STUDIA MATHEMATICA, T. XLII. (1972)

Stable probability measures on Banach spaces

by

A. KUMAR and V. MANDREKAR* (East Lansing, Mich.)

Abstract. In this paper stable probability measures on a real separable Banach space are defined and several characterizations of these measures are established using a generalization of the convergence types theorem. These results are used to identify stable probability measures as limit laws of certain normed sums of independent identically distributed Banach space valued random variables. These limit laws possess a Lévy–Khinchine representation that can be characterized on certain Orlicz spaces in terms of the representing Lévy–Khinchine measure.

0. Introduction. In this paper, we consider stable probability measures (laws) on a real separable Banach space. Using a generalization of the convergence types theorem ([2], p. 174) we establish several characterization of stable probability measures and deduce as corollaries extensions to Banach space of known results on stable laws ([2], p. 199; [5], p. 64; [8], p. 327). These results allow us to identify stable probability measures on Banach space as the limit laws of certain normed sums of independent, identically distributed Banach space valued random variables. Finally, we characterize stable probability measures on certain Orlicz spaces in terms of their Levy–Khinchine representation given in ([7], p. 71).

In Section 1, following the preliminaries, the convergence type theorem is established. Section 2 presents the characterizations of stable laws and final section characterizes the Levy–Khinchine representation of the stable laws on Orlicz spaces.

The lemmas in Section 2 are suggested by some recent work of Jajte [5]. The proofs in [5] in the Hilbert case treated there are incomplete and the main theorem in ([5], p. 64) which is extended here to certain Orlicz spaces, contains a lacuna ([5], p. 70).

1. Preliminaries and notations. In this section we present notations needed in this paper. We shall denote by E a real separable Banach space with norm $\|\cdot\|$ and by R the space of real numbers with the usual topology. The elements of E will be denoted by $x, y, z, \ldots$ and of R by $a, b, c, \ldots$ etc. E^* will denote the (topological) dual of E. For a probability measure μ

* Supported in part by NSF grant GP 11626 and NSF grant GP 28658.

on $\mathfrak{B}(E)$, where $\mathfrak{B}(E)$ denotes the Borel subsets of E, the characteristic functional (ch. f.) of μ denoted by $\hat{\mu}$ is a function on E^* defined by $\hat{\mu}(y) = \int_E e^{i(x,y)}\mu(dx)$, where $(\cdot, y)\in E^*$. It is well known ([4], p. 37) that for a real separable Banach space, $\hat{\mu}(y)$ uniquely determines the measure μ on $\mathfrak{B}(E)$. For two probability measures μ and ν on $\mathfrak{B}(E)$, we shall denote by $\mu*\nu$ the convolution of μ and ν ([9], p. 56). For any probability measure μ on $\mathfrak{B}(E)$ and $a\in R$, $T_a\mu$ is defined to be the probability measure on $\mathfrak{B}(E)$ given by $T_a\mu(B) = \mu^{(a^{-1}B)}$ for every $B\in\mathfrak{B}(E)$ and for $a = 0$ we define $T_a\mu = \delta_0$, where for each $B\in\mathfrak{B}(E)$

$$\delta_x(B) = \begin{cases} 1 & \text{if } x\in B, \\ 0 & \text{otherwise.}\end{cases}$$

We shall call δ_x the probability measure degenerate at x. We need the following definitions.

1.1 Definition. (a) A sequence $\{\mu_n\}$ of probability measures on $\mathfrak{B}(E)$ is said to *converge weakly* to a probability measure μ on $\mathfrak{B}(E)$, if for every bounded continuous real valued function f on E, $\int_E f\,d\mu_n \to \int_E f\,d\mu$ and is denoted by $\mu_n \Rightarrow \mu$.

(b) A sequence $\{\mu_n\}$ of probability measures on $\mathfrak{B}(E)$ is said to be *compact* if every subsequence $\{\nu_m\}$ of $\{\mu_n\}$ has a weakly convergent subsequence.

The following theorem will be used repeatedly and is stated here for further reference.

1.2. Theorem. ([9], p. 58). *Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ be three sequences of probability measures on $\mathfrak{B}(E)$ such that $\lambda_n = \mu_n*\nu_n$ for each n. If the sequence $\{\lambda_n\}$ and $\{\mu_n\}$ are compact, then so is the sequence $\{\nu_n\}$.*

1.3. Lemma. *Let $\{\mu_n\}$ and μ be probability measures on $\mathfrak{B}(E)$ and $\{a_n\}$, $a\in R$. Then $\mu_n \Rightarrow \mu$ and $a_n \to a$ implies $T_{a_n}\mu_n \Rightarrow T_a\mu$.*

Proof of the lemma is immediate from [1], p. 34.

Before we prove the main theorem of this section, we need the following lemma.

1.4. Lemma. *Let $\hat{\mu}(\cdot)$ be the ch. f. of a probability measure on $\mathfrak{B}(E)$ such that for some $\delta > 0$, $|\hat{\mu}(y)| = 1$ whenever $\|y\|_{E^*} \leqslant \delta$. Then $\mu = \delta_x$ for some $x\in E$.*

Proof. Let $\Delta = \{y\in E^*\colon \|y\|_{E^*} \leqslant \delta\}$. Consider the random variable $(\cdot, y)$ defined on E for each fixed y in E^*. Then by ([21], p. 202, $(\cdot, y)$ is degenerate say at $\theta(y)$. Hence $\mu\hat{\ }(y) = e^{i}\theta(y)$.

Let $\Delta_y = \{x\colon (x, y) = \theta(y)\}$. Then Δ_y is closed and $\mu(\Delta_y) = 1$, for every y in E^*. Consequently, the support c_μ of μ (see [23], p. 27) is contained in $\bigcap_{y\in E^*}\Delta_y$.

Suppose there exist two points x_1 and x_2 in c_μ. Then $(x_1, y) = (x_2, y)$ for every $y\in E^*$. Hence $x_1 = x_2$. Thus the support of μ contains only one point. This completes the proof of the lemma.

1.5. Theorem (Convergence of types theorem) *Let $\{\mu_n\}$ and μ be probability measures on $\mathfrak{B}(E)$ such that $\mu_n \Rightarrow \mu$, and there exist positive constants a_n's and a sequence $\{x_n\}$ in E such that $T_{a_n}\mu_n*\delta_{x_n} \Rightarrow \mu'$, where μ and μ' are non-degenerate probability measures on $\mathfrak{B}(E)$. Then there exists an $a\in R$ and an $x\in E$ such that $\mu' = T_a\mu*\delta_x$, $a_n \to a$ and $\|x_n - x\| \to 0$ as $n\to\infty$.*

Proof. Suppose $\overline{\lim}\, a_n = \infty$. Then there exists a subsequence $\{m\}\subseteq\{n\}$ such that $a_m\to\infty$. Let $c_m = a_m^{-1}$. Then $\mu_m = \{T_{c_m}(T_{a_m}\mu_m*\delta_{x_m})\}*\delta_{-c_m x_m} \Rightarrow \mu$. Since $T_{a_m}\mu_m*\delta_{x_m} \Rightarrow \mu'$, therefore by Lemma 1.3 $T_{c_m}(T_{a_m}\mu_m*\delta_{x_m}) \Rightarrow \delta_0$. Hence by Theorem 1.2, $\{-c_m x_m\}$ being compact on E converges to some x_0 belonging to E by ([1], p. 37). Hence μ is degenerate, contradicting the hypothesis. Hence, $\overline{\lim}\, a_n < \infty$.

Suppose now $\{a_m\}$ and $\{a_l\}$ be two subsequences of $\{a_n\}$, such that $a_m\to a$, $a_l\to a'$, where $a\neq a'$. We note that neither a nor a' can be zero, since μ' is non-degenerate. Also, we have

$$\mu_m = \{T_{a_m^{-1}}(T_{a_m}\mu_m*\delta_{x_m})\}*\delta_{-x_m a_m^{-1}} \Rightarrow \mu$$

and

$$\mu_l = \{T_{a_m^{-1}}(T_{a_l}\mu_l*\delta_{x_l})\}*\delta_{-x_l a_l^{-1}} \Rightarrow \mu.$$

Now by Lemma 1.3 and the hypothesis we get

$$\mu = T_a\mu'*\delta_{x_1} = T_{a'}\mu'*\delta_{x_2},$$

where $x_1 = \lim\limits_{m\to\infty} -x_m a_m^{-1}$, and $x_2 = \lim\limits_{l\to\infty} -x_l a_l^{-1}$.

Therefore

$$|\hat{\mu}'(ay)| = |\hat{\mu}'(a'y)| \quad \text{for every} \quad y\in E^*. \tag{1.6}$$

Without loss of generality we can assume $b = \dfrac{a}{a'} < 1$. Hence, by iteration $|\hat{\mu}'(y)| = |\hat{\mu}'(by)| = \ldots = |\hat{\mu}'(b^n y)|$. Letting $n\to\infty$ we get $|\hat{\mu}'(y)| = 1$, for every $y\in E^*$. Hence by Lemma 1.4 μ' is degenerate, contradicting the hypothesis. This proves that $a_n\to a$ and $0 < a < \infty$.

Now it follows that $T_{a_n}\mu_n \Rightarrow T_a\mu$ by Lemma 1.3, and from hypothesis $T_{a_n}\mu_n*\delta_{x_n} \Rightarrow \mu'$. Therefore, by Theorem 1.2 $\{x_n\}$ is compact in E and hence by ([1], p. 37) x_n converges to some x in E. Hence, $T_{a_n}\mu_n*\delta_{x_n} \Rightarrow T_a\mu*\delta_x$. Thus $\mu' = T_a\mu*\delta_x$, which completes the proof.

2. Stable probability measure on a Banach space. In this section we define a stable probability measure on a real separable Banach space following Loève ([8], p. 326). (See also ([2], p. 199), ([5], p. 64).

2.1 Definition. Let μ be a probability measure on the Borel subsets $\mathfrak{B}(E)$ of a real separable Banach space E. We say that μ is a *stable probability measure* if for each positive real number a and b there exists a positive real number c and an $x \in E$, such that

$$T_a\mu * T_b\mu = T_c\mu * \delta_x. \tag{2.2}$$

Our main effort in this section will be to prove various characterizations of the stable probability measures which will be useful in studying stable probability measures as limit laws of the sums of independent random variables. For this we need the following lemmas.

2.3. Lemma. *If μ is a stable probability measure on $\mathfrak{B}(E)$, then there exists a sequence $\{a_n\}$ of positive numbers and a sequence $\{x_n\}$ of elements of E such that $T_{a_n}\mu^{*n} * \delta_{x_n} \Rightarrow \mu$.*

Proof. We shall prove the lemma by showing that for each n, there exist a_n and x_n such that $\mu = \delta_{x_n} * T_{a_n}\mu^{*n}$.

For $n = 1$, take $x_1 = 0$, and $a_1 = 1$. Suppose that we have $x_1, \ldots, x_{m-1}$ and $a_1, a_2, \ldots, a_{m-1}$ such that

$$\mu = \delta_{x_i} * T_{a_i}\mu^{*i} \quad \text{for} \quad i = 1, 2, \ldots, m-1.$$

Then $\mu^{*m-1} = T_{a_{m-1}^{-1}}(\mu * \delta_{-x_{m-1}})$. Hence,

$$\mu^{*m} = T_{a_{m-1}^{-1}}\,\mu * \delta_{-x_{m-1}, a_{m-1}^{-1}} * \mu.$$

Now we use the fact that μ is stable to conclude that

$$\mu^{*m} = T_c\mu * \delta_{x - a_{m-1}^{-1}, x_{m-1}} \qquad \text{for some } c > 0 \quad \text{and} \quad x \in E.$$

Consequently

$$\mu = T_{c^{-1}}\mu^{*m} * \delta_{c_m^{-1}(a_{m-1}^{-1}, x_{m-1} - x)}.$$

Define, $a_m = c^{-1}$, $x_m = c^{-1}(a_{m-1}^{-1}x_{m-1} - x)$. Thus we have shown by induction that $\mu = \delta_{x_m} * T_{a_m}\mu^{*m}$ for every m. This completes the proof of the lemma.

2.4. Lemma. *If for some sequence of positive real numbers $\{a_n\}$ and a sequence $\{x_n\}$ of elements of the space E, we have*

$$\nu = \lim_{n\to\infty}(\delta_{x_n} * T_{a_n}\mu^{*n}),$$

where ν is non-degenerate, then $a_n \to 0$, $a_n/a_{n+1} \to 1$ as $n \to \infty$.

Proof. Suppose $a_n \nrightarrow 0$. Then there exists a subsequence $\{a_m\}$ of $\{a_n\}$ such that $a_m^{-1} \to a < \infty$. Therefore by Lemma 1.3

$$\delta_{y_m} * \mu^{*m} = T_{a_m^{-1}}(\delta_{x_m} * T_{a_m}\mu^{*m}) \to T_a\nu,$$

where $y_m = x_m/a_m$. Hence, $[\hat{\mu}(h)]^m e^{i(y_m, h)} \to \hat{\nu}(ah)$ for every $h \in E^*$. Since $\hat{\nu}$ is continuous at the origin, therefore $|\hat{\nu}(ah)| > 0$ for those h with $\|h\| < \delta$

for some $\delta > 0$. Consequently,

$$\hat{\mu}(h)e^{i(y_m/m, h)} \to 1 \quad \text{on} \quad \|h\| < \delta.$$

Thus $|\hat{\mu}(h)| = 1$ on $\|h\| < \delta$ and hence by Lemma 1.4, μ is degenerate. Consequently, ν is degenerate which contradicts the assumption and hence $a_n \to 0$.

Suppose $a_n/a_{n+1} \nrightarrow 1$. Then there exists a subsequence $\{m\}$ of $\{n\}$ such that $a_m/a_{m+1} \to a$ where $a \neq 1$.

If $a = \infty$. Then $c_m = a_{m+1}/a_m \to 0$, and

$$\delta_{x_{m+1}} * T_{a_{m+1}}\mu^{*m} = \{T_{c_m}(\delta_{x_m} * T_{a_m}\mu^{*m})\} * \delta_{x_{m+1} - c_m x_m}.$$

$$\widehat{\delta_{x_{m+1}} * T_{a_{m+1}}\mu^{*m+1}}(y) = \frac{e^{i(x_{m+1}, h)}\big(\hat{\mu}(a_{n+1}h)\big)^{m+1}}{\hat{\mu}(a_{m+1}h)} \to \hat{\nu}(h) \tag{2.5}$$

because for every h, $\hat{\mu}(a_{m+1}h) \to 1$ and $e^{i(x_{m+1}, h)}\big(\hat{\mu}(a_{m+1}h)\big)^{m+1} \to \hat{\nu}(h)$. Since $T_{c_m}(\delta_{x_m} * T_{a_m}\mu^{*m}) \to \delta_0$ by Lemma 1.3, we conclude that $|\hat{\nu}(h)| = 1$ for every $h \in E$. Hence by Lemma 1.4, ν is degenerate which contradicts the assumption.

Now suppose $a < \infty$. Then $d_m = a_m/a_{m+1} \to a$, and

$$\delta_{x_m} * T_{a_m}\mu^{*m+1} = \{T_{d_m}(\delta_{x_{m+1}} * T_{a_{m+1}}\mu^{m+1})\} * \delta_{x_m - d_m x_{m+1}}.$$

$$\widehat{\delta_{x_m} * T_{a_m}\mu^{*m+1}}(y) = \frac{e^{i(x_m, h)} * [\hat{\mu}(a_m h)]^m}{\hat{\mu}(a_m h)} \to \hat{\nu}(h)$$

with reasoning similar to one following (2.5). But $T_{d_m}(\delta_{x_{m+1}} * T_{a_{m+1}}\mu^{*m+1}) \to T_a\nu$, hence $|\hat{\nu}(h)| = |\hat{\nu}(ah)|$ for every $h \in E^*$. Without loss of generality we can assume $a < 1$. Hence by the same argument as in (1.6) we conclude that ν is degenerate which contradicts the hypothesis. Hence $a_n/a_{n+1} \to 1$.

2.6. Lemma. *Let for every positive integer n, $x_n \in E$, $a_n \in R$, and*

$$\delta_{x_n} * T_{a_n}\mu^{*n} \Rightarrow \nu,$$

where μ and ν are probability measures on $\mathfrak{B}(E)$. Then there exists an $r > 0$ and a function z of two variables defined for every pair of non-negative numbers a and b with values in E, such that

$$\hat{\nu}(ah)\hat{\nu}(bh) = e^{i(z(a,b), h)}\hat{\nu}\big((a^r + b^r)^{1/r}h\big) \qquad \text{for every} \qquad h \in E^*.$$

Hence in particular, ν is stable.

Proof. If ν is degenerate, then there is nothing to prove. So assume ν is non-degenerate. Now by Lemma 2.4, for any arbitrary pair of positive numbers a and b, there exists subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ of $\{a_n\}$ such that

$$\omega_k = \frac{a_{n_k}}{a_{m_k}} \to \frac{b}{a} \qquad \text{(Loève [7], p. 323)}$$

Suppose $\overline{\lim}\, a_{n_k}/a_{n_k+m_k} = s = \infty$. Then the sequence $\{c_k\}$, where $c_k = a_{n_k+m_k}/a_{n_k}$, will have a subsequence $\{c_{k'}\}$ such that $c_{k'} \to 0$.

(2.7) $$\delta_{x_{n_{k'}+m_{k'}}} * T_{a_{n_{k'}+m_{k'}}} \mu^{*n_{k'}+m_{k'}} = \{T_{c_{k'}}(\delta_{x_{n_{k'}}} * T_{a_{n_{k'}}} \mu^{*n_{k'}})\} * \{T_{c_{k'}} \cdot \omega_{k'}(\delta_{x_{m_{k'}}} * T_{a_{m_{k'}}} \mu^{*m_{k'}})\} * \delta_{z_{k'}}$$

for suitable $z_{k'}$. By the hypothesis we conclude that $\delta_{a_{n'}k+m'_k} * T_{a_{n_{k'}+m_{k'}}} \mu^{*m_{k'}+n_{k'}} \to \nu$. The terms in the parenthesis of (2.7) converge weakly to δ_0. Hence, $|\hat{\nu}(y)| = 1$ for every $y \in E^*$. Thus by Lemma 1.4, ν is degenerate, which is a contradiction. Hence $s < \infty$.

Suppose there exists two subsequences $\{c_{k'}^{-1}\}$, $\{c_{k''}^{-1}\}$ of $\{c_k^{-1}\}$ converging to b' and b'' respectively where $b' \neq b''$. Making use of the following equality

(2.8) $$[T_a(\delta_{x_{n_k}} * T_{a_{n_k}} \mu^{*n_k})] * [T_a \omega_k (\delta_{x_{m_k}} * T_{a_{m_k}} \mu^{*m_k})] = T_{a a_{n_k}} \mu^{*n_k+m_k} * \delta_{z_k} = [T_{a c_k^{-1}}(\delta_{x_{n_k+m_k}} * T_{a_{n_k+m_k}} \mu^{*n_k+m_k})] * \delta_{z'_k}$$

for suitable z_k and $z_{k'}$ we conclude that

$$|\hat{\nu}(ah)| \cdot |\hat{\nu}(bh)| = |\hat{\nu}(ab'h)| = |\hat{\nu}(ab''h)| \quad \text{for every} \quad h \in E^*.$$

Since $b' \neq b''$ and both are finite, we can conclude by the same reasoning as in (1.6), that ν is degenerate. This contradicts the assumption. Hence, $s = \lim a_{n_k}/a_{n_k+m_k}$.

Now we make use of equation (2.8) and Theorem 1.2 to conclude that there exists a function $z(a, b)$ which is the limit of $z_{k'} = ax_{n_k} + a_{n_k}/a_{m_k} x_{m_k} - a(a_{n_k}/a_{n_k+m_k}) \cdot x_{m_k+n_k}$ and satisfies the equation

$$\hat{\nu}(ah)\hat{\nu}(bh) = e^{i(z(a,b),h)}\hat{\nu}(sh) \quad \text{for every} \quad h \text{ in } E^*.$$

Define a function $g(.,.)$ on $[0, \infty) \times [0, \infty)$ as follows

$$g(x, y) = xy,\ x, y > 0; \qquad g(x, 0) = x,\ x \geqslant 0; \qquad g(0, y) = y,\ y \geqslant 0;$$

then the equality

(2.9) $$\hat{\nu}(xh)\hat{\nu}(yh) = e^{i(z(x,y),h)}\nu(g(x, y) \cdot h) \text{ holds for all } h \text{ in } E^*,$$

and for all $x, y \geqslant 0$. We shall prove in this part that g is the only function which satisfies (2.9).

Suppose not. Then there exist g_1 and g_2 satisfying (2.9) and for some x_0 and y_0, $g_1(x_0, y_0) < g_2(x_0, y_0)$. Let $u = \dfrac{g_1(x_0, y_0)}{g_2(x_0, y_0)}$. Then $u < 1$. Using (2.9) for g_1 and g_2, we get

$$e^{i(z(x_0,y_0),h/g_2(x_0,y_0))}\hat{\nu}(uh) = e^{i(z(x_0,y_0)h/g_2(x_0,y_0))}\hat{\nu}(h).$$

Thus $|\hat{\nu}(uh)| = |\hat{\nu}(h)|$. Hence the same argument as in (1.6) yields that ν is degenerate. This contradicts the assumption. Hence the uniqueness of g has been proved.

The function g so defined is continuous. Let $x_n \to x$; $y_n \to y$. Then we shall prove that $t = \overline{\lim}\, g(x_n, y)$ is finite. Suppose not, then there exists a subsequence $\{n'\}$ of $\{n\}$ such that $g^{-1}(x_{n'}, y_{n'}) \to 0$. Therefore $e_{n'} = x_{n'} g^{-1}(x_{n'}, y_{n'})$ and $k_{n'} = y_{n'} g^{-1}(x_{n'}, y_{n'}) \to 0$. Hence from (2.9) we get

$$|\hat{\nu}(e_{n'} \cdot h)| \cdot |\hat{\nu}(k_{n'} \cdot h)| = |\hat{\nu}(h)| \quad \text{for every} \quad h \text{ in } E^*.$$

By letting $n \to \infty$, in view of Lemma 1.4 we get ν is degenerate. Thus $t < \infty$.

To conclude that $t = \lim g(x_n, y_n)$, we shall show that no two distinct subsequences of $g(x_n, y_n)$ can converge to two different limits. If not, let $t' = \lim g(x_{n'}, y_{n'})$ and $t'' = \lim g(x_{n''} y_{n''})$, where $t' \neq t''$. Consequently, from (2.9) we get $|\hat{\nu}(t'h)| = |\hat{\nu}(t''h)|$ for every h in E^*. Since $t' \neq t''$, therefore from the same reasoning as in (1.6) we conclude that ν is degenerate. Hence, $t = \lim g(x_n, y_n)$. Now we make use of (2.9) again to conclude that

$$\hat{\nu}(xh) \cdot \hat{\nu}(yh) = e^{i(z(x,y),h)}\hat{\nu}(th) \quad \text{for every} \quad h \text{ in } E^*.$$

Since g is unique, therefore $t = g(x, y)$. Thus g is continuous.

It can be verified that the function g satisfies the hypothesis of Theorem 4.1 of ([3], p. 632). Hence by ([3], p. 632)

$$g(x, y) = (x^r + y^r)^{1/r} \quad \text{for some} \quad 0 < r < \infty,$$

and

$$\hat{\nu}(ah)\hat{\nu}(bh) = e^{i(z(a,b),h)}\hat{\nu}\big((a^r + b^r)^{1/r}h\big) \quad \text{for every} \quad h \in E^*,$$

where a, b are positive real numbers. This completes the proof.

2.10. THEOREM (CHARACTERIZATIONS OF STABLE PROBABILITY MEASURES). *Let E be a real separable Banach space and μ be a probability measure on $\mathfrak{B}(E)$. Then the following are equivalent.*

(a) *μ is stable.*

(b) *There exists a sequence a_n of positive real numbers and $\{x_n\} \subseteq E$ such that $\delta_{x_n} * T_{a_n} \mu^{*n} \Rightarrow \mu$.*

(c) *For each integer n, there exists a $y_n \in E$ and $c_n > 0$ such that $\mu^{*n} = \delta_{y_n} * T_{c_n} \mu$.*

Proof. The equivalence of (a) and (b) follows from Lemma 2.3 and Lemma 2.6. We note that for each n,

$$\mu = T_{c_n^{-1}}(\mu^{*n} * \delta_{-y_n}) = T_{c_n^{-1}} \mu^{*n} * \delta_{-c_n^{-1} y_n}.$$

Hence (c) implies (b). Also if μ is degenerate clearly (b) implies (c). Assume that (b) holds and μ is not degenerate. Then for every $k = 1, 2, \ldots$

$$\delta_{x_{nk}} * T_{a_{nk}} \mu^{*nk} = \mu_{nk} \Rightarrow \mu, \text{ where } \mu_n = \delta_{x_n} * T_{a_n} \mu^{*n}.$$

Hence,

$$\underbrace{(\delta_{x_n} * T_{a_n}\mu^{*n}) * \ldots * (\delta_{x_n} * T_{a_n}\mu^{*n})}_{\text{k factors}} = \delta_{k.x_n} * T_{a_n}\mu^{*n.k}$$

$$= (T_{a_n/a_{nk}}\mu_{nk}) * \delta_{kx_n(a_n/a_{nk})x_{nk}}$$

$$= (T_{d_{n_k}}\mu_{nk}) * \delta_{c_{n_k}},$$

where $d_{n_k} = a_n/a_{nk}$ and $c_{n_k} = kx_n - (a_n/a_{nk})x_{nk}$. Let $n \to \infty$ above. Then, $T_{d_{n_k}}\mu_{nk} * \delta_{z_{n_k}} \Rightarrow \mu^{*k}$. Since μ is not degenerate by Theorem 1.5 we conclude that $d_{n_k} \to d$, $z_{n_k} \to z \in E$, as $n \to \infty$, and $\mu^{*k} = T_d\mu * \delta_z$. Thus (b) $\Rightarrow$ (c) which completes the proof of the theorem.

2.11. Corollary (Proposition 9.25) of ([2], p. 199)). *Let $X_1, X_2, \ldots$ be identically distributed, non-degenerate, independent, random variables taking values in E. Then $\mu = \lim_n \mu_{S_n}$, where $S_n = \dfrac{X_1 + \ldots + X_n}{A_n} - y_n$, for some sequence A_n of positive real numbers and $\{y_n\} \subset E$ iff for each non-negative integer n, there exists a $c_n > 0$ and $z_n \in E$ such that $\mu^{*n} = \delta_{z_n} * T_{c_n}\mu$.*

Proof. Follows from the equivalence of (b) and (c) in Theorem 2.10. In particular this shows that limit laws of the normed sums given in (b) of Theorem 2.10, are infinitely divisible (i.d.).

2.12. Corollary ([8], p. 327). *Class of stable probability measures on $\mathfrak{B}(E)$ coincides with the limit laws of normed sums of independent and identically distributed random variables taking values in E.*

Proof. Follows from the equivalence of (a) and (b). The following corollary is now obvious.

2.13. Corollary ([5], p. 64). *Every stable law on a real separable Banach space is i. d.*

Corollaries 2.11, 2.12 and 2.13 relate stable probability measures to a certain subclass of i. d. measures. Recently, J. Kuelbs and V. Mandrekar ([7], p. 71) have obtained Levy–Khinchine representation for i. d. measures on certain orlicz spaces extending the work of S. R. S. Vardhan ([11], p. 227) on Hilbert space. In the next section we obtain a characterization of stable probability measures as a subclass of these i. d. measures in terms of the Levy–Khinchine representation of their ch. f. s. This result will generalize the recent work of Jajte ([5], p. 64) to these Orlicz spaces.

3. Levy–Khinchine representation of stable measures on certain Orlicz spaces. The Levy–Khinchine representation for the characteristic functional of a stable probability measure on Hilbert spaces has been studied by Jajte in [5]. In this section we shall obtain similar representation for stable probability measures on certain Orlicz spaces. We remark that the proof of the main theorem in [5] is incomplete and contains a lacuna which can be corrected (Cf. Lemma 3.6).

We recall for further reference some notation and results on Orlicz spaces. The function α used in this section will have the following properties.

(3.1)
$$\begin{cases} \text{(a)} & \alpha \text{ is defined on } [0, \infty) \text{ into } [0, \infty), \\ \text{(b)} & \alpha(0) = 0,\ \alpha(s) > 0 \text{ for } s > 0, \\ \text{(c)} & \alpha \text{ is convex and strictly increasing on } [0, \infty), \\ \text{(d)} & \alpha(2s) \leqslant M\alpha(s) \text{ for all } s \in [0, \infty), \text{ where } M \text{ is a finite positive constant independent of } s, \\ \text{(e)} & \int_{-\infty}^{\infty} \alpha(u^2)\,d\nu(u) \leqslant c\alpha\left[\int_{-\infty}^{\infty} u^2\,d\nu(u)\right] \text{ for all Gaussian measures } \nu \text{ on } (-\infty, \infty) \text{ with mean zero, where } c \text{ is a constant.} \end{cases}$$

3.2. Definition. The space of real sequence $x = (x_1, x_2, x_3, \ldots)$ satisfying $\sum_{i=1}^{\infty} \alpha(x_i^2) < \infty$ is denoted by E_α.

The Orlicz space S_Γ given by $\Gamma(t) = \alpha(t^2)$, $t \in [0, \infty)$, is isomorphically isometric to E_α. Throughout this section we use this identification for E_α, ([7], p. 61).

Let α_c be the function complementary to α in the sense of Young ([12], p. 77) and S_{α_c} be the Orlicz space corresponding to α_c ([12], p. 79). Then for each λ in the positive cone of S_{α_c} (except when $E_\alpha = l_2$), whose norm is less than or equal to one half, define; $\|x\|_\lambda^2 = \sum_{i=1}^{\infty} \lambda_i x_i^2$ and if $E_\alpha = l_2$, then $\|x\|_\lambda^2 = \sum x_i^2$. The space of sequences with property that $\|x\|_\lambda < \infty$ will be denoted by H_λ. Obviously $E_\alpha \subseteq H_\lambda$ by Young's inequality ([12], p. 77). In fact, H_λ is a Hilbert space containing E_α as its measureble subset ([7], p. 62).

3.3 Theorem. *Let μ be a probability measure on the Orlicz space E_α, where α satisfies (3.1). Then μ is stable on E_α iff either*

$$\hat{\mu}(y) = \exp[i(x_0, y) - \tfrac{1}{2}T(y, y)] \quad \text{for all} \quad y \in E_\alpha^*, \tag{3.4}$$

where $x_0 \in E_\alpha$ and T is an α-operator ([7], p. 16) (i. e. *φ is the characteristic functional of Gaussian measure) OR*

$$\hat{\mu}(y) = \exp\Big[i(x_0, h) + \int_U \Big(e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\lambda^2}\, dF(x)\Big) + \int_{E_\alpha - U} \Big(e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|x\|_\Gamma^2}\Big)\, dF(x), \tag{3.5}$$

where $x_0 \in E_\alpha$, $\|x\|_\Gamma$ is the norm of x in S_Γ, $U = \{x \in E_\alpha : \sum_{i=1}^{\infty} \alpha(x_1^2) \leqslant 1\}$, F is a σ-finite measure on E_α, finite on the complement of every nbhd of zero in

E_a and such that $\sum_{i=1}^{\infty} a\left(\int_u x_i^2 dF(x)\right) < \infty$, and there exists a r $(0 < r < 2)$ such that $T_a F = a^r F$ for every positive real a. (Stable probability measure of index r.)

Proof. Let (3.4) hold. Then

$$\hat{\mu}(ay)\cdot\hat{\mu}(by) = \exp[i(x_0, y)(a+b) - \tfrac{1}{2}T(y, y)(a^2+b^2)]$$

for every a and b positive and therefore

$$\hat{\mu}(ay)\cdot\hat{\mu}(by) = \hat{\mu}\left((a^2+b^2)^{\frac{1}{2}}y\right) \exp\left[i(x_0, y)\left((a+b)-(a^2+b^2)^{\frac{1}{2}}\right)\right].$$

Hence by ([4], p. 37), $T_a\mu * T_b\mu = T_c\mu * \delta_x$, where $c = (a^2+b^2)^{\frac{1}{2}}$ and $x = \left((a+b)-(a^2+b^2)^{\frac{1}{2}}\right)x_0 \in E_a$. Consequently, μ is stable. If (3.5) holds, then by ([7], p. 71), μ is i.d. on E_a. Hence, there exists a sequence of finite measures F_n on E_a such that $F_n \uparrow F$ and a sequence $x_n \in E_a$ such that $e(F_n)*\delta_{x_n} \Rightarrow \mu$ on E_a. We can regard F_n's and μ as measures on H_λ ([7], p. 62).

Since every bounded and continuous function on H_λ is also bounded and continuous when restricted to E_a by [7] (p. 65), we conclude that $e(F_n)*\delta_{x_n} \Rightarrow \mu$ on H_λ. Hence by ([11], p. 224) we get $\hat{\mu}(y) = \exp[i(x_1, y) + \int_{H_\lambda}\left(e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|_\lambda^2}\right)dF(x)]$ for every $y \in H_\lambda^*$, where $x_1 \in H_\lambda$, $\int_{\|x\|_\lambda \leq 1} \|x\|_\lambda^2 dF(x) < \infty$ and F is the σ-finite measure as before. Since $T_a F = a^\lambda F$ on H_λ, therefore by ([5] p. 65) μ is stable on H_λ. Consequently, for every $a, b > 0$ there exists a $c > 0$ and $z \in H_\lambda$ such that $T_a\mu * T_b\mu = T_c\mu * \delta_z$. To prove μ is stable on E_a, it would be enough to show that $z \in E_a$. Denote $z = (z_1, z_2, z_3, \ldots)$.

Define $\mu_n = \mu P_n^{-1}$, where $P_n(x) = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$ on E_a. Obviously $\mu_n \Rightarrow \mu$ on E_a with argument similar to one in ([6] p. 221). Now it is easy to see that $T_a\mu_n * T_b\mu_n = T_c\mu_n * \delta_{\tau_n}$, where $\tau_n = (z_1, z_2, \ldots \ldots, z_n, 0, 0, 0, \ldots)$. We note that $\tau_n \in E_a$ for every n and $\mu_n \Rightarrow \mu$ implies for any real d, $T_d\mu_n \Rightarrow T_d\mu$. Hence by Theorem 1.2, $\{\tau_n\}$ is compact on E_a. Consequently, $\tau_n \to z_0 \in E_a$ by ([1], p. 37). Hence $\|\tau_n - z_0\|_\lambda \to 0$ by ([7] p. 71), therefore $z = z_0$. Hence, for all a and $b > 0$ there exists a $c > 0$ and $z_0 \in E_a$ such that $T_a\mu * T_b\mu = T_c\mu * \delta_{z_0}$. This completes the proof of sufficiency.

Suppose μ is stable on E_a. Then by Corollary 2.13, μ is i. d. on E_a. Consequently, by ([7] p. 71), $\mu = \nu * \beta$ where β is the Gaussian part of μ on E_a and $\nu = \lim_{n\to\infty} e(F_n)*\delta_{x_n}$ where F_n's are increasing sequence of finite measures on E_a and $x_n \in E_a$ for all n. We can regard F_n's, μ, ν and β as measures on H_λ ([7], p. 62). Since an a-operator on E_a is also a trace class operator on H_λ, therefore by ([11], p. 226) β is Gaussian on H_λ. Thus

$\mu = \nu * \beta$ where β is Gaussian on H_λ, and $\nu = \lim_{n\to\infty} e(F_n)*\delta_{x_n}$ on H_λ. Since μ is stable on H_λ, therefore by ([5], p. 64), $\mu = \beta$ or $\mu = \nu$ where $\hat{\nu}(y) = \exp\left[i(z, y) + \int_{E_a}\left(e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|_\lambda^2}\right)dF(x)\right]$ for all $y \in H_\lambda^*$, and where $z \in H_\lambda$, $F = \lim_{n\to\infty} F_n$, $\int_{\|x\|_\lambda \leq 1} \|x\|_\lambda^2 dF(x) < \infty$ and for some $0 < r < 2$, $T_a F = a^r F$ for all positive a.

Since $\nu = \lim_{n\to\infty} e(F_n)*\delta_{x_n}$ on E_a, the result follows from ([7], p. 66). This completes the proof of the theorem.

To correct the proof of the theorem in ([5], p. 64) we need the following Lemma.

3.6 Lemma. *Let H be a Hilbert space and F a σ-finite measure on H satisfying for every a and b positive*

$$T_a F + T_b F = T_{(a^\lambda+b^\lambda)^{1/\lambda}} F, \quad 0 < \lambda < \infty. \tag{3.7}$$

Then F is necessarily of the form $T_a F = a^\lambda F$ for every positive a.

Proof. Since F is σ-finite, therefore it is enough to prove the above result for finite measure. So assume without loss of generality that F is a finite measure.

Let $B \in \mathfrak{B}(H)$ such that $\partial(B)$, the boundary of B, has F measure zero. Then $T_a F(B)$ is a continuous function on $(0, \infty)$, by Lemma 1.3 and ([9], p. 40). From (3.7) we get

$$F(a^{-1}B) + F(b^{-1}B) = F\left((a^\lambda + b^\lambda)^{-1/\lambda}B\right)$$

for all a and $b > 0$. Since the above is true for all a and $b > 0$, therefore, we get

$$F(a^{-1/\lambda}B) + F(b^{-1/\lambda}B) = F\left((a+b)^{-1/\lambda}B\right).$$

Let $F(a^{-1/\lambda}B) = g(a)$. Then g is continuous on $(0, \infty)$ and $g(a)+g(b) = g(a+b)$, for all a and $b > 0$. Therefore $g(a) = c\cdot a$ for $a > 0$, where c is a constant depending on B. Hence $g(a^\lambda) = a^\lambda c$. Thus $F(a^{-1}B) = a^\lambda c$. Let $a = 1$. Then $c = F(B)$. Hence $F(a^{-1}B) = a^\lambda F(B)$. Since the class $\{B\colon B \in \mathfrak{B}(H),\ F(\partial B) = 0\}$ is a field by ([1], p. 16), therefore by Caratheodory extension theorem

$$F(a^{-1}B) = a^\lambda F(B) \quad \text{for every } B \in \mathfrak{B}(H).$$

References

[1] P. Billingsley, *Convergence of probability measures*, 1968.

[2] L. Brieman, *Probability*, 1968.

[3] F. Bohnenblust, *An axiomatic characterization of L_p-spaces*, Duke Math. Journal 6 (1940), pp. 627–640.

[4] K. Ito and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka J. Math. 5 (1968), pp. 35–48.

[5] R. Jajte, *On stable distributions in Hilbert space*, Studia Math. 30 (1964), pp. 63–71.
[6] J. Kuelbs and V. Mandrekar, *Harmonic analysis on certain vector spaces*, Trans. Amer. Math. Soc. 14a (1970), pp. 213–231.
[7] — — *Harmonic analysis on F-spaces with a basis*, (ibidem, to appear).
[8] M. Loève, *Probability Theory*, 3rd ed., New York 1963.
[9] K. S. Parthasarathy, *Probability measures in metric spaces*, 1967.
[10] K. Urbanik, *Generalized convolutions*, Studia Math. 23 (1964), pp. 217–245.
[11] S. R. S. Varadhan, *Limit theorems for sums of independent random variables with values in a Hilbert space*, Sankhya 24 (1962), pp. 213–238.
[12] A. C. Zaanen, *Linear Analysis*, New York, 1953.

Received February 2, 1971 (293)

STUDIA MATHEMATICA, T. XLII. (1972)

Unitary representations induced from compact subgroups

by

MARC A. RIEFFEL* (Berkeley)

Abstract. It is shown, for the case in which the subgroup is compact, that the induced representations of Mackey and Mautner can be defined in terms of certain Hilbert module tensor products, or, alternatively, certain spaces of Hilbert–Schmidt intertwining operators. These definitions are used to derive basic properties of induced representations, and the connection with Blattner's approach in terms of positive type measures is discussed.

Let H be a compact subgroup of a locally compact group G. Mackey ([13], [14], [15]) and Mautner ([18], [19]), using definitions involving certain spaces of measurable vector valued functions, showed how to induce representations of H up to G. (Mackey, in fact, treated the more general case in which H need not be compact). In the present paper we show how these induced representations can be defined in terms of certain Hilbert module tensor products, or, alternatively, in terms of certain spaces of Hilbert–Schmidt intertwining operators. Such definitions enable us to give convenient derivations of the basic properties of induced representations along lines which follow fairly closely the theory of induced representations as it is developed for finite groups (for which see [2]). Our approach is also quite similar to that for induced Banach space representations which we gave in [22].

The exposition is organized in the following way. In Section 1 we consider the basic properties of the Hilbert space tensor product. The principal result is that this tensor product provides the left adjoint for the construction of spaces of Hilbert–Schmidt operators. We believe that this result is new, although the interconnection between the Hilbert space tensor product and Hilbert–Schmidt operators is found implicitly in a number of papers. In Section 2 the results of Section 1 are extended to the setting of Hilbert spaces which are modules over sets, and in Section

* This research was partially supported by National Science Foundation grant GP–12997.