Differentiation of trigonometric series

by

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Abstract. Necessary and sufficient conditions are given in order that a termwise integrated trigonometric series can be differentiated a.e.

1. Let $\sum A_n(x), A_n(x) = \frac{a_n}{2}, A_n(x) = a_n \cos nx + b_n \sin nx$, be a trigonometric series. We write $s_n(x) = \sum_{i=1}^{n} A_i(x)$ and $\tilde{s}_n(x) = \sum_{i=1}^{n} B_i(x)$, where $B_i(x) = a_i \sin nx - b_i \cos nx$. If $\sum_{n=1}^{\infty} B_n(x)$ converges on $E$, $|E| > 0$, then the termwise integrated series $\sum_{n=1}^{\infty} B_n(x)$ is the Fourier series of a function $f \in L^p$ and $f$ has an approximate derivative a.e. in $E$. This result [9, p. 325] has been extended by M. Weiss [7] who has shown that $f$ in fact has at a.e. $x \in E$ a derivative in $L^p$ for every $p < \infty$, i.e., for a.e. $x \in E$

$$\int_{-\infty}^{\infty} |f(x+t) - f(x) - at|^p dt = o(t^{p+1})$$

for some $a = a(x)$. There are examples of trigonometric series $\sum A_n(x)$ converging a.e. and for which $\sum B_n(x)$ is ordinarily $(p = \infty)$ derivable almost nowhere [11], p. 99]. The purpose of this paper is to present necessary and sufficient conditions on the sequence $(s_n(x))$ so that $\sum B_n(x)$ be ordinarily derivable.

2. In this and the following section we will collect some lemmas and remarks needed later.

Lemma 1. Let $s_n(x) = O(1), x \in E$, $|E| > 0$. Then

(1) $\phi_n = \sqrt{a_n^2 + b_n^2} = O(1)$.

(2) $\tilde{s}_n(x) = O(1)$, for a.e. $x \in E$.

(3) $\sum_{n=1}^{\infty} A_n(x), \sum_{n=1}^{\infty} B_n(x)$ are Fourier series of functions in $L^2$ which converge to these functions a.e.
Proof. The assertion (1) can be found in [9, p. 317] and (2) is a special case of a general theorem in [3]. That the two series in (3) are Fourier series of functions in \( L^1 \) follows from the Riesz-Fischer theorem, and the convergence a.e. of their \((c,1)\) summability and the fact that the terms are \( O \left( \frac{1}{n^m} \right) \) by (1) [97, p. 78].

**Theorem 1.** Let \( s_n(x) = O(1), x \in E, |E| > 0 \). Let \( x \in E \) be a point at which \( \sum \frac{1}{n} A_n(x) \) converges. Then there exists a measurable set \( Q(x) \) having at 0 positive lower density such that \( L(x + h) - L(x - h) = O(h), h \cdot Q(x), \) where \( L(x) = \sum \frac{1}{n} B_n(x) \).

The proof is, with only obvious modifications, the same as the one in [97, p. 324].

For a.e. \( x \) we have

\[
\frac{1}{2h} A^* L(x, 2h) = \frac{1}{2h} \left[ L(x + 2h) + L(x - 2h) - 2L(x) \right] = -2 \sum B_n(x) \sin^2 \frac{n\pi h}{h}.
\]

Summation by parts gives

\[
\frac{1}{2h} A^* L(x, 2h) = -2 \sum \delta_n(x) \left\{ \frac{\sin^2 \frac{n\pi h}{h}}{n\pi h} + \sin^2 \left( \frac{n\pi (n+1)h}{(n+1)h} \right) \right\}.
\]

The expression in \( \{ \} \) is

\[
\frac{\sin^2 \frac{n\pi h}{h}}{n\pi (n+1)h} + \sin^2 \left( \frac{n\pi (n+1)h}{(n+1)h} \right)
\]

and

\[
\sin^2 \frac{n\pi h}{h} - \sin^2 \left( \frac{n\pi (n+1)h}{(n+1)h} \right) = -\sin h \sin \left( 2n + 1 \right) h.
\]

**Lemma 2.** If \( s_n(x) = O(1), x \in E, |E| > 0 \), then

\[
\frac{1}{2h} A^* L(x, 2h) = -2 \sum \frac{s_n(x)}{n(n+1)h} \sin^2 \frac{n\pi h}{h} + \frac{2\sin h}{h} \sum \frac{s_n(x)}{n+1} \sin \left( 2n + 1 \right) h
\]

for a.e. \( x \in E \). The same formula holds a.e. if \( \sum A_n(x) \) is the Fourier series of a function in \( L^1 \).

Proof. The first part is a consequence of the above calculations, and the second part follows from the well-known fact that both \( s_n(x) \) and \( \delta_n(x) \) are \( o(\log n) \) for a.e. \( x \).

3. We have occasion to use the following results on the differentiability of a measurable function \( f: [a, b] \rightarrow \text{reals} \).

**Theorem 2.** Let \( E \subset [a, b] \) be a measurable set such that (1) for \( x \in E \) there is a measurable set \( Q(x) \) having 0 as a point of positive lower density and \( f(x+h) - f(x-h) = O(h), h \cdot Q(x), \) and (2) for \( x \in E \) there is a set \( N_x \) such that \( |N_x| = 0 \) and \( A^* f(x, h) = O(h) \) as \( h \rightarrow 0, h \cdot N_x \). Then \( f \) is equivalent with a function which is differentiable a.e. in \( E \).

Proof. Since \( 2f(x+h) - f(x) = f(x+h) - f(x-h) + A^* f(x, h) \), we have \( f(x+h) - f(x) = O(h) \), \( h \rightarrow 0 \) on a set having positive lower density at 0. Application of [6, p. 290] shows that the approximate derivative of \( f \) exists a.e. in \( E \). Blumberg's upper boundary of \( f(2) \) defined by

\[
g(x) = \inf \{ y: [E_x \cap \{ x \} = O(|\{ x \}|), \text{ as } |\{ x \}| \rightarrow 0, x \in \{ x \} \},
\]

where \( E_x = \{ x: f(x) > y \} \), possesses the desired properties. The proof of this is in [5].

4. Let \( L^\infty \) be the space of infinitely differentiable functions \( \psi \) of period \( 2\pi \) with the usual topology. We let \( c_0 \) be the distribution \( c_0(\psi) = \frac{1}{2\pi} \int_0^{2\pi} \cos(2\pi n + 1) \psi(t) dt \). If \( s_n(x) = O(1), x \in E \), or if \( \{ s_n(x) \} \) is the sequence of partial sums of the Fourier series of a function in \( L^1 \), we can consider the distribution

\[
\delta_n(x) = \sum s_n(x) c_0(\psi).
\]

For \( 0 < \eta < \pi \) we let \( \mathcal{C}_0^\infty = \{ \psi \in \mathcal{C}_0^\infty: \text{supp} \psi[\eta, \eta] \}, \) and we introduce in \( \mathcal{C}_0^\infty \) the norm \( \| \psi \| = \| \psi \| + \| \psi \| \). The completion of \( \mathcal{C}_0^\infty \) with respect to \( \| \cdot \| \) is the space \( W_0 \) of all absolutely continuous functions of period \( 2\pi \) supported in \( [-\eta, \eta] \).

**Theorem 3.** Let \( s_n(x) = O(1), x \in E, |E| > 0 \). Then \( L(x) = \sum \frac{1}{n} B_n(x) \) is equivalent with a function differentiable a.e. in \( E \) if and only if for a.e. \( x \in E \) there is \( \eta = \eta_x > 0 \) such that \( |\delta_n(x)| \leq M_x \| \psi \|, \psi \mathcal{C}_0^\infty \).

Proof. By Theorems 1 and 2 we only need to show that \( A^* L(x+h) = O(h), h \cdot N_x, |N_x| = 0, x \in E, \) if and only if for a.e. \( x \in E \) there is \( \eta = \eta_x > 0 \) such that \( |\delta_n(x)| \leq M_x \| \psi \|, \psi \mathcal{C}_0^\infty \).

Let \( x \in E \) such that \( |\delta_n(x)| \leq M_x \), \( n = 1, 2, \ldots \), and at which the formula for \( \frac{1}{2h} A^* L(x, 2h) \) of Lemma 2 holds. This is true for a.e. \( x \in E \).

By [97, p. 10 (4.17)],

\[
\sum s_n(x) \sin^2 \frac{n\pi h}{h} \leq K < \infty \text{ for all } h.
\]

We assume now that there is \( |N_x| = 0 \) such that \( A^* L(x+h) = O(h), h \cdot N_x \). Then there is \( \eta = \eta_x > 0 \) such that \( \sum \frac{s_n(x)}{2n+1} \sin \left( 2n + 1 \right) h = f(h) \) is essentially bounded on \( [-\eta, \eta] \), and \( f \in L^1 \). As a distribution, for \( \psi \mathcal{C}_0^\infty, f(\psi) = -\sum s_n(x) c_0(\psi) = \delta_n(\psi). \) Since \( f(\psi) = \int f(\psi) \)

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see that \( |\tilde{S}_f(\sigma)| \leq M_n(\|\sigma\|_1) \), where \( M_n \) is the essential bound of \( f \) on \([-\eta, \eta]\).

Conversely, let \( f \) be the distribution in \( L^1 \) whose Fourier series is
\[
\sum_{2n+1} \hat{f}(k) \sin(2\pi + 1)k, \quad x \text{ fixed.}
\]
Then \( f(\cdot) = \tilde{S}_f(\cdot) \), so that \( |f(\sigma)| \leq M_n(\|\sigma\|_1) \), \( \sigma \in C_0^\infty \). If \( \sigma \in C_0^\infty \) and \( \|\sigma\|_1 = 0 \), then \( \psi(\cdot) = \int_{-\eta}^{\eta} \psi \) is in \( C_0^\infty \) and hence \( |f(\sigma)| \leq M_n(\|\sigma\|_1) \).

We will show that \( f \) is essentially bounded on \((0, \eta)\). Let \( t_0 \in (0, \eta) \) be a Lebesgue point of \( f \) and let \( f(t_0) = L \). Let \( t_1, t_2, \ldots, t_n \) be another Lebesgue point of \( f \), say \( t_1 < t_2, \ldots, \) and finally let \( t = \min(t_1, t_2, \ldots, t_n) \).

We introduce the collection \( \Phi = \{\psi \in C_0^\infty : \psi \geq 0 \text{ and } \int \psi = 1\} \), and for \( \sigma \in \Phi \) we let \( f'(t) = \int_{-\eta}^{\eta} \psi(\cdot) d\phi \). Then \( \int_{-\eta}^{\eta} \psi = 0 \) and \( \sigma \in C_0^\infty \). Hence
\[
\int_{-\eta}^{\eta} f'(t) \psi(t) d\phi < K < \infty, \quad \psi \in \Phi.
\]
From this we obtain
\[
\int_{-\eta}^{\eta} F(u) \psi(u) du < K, \quad \text{where } F(u) = f(t_2 + u) - f(t_1 + u).
\]

For \( \psi \in \Phi \) we denote by \( \phi_h(t) = \psi(\cdot - a, a) \) of \( F \) we have
\[
\int_{-\eta}^{\eta} F(t) \phi_h(t) dt = \int_{-\eta}^{\eta} F(t) \psi(t) dt \psi(h) \psi(h) dt,
\]
so that \( |F(t)|_1 \leq K \). If we apply this to \( \eta = 0 \) we obtain \( \int_{-\eta}^{\eta} f(\cdot) \psi(\cdot) d\phi \leq K \) from which \( |f(\eta)|_1 \leq L + K \). The proof is now complete.

5. The hypothesis that \( \sigma_n(x) = O(1), \sigma \in E \), in Theorem 3 is not satisfied for Fourier series of functions in \( L^1 \). However, in this case the sequence \( \{\sigma_n(x)\} \) of \( (C, 1) \) means is bounded a.e. (in fact converges to \( f \) a.e.). We shall present a version of Theorem 3 in terms of \( \sigma_n \).

We assume that \( \{\sigma_n(x)\} \) is bounded for \( \sigma \in E \), \( |E| > 0 \). Then \( \|\sigma_n(x)\|_1 \) is \( (C, 1) \) summable to \( L(x) \), and \( L \) is equivalent with a function differentiable a.e. in \( E \) if and only if for a.e. \( \sigma \in E \) there is \( \eta = \eta_{\sigma} > 0 \) such that
\[
\|\tilde{S}_f(\sigma)\|_1 \leq M_n(\|\sigma\|_1) \psi(\sigma) C_0^\infty.
\]

**Proof.** The boundedness of \( \{\tilde{S}_f(\sigma)\} \) implies the \( (C, 2) \) summability of \( \sum \sigma_n(x) \), and this in turn implies the \( (C, 1) \) summability of \( \sum \frac{1}{n} B_n(x) \) \[8\]. The proof now is the same as before if one first applies an additional summation by parts to the formula for \( \frac{1}{2\pi} \int_{2\pi} f(x, 2\pi) \) of Lemma 2.

**References**


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