-\lambda R(\lambda;A)]f_0 d\lambda = \int_0^\infty p_\alpha(\lambda/N)R(\lambda;A)h_0 d\lambda.$$

Letting $N \to \infty$ the right-hand side of the last equation tends to $C_\alpha h_0$ in norm, giving $f_0 \in \mathsf{D}((-A)^\alpha)$ and $(-A)^\alpha f_0 = h_0$. This proves that $(-A)^\alpha$ is closed.

Remark. Having shown some properties of the operator $(-A)^\alpha$ defined in (2.1) it is now obvious that it coincides with the fractional power $(-A)^\alpha_B$ of Balakrishnan [2]. Anyhow, this is clear for the restrictions of the operators on $\mathsf{D}(A)$, and therefore

$$(-A)^\alpha_B \lambda R(\lambda;A)f = (-A)^\alpha \lambda R(\lambda;A)f \quad (f \in \mathsf{X}),$$

giving as $\lambda \to \infty$ the desired result

$$(-A)^\alpha_B = (-A)^\alpha$$

because $(-A)^\alpha_B$ and $(-A)^\alpha$ are both closed.

3. Inversion formulae. We defined the fractional power $(-A)^\alpha$ as limit of a family of integral operators constructed by means of the difference $[I-\lambda R(\lambda;A)]$. Conversely, we now want to give a representation of just this difference in case $f \in \mathsf{D}((-A)^\alpha)$. This will lead to the inclusion

$$\mathsf{D}((-A)^\alpha) \subset \mathsf{D}((-A)^\beta) \quad (\beta < \alpha),$$

which will be needed for the proof of the additivity law for fractional powers.

LEMMA 3.1. *If* $f \in \mathsf{X}$, $\lambda > 0$, *then*

(3.1) $$[I-\lambda R(\lambda;A)]f = (-A)^\alpha \lambda^{-\alpha} \int_0^\infty p_\alpha(u/\lambda)R(u;A)[I-uR(u;A)]f du$$

and

(3.2) $$[I-\lambda R(\lambda;A)]f = (-A)^\alpha C_\alpha^{-1} \int_0^\infty u^{-\alpha} R(u;A)[I-\lambda R(\lambda;A)]f du.$$

Proof. Replacing the integration variable λ by u and N by λ in formula (2.13) and then differentiating it with respect to λ, (3.1) is easily obtained. Indeed, one deduces for $h > 0$ from (2.13)

$$(1/h)\int_\lambda^{\lambda+h} u^{\alpha-1}[I-uR(u;A)]f du$$
$$= (-A)^\alpha \int_0^\infty p_\alpha(u)R(u(\lambda+h);A)[I-\lambda uR(\lambda u;A)]f du,$$

which yields (3.1) as $h \to 0+$.

Concerning the second formula we can prove it by verifying the following identity by means of the Stieltjes convolution theorem, namely Lemma 2.3:

$$[I-\lambda R(\lambda;A)]\int_0^\infty p_\alpha(x/N)R(x;A)f dx$$
$$= \frac{1}{C_\alpha}\int_0^N x^{\alpha-1}[I-xR(x;A)]dx \int_0^\infty u^{-\alpha}R(u;A)[I-\lambda R(\lambda;A)]f du \quad (f \in \mathsf{X}).$$

This gives the desired representation in (3.2) by letting $N \to \infty$.

From both formulae we may conclude an approximation theorem for the family of resolvent operators $\{\lambda R(\lambda;A); \lambda > 0\}$ in case $f \in \mathsf{D}((-A)^\alpha)$.

PROPOSITION 3.2. *If* $f \in \mathsf{D}((-A)^\alpha)$, $0 < \alpha < 1$, *then*

$$\|[I-\lambda R(\lambda;A)]f\| = o(\lambda^{-\alpha}) \quad (\lambda \to \infty).$$

Proof. As $f \in \mathsf{D}((-A)^\alpha)$, the closed operator $(-A)^\alpha$ in (3.1) may be interchanged with the integral sign. After having multiplied the identity by λ^α we may continue in the same manner as in the proof of (2.15) of Proposition 2.5.

Form this result we obtain immediately

PROPOSITION 3.3. *If* $f \in \mathsf{D}((-A)^\alpha)$ *for some* α, $0 < \alpha < 1$, *then* $f \in \mathsf{D}((-A)^\beta)$ *for each* β, $0 < \beta < \alpha$, *and*

$$(-A)^\beta f = C_\beta^{-1}\int_0^\infty u^{\beta-1}[I-uR(u;A)]f\, du.$$

Proof. By Proposition 3.2 the integrand in norm of the latter integral is of order $O(u^{\beta-\alpha-1})$ for large u, proving that the integral exists and by definition is equal to $(-A)^\beta f$.

Let us look once more at formula (3.2). Multiplying it by λ and noting that $[I-\lambda R(\lambda;A)] = (-A)R(\lambda;A)$, and that, by definition of $(-A)^{1-\alpha}$,

$$C_\alpha^{-1}\int_0^\infty u^{-\alpha}(-A)R(u;A)\lambda R(\lambda;A)f du = (-A)^{1-\alpha}\lambda R(\lambda;A)f \quad (f \in \mathsf{X})$$

since $C_\alpha = C_{1-\alpha}$ and $R(\lambda; A)f \in \mathsf{D}(A)$ $(f \in \mathsf{X})$, we have that

$$(3.3)\qquad (-A)\lambda R(\lambda; A)f = (-A)^\alpha(-A)^{1-\alpha}\lambda R(\lambda; A)f \qquad (f\in\mathsf{X}).$$

For $\lambda \to \infty$ this yields a particular case of the power rule

$$(3.4)\qquad (-A) = (-A)^\alpha(-A)^{1-\alpha}.$$

Indeed, $f\in\mathsf{D}(A)$ implies $f \in \mathsf{D}\big((-A)^{1-\alpha}\big)$ and (3.3) can be written as

$$\lambda R(\lambda; A)(-A)f = (-A)^\alpha \lambda R(\lambda; A)(-A)^{1-\alpha}f,$$

giving $(-A)f = (-A)^\alpha(-A)^{1-\alpha}f$ for $\lambda\to\infty$ because $(-A)^\alpha$ is closed. Conversely, if $f\in\mathsf{D}\big((-A)^\alpha(-A)^{1-\alpha}\big)$, $R(\lambda; A)$ and $(-A)^\alpha(-A)^{1-\alpha}$ may be interchanged on the right of (3.3), and because $(-A)$ is closed it follows that for $\lambda\to\infty$ $(-A)f = (-A)^\alpha(-A)^{1-\alpha}f$.

Next we want to generalize (3.4) to

$$(3.5)\qquad (-A)^{\alpha+\beta} = (-A)^\alpha(-A)^\beta$$

if $0<\alpha, \beta$ and $\alpha+\beta<1$.

4. Additivity and multiplicativity. To prove the additivity of fractional powers, namely (3.5), we may verify the following identity

$$\int_0^N u^{\alpha-1}[I-uR(u; A)]\,du \int_0^M x^{\beta-1}[I-xR(x; A)]\,dx \int_0^\infty p_{\alpha+\beta}(y/P)R(y; A)\,dy$$
$$= \int_0^P y^{\alpha+\beta-1}[I-yR(y; A)]\,dy \int_0^\infty p_\beta(x/M)R(x; A)\,dx \int_0^\infty p_\alpha(u/N)R(u; A)f\,du$$

for positive real numbers M, N and P, and each $f\in\mathsf{X}$. Of course when carrying this out in the usual manner we have to apply the Stieltjes convolution theorem twice to each side of the identity because of three iterated integrals being there.

On the other hand we may proceed as in the proof of (3.4) using density and the fact that $(-A)^\alpha$ is closed.

LEMMA 4.1. *If $f\in\mathsf{D}(A^2)$, $0<\alpha, \beta$, $\alpha+\beta<1$, then*

$$(4.1)\qquad (-A)^{\alpha+\beta}f = (-A)^\alpha(-A)^\beta f.$$

Proof. $f\in\mathsf{D}(A^2)$ implies $(-A)^\beta f\in\mathsf{D}(A)$ and

$$(-A)^\alpha(-A)^\beta f = \frac{1}{C_\alpha C_\beta}\int_0^\infty u^{\alpha-1}[I-uR(u; A)]\,du \int_0^\infty v^{\beta-1}[I-vR(v; A)]f\,dv,$$

which shall be shown to be equal to

$$\frac{1}{C_{\alpha+\beta}}\int_0^\infty u^{\alpha+\beta-1}[I-uR(u; A)]f\,du,$$

representing $(-A)^{\alpha+\beta}f$ if $f\in\mathsf{D}(A)$ and therefore all the more if $f\in\mathsf{D}(A^2)$. But this follows by a direct calculation already carried out by Balakrishnan [2, p. 422].

Now for $\lambda>0$, $[\lambda R(\lambda; A)]^2$ is a family of operators from X in $\mathsf{D}(A^2)$ approximating the identity I for $\lambda\to\infty$. Therefore, applying Lemma 4.1, we have

$$(-A)^\alpha(-A)^\beta[\lambda R(\lambda; A)]^2 f = (-A)^{\alpha+\beta}[\lambda R(\lambda; A)]^2 f$$

for each $f\in\mathsf{X}$. Continuing as in the proof of (3.4), carried out in the foregoing section in detail, we obtain the desired power property by letting λ tend to infinity. Note that the inclusion $\mathsf{D}\big((-A)^{\alpha+\beta}\big)\subset\mathsf{D}\big((-A)^\beta\big)$, which is clear from Proposition 3.3, is essential in that argument. Thus we have

THEOREM 4.2. *Let $\alpha, \beta>0$ and $\alpha+\beta\leqslant 1$. An element $f\in\mathsf{X}$ belongs to $\mathsf{D}\big((-A)^\alpha(-A)^\beta\big)$ if and only if $f\in\mathsf{D}\big((-A)^{\alpha+\beta}\big)$. In this event*

$$(-A)^\alpha(-A)^\beta f = (-A)^{\alpha+\beta}f.$$

We now turn to the multiplicativity of fractional powers

$$\big((-A)^\alpha\big)^\beta = (-A)^{\alpha\beta}.$$

For this purpose we must show that the operator $B = -(-A)^\alpha$ is of class $\mathscr{K}$. To verify this we need a representation of the resolvent $R(\lambda; B)$ of B for $\lambda>0$.

First for positive numbers λ, u and $0<\alpha<1$ we define the function $g_\lambda(u)$ by

$$g_\lambda(u) = C_\alpha^{-1}u^\alpha[\lambda^2+2\lambda u^\alpha\cos\pi\alpha+u^{2\alpha}]^{-1}$$

which has the properties $g_\lambda(u)\geqslant 0$, $g_\lambda(u)\in\mathsf{L}(0,\infty)$ for $\lambda>0$ and $\int_0^\infty u^{-1}g_\lambda(u)\,du = \lambda^{-1}$. Moreover,

$$\mathfrak{S}[g_\lambda](s) = (\lambda+s^\alpha)^{-1} \qquad (s\geqslant 0, \lambda>0).$$

Next we introduce the operator R_λ by

$$(4.2)\qquad R_\lambda f = \int_0^\infty g_\lambda(u)R(u; A)f\,du \qquad (f\in\mathsf{X}; \lambda>0)$$

which is bounded and linear on X to itself and satisfies

$$(4.3)\qquad \|R_\lambda f\| < M_0\lambda^{-1}\|f\|.$$

Since $(-A)^\alpha$ is closed we have for $f\in\mathsf{D}\big((-A)^\alpha\big)$ $R_\lambda f\in\mathsf{D}\big((-A)^\alpha\big)$ and

$$(4.4)\qquad R_\lambda[\lambda I+(-A)^\alpha]f = [\lambda I+(-A)^\alpha]R_\lambda f \qquad \big(f\in\mathsf{D}\big((-A)^\alpha\big); \lambda>0\big).$$

Now A. V. Balakrishnan [2] established the relation

$$\Big\{\lambda I + C_\alpha^{-1}\int_0^\infty x^{\alpha-1}[I-xR(x; A)]\,dx\Big\}\int_0^\infty g_\lambda(u)R(u; A)f\,du = f \qquad (f\in\mathsf{D}(A); \lambda>0)$$

by a direct calculation, which in our terminology reads

$$[\lambda I+(-A)^{\alpha}]R_{\lambda}f = f \quad (f\in \mathsf{D}(A);\ \lambda>0). \tag{4.5}$$

This leads us to

PROPOSITION 4.3. *If $0<\alpha<1$, the operator $B = -(-A)^{\alpha}$ is of class $\mathscr{K}$ and*

$$R(\lambda; B) = R_{\lambda} \quad (\lambda>0). \tag{4.6}$$

Proof. For a given $f\in \mathsf{X}$ we choose a suitable sequence $\{f_n\}_{n=1}^{\infty}$ of elements in $\mathsf{D}(A)$ converging to f. Then (4.5) yields $[\lambda I-B]R_{\lambda}f_n = f_n$ for $\lambda>0$, and passing to the limit in norm for $n\to\infty$ we have

$$\operatorname*{s-lim}_{n\to\infty}[\lambda I-B]R_{\lambda}f_n = f.$$

Now, since $\operatorname*{s-lim}_{n\to\infty} R_{\lambda}f_n = R_{\lambda}f$ and $[\lambda I-B]$ is closed, we obtain $R_{\lambda}f\in\mathsf{D}(B)$ and

$$[\lambda I-B]R_{\lambda}f = f \quad (f\in\mathsf{X};\ \lambda>0),$$

implying that

$$R_{\lambda}[\lambda I-B]f = f \quad (f\in\mathsf{D}(B);\ \lambda>0)$$

because of (4.4). By these two equations it follows that the operator R_{λ} is the resolvent of B for $\lambda>0$, and thus B is obviously of class $\mathscr{K}$.

We remark that Kato [7] defined fractional powers by stipulating the operator R_{λ} given in (4.2) to be the resolvent of $(-A)^{\alpha}$, $0<\alpha<1$.

The lemma to follow will again deal with a fundamental identity from which we may conclude the rule of multiplicativity.

LEMMA 4.4. *In case $0<\alpha,\beta<1$ and $M,N>0$ we have for each $f\in\mathsf{X}$*

$$\int_0^N \lambda^{\beta-1}[I-\lambda R(\lambda;\ -(-A)^{\alpha})]d\lambda\int_0^{\infty} p_{\alpha\beta}(u/M)R(u;\ A)f\,du$$
$$= \int_0^M u^{\alpha\beta-1}[I-uR(u;A)]\,du\int_0^{\infty} p_{\beta}(\lambda/N)R(\lambda;\ -(-A)^{\alpha})f\,d\lambda. \tag{4.7}$$

Proof. Setting $h_1(u) = \int_0^N \lambda^{\beta}g_{\lambda}(u)\,d\lambda$, $h_2(u) = \int_0^{\infty} p_{\beta}(\lambda/N)g_{\lambda}(u)\,d\lambda$ and $b_M^{\alpha\beta}(\lambda) = \lambda^{\alpha\beta}\ (0\leqslant\lambda\leqslant M)$, $b_M^{\alpha\beta}(\lambda) = 0\ (\lambda>M)$, we obtain that the Stieltjes convolutions $H_1(u) = \left[h_1(\cdot)\overset{*}{*}p_{\alpha\beta}\left(\frac{\cdot}{M}\right)\right](u)$ and $H_2(u) = [h_2(\cdot)\overset{*}{*}b_M^{\alpha\beta}(\cdot)](u)$ exist almost everywhere in $(0,\infty)$ with $u^{-1}H_i(u)\in\mathsf{L}(0,\infty)$ $(i=1,2)$. A detailed proof of this result is given in Hövel [6]. By means of Proposition 4.3, Lemma 2.3 and (4.2) each side of (4.7), denoted by $I_1(f)$ and $I_2(f)$, respectively, may be rewritten in the form

$$I_i(f) = \int_0^{\infty} Q_i(u)R(u;A)f\,du \quad (i=1,2)$$

where

$$Q_1(u) = \beta^{-1}N^{\beta}p_{\alpha\beta}(u/M)-H_1(u), \quad Q_2(u) = (\alpha\beta)^{-1}M^{\alpha\beta}h_2(u)-H_2(u).$$

Moreover, replacing $R(u;A)f$ by $(u+s)^{-1}$, it is easily to be seen that

$$\mathfrak{S}[Q_1](s) = \mathfrak{S}[Q_2](s) \quad (s\geqslant 0),$$

and the uniqueness theorem of the Stieltjes transform gives $Q_1 = Q_2$. This proves the lemma.

THEOREM 4.5. *Let $0<\alpha,\beta<1$. $f\in\mathsf{X}$ is an element of the domain of $((-A)^{\alpha})^{\beta}$ if and only if it belongs to $\mathsf{D}((-A)^{\alpha\beta})$. In this event*

$$((-A)^{\alpha})^{\beta}f = (-A)^{\alpha\beta}f.$$

Proof. Passing to the limit in norm for $N\to\infty$ in (4.7) we have

$$((-A)^{\alpha})^{\beta}\int_0^{\infty} p_{\alpha\beta}(u/M)R(u;A)f\,du = \int_0^M u^{\alpha\beta-1}[I-uR(u;A)]f\,du \quad (f\in\mathsf{X})$$

yielding the desired statement for $M\to\infty$.

The restriction of our investigations to real exponents α in the intervall $0<\alpha<1$ is not essential. The method may be extended to any real $\alpha>1$. Indeed, for $n<\alpha<n+1$ $(n = 0,1,2,\ldots$ fixed) it is natural to define $(-A)^{\alpha}$ iteratively by $(-A)^{\alpha} = (-A)^{\alpha-n}(-A)^n$. Since $0<\alpha-n<1$, $(-A)^{\alpha-n}$ is defined by the method of our paper. Thus

$$(-A)^{\alpha}f = \operatorname*{s-lim}_{N\to\infty} C_{\alpha-n}^{-1}\int_0^N \lambda^{\alpha-n-1}[I-\lambda R(\lambda;A)](-A)^n f\,d\lambda$$

whenever this limit exists. Another method to define such fractional powers would be

$$(-A)^{\alpha}f = \operatorname*{s-lim}_{N\to\infty} C_{\alpha}^{-1}\int_0^N \lambda^{\alpha-1}[I-\lambda R(\lambda;A)]^{n+1}f\,d\lambda.$$

Whereas the former procedure is complete as it stands, the latter approach, considered ab initio, would need a parallel treatment along the lines of this paper. In any case, both definitions are equivalent by the standard closure arguments.

The method of this paper may of course be extended to complex values of α with $\operatorname{Re}\alpha>0$.

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141_4	$\exp[i(x_0, h) + \int\limits_U \big(e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|_\lambda^2} dF(x)\big)$	$\exp\Big[i(x_0, h) + \int\limits_U \Big(e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|_\lambda^2}\Big) dF(x)$
141_3	$\int\limits_{E_a-U} \Big(e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|_\Gamma^2}\Big) dF(x),$	$\int\limits_{E_u-U} \Big(e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|_\Gamma^2}\Big) dF(x)\Big],$