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Differentiation of trigonometric series

by

C. J. NEUGEBAUER (Lafayette, Ind.)

Abstract. Necessary and sufficient conditions are given in order that a termwise integrated trigonometric series can be differentiated a.e.

1. Let $\sum A_n(x)$, $A_0(x) = \frac{a_0}{2}$, $A_n(x) = a_n \cos nx + b_n \sin nx$, be a trigonometric series. We write $s_n(x) = \sum_1^n A_j(x)$ and $\tilde{s}_n(x) = \sum_1^n B_j(x)$, where $B_j(x) = a_j \sin jx - b_j \cos jx$. If $\sum A_n(x)$ converges on E, $|E| > 0$, then the termwise integrated series $\sum \frac{B_n(x)}{n}$ is the Fourier series of a function $f \in L^2$ and f has an approximate derivative a.e. in E. This result [9_I, p. 325] has been extended by M. Weiss [7] who has shown that f in fact has at a.e. $x \in E$ a derivative in L^p for every $p < \infty$, i.e., for a a.e. $x \in E$

$$\int_{-h}^{h} |f(x+t) - f(x) - at|^p \, dt = o(h^{p+1})$$

for some $a = a(x)$. There are examples of trigonometric series $\sum A_n(x)$ converging a.e. and for which $\sum \frac{B_n(x)}{n}$ is ordinarily ($p = \infty$) derivable almost nowhere [1_{II}, p. 99]. The purpose of this paper is to present necessary and sufficient conditions on the sequence $\{\tilde{s}_n(x)\}$ so that $\sum \frac{B_n(x)}{n}$ be ordinarily derivable.

2. In this and the following section we will collect some lemmas and remarks needed later.

LEMMA 1. *Let* $s_n(x) = O(1)$, $x \in E$, $|E| > 0$. *Then*

(1) $\varrho_n = \sqrt{a_n^2 + b_n^2} = O(1)$.

(2) $\tilde{s}_n(x) = O(1)$, *for a.e.* $x \in E$.

(3) $\sum \frac{1}{n} A_n(x)$, $\sum \frac{1}{n} B_n(x)$ *are Fourier series of functions in* L^2 *which converge to these functions a.e.*

Proof. The assertion (1) can be found in [9_I, p. 317] and (2) is a special case of a general theorem in [3]. That the two series in (3) are Fourier series of functions in L^2 follows from the Riesz–Fischer theorem, and the convergence a.e. is a consequence of their $(C, 1)$ summability and the fact that the terms are $O\left(\frac{1}{n}\right)$ by (1) [9_I, p. 78].

THEOREM 1. *Let $s_n(x) = O(1)$, $x\in E$, $|E|>0$. Let $x\in E$ be a point at which $\sum \frac{1}{n} A_n(x)$ converges. Then there exists a measurable set $Q(x)$ having at 0 positive lower density such that $L(x+h)-L(x-h) = O(h)$, $h\in Q(x)$, where $L(x) = \sum \frac{1}{n} B_n(x)$.*

The proof is, with only obvious modifications, the same as the one in [9_I, p. 324].

For a.e. x we have

$$\frac{1}{2h}\Delta^2 L(x, 2h) = \frac{1}{2h}[L(x+2h)+L(x-2h)-2L(x)]$$
$$= -2\sum B_n(x)\frac{\sin^2 nh}{nh}.$$

Summation by parts gives

$$\frac{1}{2h}\Delta^2 L(x, 2h) = -2\sum \tilde{s}_n(x)\left\{\frac{\sin^2 nh}{nh} - \frac{\sin^2(n+1)h}{(n+1)h}\right\}.$$

The expression in $\{\ \} = \dfrac{\sin^2 nh}{n(n+1)h} + \dfrac{\sin^2 nh - \sin^2(n+1)h}{(n+1)h}$ and $\sin^2 nh - \sin^2(n+1)h = -\sin h \sin(2n+1)h$.

LEMMA 2. *If $s_n(x) = O(1)$, $x\in E$, $|E|>0$, then*

$$\frac{1}{2h}\Delta^2 L(x, 2h) = -2\sum \frac{\tilde{s}_n(x)}{n(n+1)h}\sin^2 nh + \frac{2\sin h}{h}\sum \frac{\tilde{s}_n(x)}{n+1}\sin(2n+1)h$$

for a.e. $x\in E$. The same formula holds a.e. if $\sum A_n(x)$ is the Fourier series of a function in L^1.

Proof. The first part is a consequence of the above calculations, and the second part follows from the well-known fact that both $s_n(x)$ and $\tilde{s}_n(x)$ are $o(\log n)$ for a.e. x.

3. We have occasion to use the following results on the differentiability of a measurable function $f\colon [a, b]\to$ reals.

THEOREM 2. *Let $E\subset [a, b]$ be a measurable set such that (1) for $x\in E$ there is a measurable set $Q(x)$ having 0 as a point of positive lower density and $f(x+h)-f(x-h) = O(h)$, $h\in Q(x)$, and (2) for $x\in E$ there is a set N_x such that $|N_x| = 0$ and $\Delta^2 f(x, h) = O(h)$ as $h\to 0$, $h\notin N_x$. Then f is equivalent with a function which is differentiable a.e. in E.*

Proof. Since $2[f(x+h)-f(x)] = f(x+h)-f(x-h)+\Delta^2 f(x, h)$, we have $f(x+h)-f(x) = O(h)$, a $h\to 0$ on a set having positive lower density at 0. Application of [6, p. 295] shows that the approximate derivative of f exists a.e. in E. Blumberg's upper boundary of f [2] defined by

$$g(x) = \inf\{y\colon |E_y\cap I| = O(|I|),\ \text{as } |I|\to 0,\ x\in I\},$$

where $E_y = \{x\colon f(x)>y\}$, possesses the desired properties. The proof of this is in [5].

4. Let C^∞ be the space of infinitely differentiable functions φ of period 2π with the usual topology. We let c_n be the distribution $c_n(\varphi) = \frac{1}{\pi}\int_0^{2\pi}\cos(2n+1)t\varphi(t)\,dt$. If $s_n(x) = O(1)$, $x\in E$, or if $\{s_n(x)\}$ is the sequence of partial sums of the Fourier series of a function in L^1, we can consider the distribution

$$\tilde{S}_x(\varphi) = \sum \tilde{s}_n(x)c_n(\varphi).$$

For $0<\eta\leqslant\pi$ we let $C_\eta^\infty = \{\varphi\in C^\infty\colon \operatorname{supp}\varphi\subset[-\eta, \eta]\}$, and we introduce in C_η^∞ the norm $\|\varphi\| = \|\varphi\|_1 + \|\varphi'\|_1$. The completion of C_η^∞ with respect to $\|\cdot\|$ is the space W_η of all absolutely continuous functions of period 2π supported in $[-\eta, \eta]$.

THEOREM 3. *Let $s_n(x) = O(1)$, $x\in E$, $|E|>0$. Then $L(x) = \sum \frac{1}{n}B_n(x)$ is equivalent with a function differentiable a.e. in E if and only if for a.e. $x\in E$ there is $\eta = \eta_x>0$ such that $|\tilde{S}_x(\varphi)|\leqslant M_x\|\varphi\|$, $\varphi\in C_\eta^\infty$.*

Proof. By Theorems 1 and 2 we only need to show that $\Delta^2 L(x, h) = O(h)$, $h\notin N_x$, $|N_x| = 0$, a.e. $x\in E$, if and only if for a.e. $x\in E$ there is $\eta = \eta_x>0$ such that $|\tilde{S}_x(\varphi)|<M_x\|\varphi\|$, $\varphi\in C_\eta^\infty$.

Let $x\in E$ such that $|\tilde{s}_n(x)|\leqslant M$, $n = 1, 2, \ldots$, and at which the formula for $\frac{1}{2h}\Delta^2 L(x, 2h)$ of Lemma 2 holds. This is true for a.e. $x\in E$. By [9_I, p. 10 (4.17)], $\left|\sum_1^\infty \frac{\tilde{s}_n(x)}{n(n+1)h}\sin^2 nh\right|\leqslant K<\infty$ for all h.

We assume now that there is $|N_x| = 0$ such that $\Delta^2 L(x, h) = O(h)$, $h\notin N_x$. Then there is $\eta = \eta_x>0$ such that $\sum \frac{\tilde{s}_n(x)}{2n+1}\sin(2n+1)h = f(h)$ is essentially bounded on $[-\eta, \eta]$, and $f\in L^2$. As a distribution, for $\varphi\in C_\eta^\infty$, $f'(\varphi) = -f(\varphi') = \sum\tilde{s}_n(x)c_n(\varphi) = \tilde{S}_x(\varphi)$. Since $f(\varphi') = \int f\varphi'$ we see that $|\tilde{S}_x(\varphi)| \leqslant M_x\|\varphi'\|_1$, where M_x is the essential bound of f on $[-\eta, \eta]$.

Conversely, let f be the distribution in L^2 whose Fourier series is $\sum \frac{\tilde{s}_n(x)}{2n+1} \sin(2n+1)h$, x fixed. Then $f'(\varphi) = \tilde{S}_x(\varphi) = -f(\varphi')$, so that $|f(\varphi')| \leqslant M_x\|\varphi\|$, $\varphi \in C_\eta^\infty$. If $\varphi \in C_\eta^\infty$ and $\int_{-\pi}^{\pi} \varphi = 0$, then $\psi(t) = \int_{-\pi}^{t} \varphi$ is in C_η^∞ and hence $|f(\varphi)| \leqslant M_x\{\|\psi\|_1 + \|\varphi\|_1\}$.

We will show that f is essentially bounded on $(0, \eta)$. Let $t_0 \in (0, \eta)$ be a Lebesgue point of f and let $|f(t_0)| = L$. Let $t_1 \in (0, \eta)$ be another Lebesgue point of f, say $t_0 < t_1$, and finally let $a = \min\left(\frac{t_1 - t_0}{3}, t_0, \eta - t_1\right)$. We introduce the collection $\Phi = \{\varphi \in C_a^\infty : \varphi \geqslant 0 \text{ and } \int_{-\pi}^{\pi} \varphi = 1\}$, and for $\varphi \in \Phi$ we let $\varphi(t) = \varphi(t-t_0) - \varphi(t-t_1)$. Then $\int_{-\pi}^{\pi} \varphi = 0$ and $\varphi \in C_\eta^\infty$. Hence

$$\Big|\int_{-\pi}^{\pi} f(t)\varphi(t)\,dt\Big| \leqslant K < \infty, \qquad \varphi \in \Phi.$$

From this we obtain

$$\Big|\int_{-\pi}^{\pi} F(u)\varphi(u)\,du\Big| \leqslant K, \quad \text{where} \quad F(u) = f(t_0+u) - f(t_1+u).$$

For $\varphi \in \Phi$ we denote by $\varphi_n(t) = n\varphi(nt)$. At every Lebesgue point $\tau_0 \in (-a, a)$ of F we have

$$\int_{-\pi}^{\pi} F(t)\varphi_n(\tau_0 - t)\,dt \to F(\tau_0),$$

so that $|F(\tau_0)| \leqslant K$. If we apply this to $\tau_0 = 0$ we obtain $|f(t_0) - f(t_1)| \leqslant K$ from which $|f(t_1)| \leqslant L + K$. The proof is now complete.

5. The hypothesis that $s_n(x) = O(1)$, $x \in E$, in Theorem 3 is not satisfied for Fourier series of functions in L^1. However, in this case the sequence $\{\sigma_n(x)\}$ of $(C, 1)$ means is bounded a.e. (in fact converges to f a.e.). We shall present a version of Theorem 3 in terms of σ_n.

We assume that $\{\sigma_n(x)\}$ is bounded for $x \in E$, $|E| > 0$. Then [3], $\{\tilde{\sigma}_n(x)\}$ is bounded for a.e. $x \in E$, and hence $\tilde{s}_n(x) = O(n)$. Consequently we can consider the distribution

$$\tilde{S}_x^* = \sum [\tilde{\sigma}_n(x) + \tilde{s}_n(x)] e_n.$$

THEOREM 4. *Assume that $\sigma_n(x) = O(1)$, $x \in E$, $|E| > 0$. Then $\sum \frac{B_n(x)}{n}$ is $(C, 1)$ summable to $L(x)$, and L is equivalent with a function differentiable a.e. in E if and only if for a.e. $x \in E$ there is $\eta = \eta_x > 0$ such that*

$$|\tilde{S}_x^*(\varphi)| \leqslant M_x\|\varphi\|, \quad \varphi \in C_\eta^\infty.$$

Proof. The boundedness of $\{\tilde{\sigma}_n(x)\}$ implies the $(C, 2)$ summability of $\sum A_n(x)$, and this in turn implies the $(C, 1)$ summability of $\sum \frac{1}{n} B_n(x)$ [8]. The proof now is the same as before if one first applies an additional summation by parts to the formula for $\frac{1}{2h} \Delta^2 L(x, 2h)$ of Lemma 2.

References

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