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Products and convolutions of vector valued set functions

by

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Abstract. Let $\mathscr{S}$ be a topological semigroup and X, Y, and Z be Banach spaces with a bilinear function from $X\times Y$ into Z. Let μ and ν be countably additive functions from the Borel sets of $\mathscr{S}$ into X and Y respectively. The notions of product and convolution of μ and ν are defined and some properties of these are investigated.

I. INTRODUCTION

The concept of the convolution of scalar valued set functions plays an important role in harmonic analysis of scalar valued functions on a semigroup. It would seem natural to investigate a concept of convolution of vector valued set functions in hopes that a similar analysis of vector valued functions would be facilitated. This paper is an attempt in that direction.

We rely heavily on previous results concerning vector valued set functions and Bochner integration [1–4, 6] as well as well-known results concerning the convolution of scalar valued set functions (see, for example, [8]).

Chapter II is concerned with the problem of beginning with two countably additive vector valued set functions on the Borel σ-algebras of two topological spaces and generating a countably additive set function on the measurable rectangles. We then treat the same problem in the case the set functions are regular; in this case, the "product" can be extended to the Borel sets. Chapter III yields a Fubini-type theorem for the product and Chapter IV considers the notion of convolution when the topological spaces are topological semigroups.

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II. PRODUCTS OF VECTOR VALUED SET FUNCTIONS

Let X, Y, and Z be normed linear spaces with a bilinear map $(x, y) \to x\cdot y$ from $X\times Y$ into Z such that there is a $k>0$ with $\|x\cdot y\| \leqslant k\|x\|\cdot\|y\|$ (X, Y, and Z will be said to form a bilinear system). Let S and T be locally compact, Hausdorff spaces with Borel σ-rings $\mathscr{B}(S)$ and $\mathscr{B}(T)$ respectively (the Borel sets form the smallest σ-ring containing the compact sets). The following is trivial.

LEMMA II. 1. *$\mathscr{B}(S)\times\mathscr{B}(T) = \{A\times B: A\in\mathscr{B}(S), B\in\mathscr{B}(T)\}$ is a semiring* ([5]; *p.* 22) *and, moreover, if $(A\times B)\cup(C\times D) = E\times F$, where $(A\times B)\cap\cap(C\times D) = \emptyset$, then either*

(a) *$A\cap C = \emptyset$ and $B = D$ or*

(b) *$B\cap D = \emptyset$ and $A = C$.*

The members of $\mathscr{B}(S)\times\mathscr{B}(T)$ are sometimes called measurable rectangles. We are now ready to define a product.

DEFINITION II. 2. *Let μ: $\mathscr{B}(S)\to X$ and ν: $\mathscr{B}(T)\to Y$ be finitely additive. Then for $A\times B\in\mathscr{B}(S)\times\mathscr{B}(T)$, we define*

$$(\mu\times\nu)(A\times B) = \mu(A)\cdot\nu(B).$$

THEOREM II. 3. *$\mu\times\nu$ is finitely additive on $\mathscr{B}(S)\times\mathscr{B}(T)$ to Z and has a unique finitely additive extension to the smallest ring $\mathscr{R}(\mathscr{B}(S)\times\mathscr{B}(T))$ containing $\mathscr{B}(S)\times\mathscr{B}(T)$.*

Proof. Since $\mathscr{B}(S)\times\mathscr{B}(T)$ is a semiring, it is sufficient to prove that $\mu\times\nu$ is 2-additive. Let $A\times B$, $C\times D$, and $E\times F$ be members of $\mathscr{B}(S)\times\mathscr{B}(T)$ and $(A\times B)\cup(C\times D) = E\times F$ with $(A\times B)\cap(C\times D) = \emptyset$; then either (a) or (b) of Lemma II. 1 must hold, say (a) (the proof for (b) is similar).

$$\begin{aligned}(\mu\times\nu)[(A\times B)\cup(C\times D)] = (\mu\times\nu)[(A\times B)\cup(C\times B)] &= (\mu\times\nu)[(A\cup C)\times B]\\ &= \mu(A\cup C)\cdot\nu(B)\\ &= [\mu(A)+\mu(C)]\cdot\nu(B) = \mu(A)\cdot\nu(B)+\mu(C)\cdot\nu(B)\\ &= (\mu\times\nu)[A\times B]+(\mu\times\nu)[C\times D].\end{aligned}$$

DEFINITION II. 4. *Let $\mathscr{S}$ be a semiring of subsets of a set U and X a normed linear space, and μ: $\mathscr{S}\to X$. For $A\subseteq U$, we define the variation $\mathscr{V}(\mu)$ by*

$$\mathscr{V}(\mu, A) = \sup\Big\{\sum_{i=1}^{n}\|\mu(B_i)\|\Big\},$$

where $\{B_i\}_1^n$ is a pairwise disjoint collection in $\mathscr{S}$ and $\bigcup_1^n B_i\subseteq A$;

$$\|\mu\| = \sup\{\mathscr{V}(\mu, A)\,|\,A\subseteq U\} = \mathscr{V}(\mu, U).$$

We recall that if μ is finitely (countably) additive, and $\|\mu\|<\infty$, then $\mathscr{V}(\mu)$ is a finitely (countably) additive, non-negative set function. If $\|\mu\|<\infty$, then μ is said to be of bounded variation. If μ is finitely additive on $\mathscr{S}$ and $\hat{\mu}$ is its unique finitely additive extension to the ring generated by $\mathscr{S}$, then $\mathscr{V}(\mu) = \mathscr{V}(\hat{\mu})$.

THEOREM II. 5. *Let μ: $\mathscr{B}(S)\to X$ and ν: $\mathscr{B}(T)\to Y$ be finitely additive with $\|\mu\|<\infty$ and $\|\nu\|<\infty$; then $\|\mu\times\nu\|<\infty$ and $\mathscr{V}(\mu\times\nu, A\times B)\leqslant k\mathscr{V}(\mu, A)\cdot\mathscr{V}(\nu, B)$. Moreover, if μ and ν are countably additive, so is $\mu\times\nu$.*

Proof. Define λ: $\mathscr{B}(S)\times\mathscr{B}(T)\to[0, \infty)$ by $\lambda(A\times B) = \mathscr{V}(\mu, A)\cdot\mathscr{V}(\nu, B)$; then λ is finitely additive and countably additive whenever μ and ν are. Now let $\hat{\lambda}$ be the unique finitely additive extension of λ to the ring generated by $\mathscr{B}(S)\times\mathscr{B}(T)$. If $\{C_i\times D_i\}_1^n$ is a finite, pairwise disjoint collection in $\mathscr{B}(S)\times\mathscr{B}(T)$ such that $\bigcup_1^n C_i\times D_i\subseteq A\times B$, then

$$\begin{aligned}\sum_i^n\|(\mu\times\nu)(C_i\times D_i)\| &= \sum_1^n\|\mu(C_i)\cdot\nu(D_i)\|\leqslant\sum_1^n k\cdot\|\mu(C_i)\|\cdot\|\nu(D_i)\|\\ &\leqslant\sum_1^n k\cdot\mathscr{V}(\mu, C_i)\cdot\mathscr{V}(\nu, D_i) = \sum_1^n k\cdot\lambda(C_i\times D_i) = k\cdot\hat{\lambda}\Big(\bigcup_1^n C_i\times D_i\Big)\\ &\leqslant k\cdot\lambda(A\times B) = k\cdot\mathscr{V}(\mu, A)\cdot\mathscr{V}(\nu, B).\end{aligned}$$

Thus, $\mathscr{V}(\mu\times\nu, A\times B)\leqslant k\cdot\mathscr{V}(\mu, A)\cdot\mathscr{V}(\nu, B)$. Let μ and ν be countably additive; then λ is countably additive. Let $\{E_i\}_1^\infty$ be a sequence in $\mathscr{B}(S)\times\times\mathscr{B}(T)$ such that $\bigcup_1^\infty E_i\in\mathscr{B}(S)\times\mathscr{B}(T)$. Let $(\mu\times\nu)^\wedge$ be the finitely additive extension of $\mu\times\nu$ to the ring generated by $\mathscr{B}(S)\times\mathscr{B}(T)$. Then

$$\begin{aligned}\Big\|\sum_1^n(\mu\times\nu)(E_i)-(\mu\times\nu)\Big(\bigcup_1^\infty E_i\Big)\Big\| &= \Big\|(\mu\times\nu)^\wedge\Big(\bigcup_1^n(E_i)\Big)-(\mu\times\nu)^\wedge\Big(\bigcup_1^\infty E_i\Big)\Big\|\\ &= \Big\|(\mu\times\nu)^\wedge\Big(\bigcup_{n+1}^\infty E_i\Big)\Big\|\leqslant\mathscr{V}\Big(\mu\times\nu, \bigcup_{n+1}^\infty E_i\Big)\leqslant k\cdot\hat{\lambda}\Big(\bigcup_{n+1}^\infty E_i\Big)\to 0.\end{aligned}$$

DEFINITION II. 6: *Let $\mathscr{S}$ be a semiring of subsets of the topological space S and μ: $\mathscr{S}\to X$. μ is said to be regular provided that for each $\varepsilon>0$ and $A\in\mathscr{S}$, there exist $B, C\in\mathscr{S}$ such that $\bar{B}$ is compact,*

$$B\subseteq\bar{B}\subseteq A\subseteq C^{\text{int.}}\subseteq C \quad\text{and}\quad \mathscr{V}(\mu, C\setminus B)<\varepsilon.$$

It is well known that a regular, finitely additive set function is countably additive and that if μ is finitely additive and $\|\mu\|<\infty$, then μ is regular if and only if $\mathscr{V}(\mu)$ is regular.

THEOREM II. 7. *If μ: $\mathscr{B}(S)\to X$ and ν: $\mathscr{B}(T)\to Y$ are finitely additive and regular, then $\mu\times\nu$: $\mathscr{B}(S)\times\mathscr{B}(T)\to Z$ is finitely additive and regular (and thus countably additive).*

Proof. We have only to prove that $\mu\times\nu$ is regular. If $A\times B\subseteq C\times D$, then $(C\times D)\setminus(A\times B)=[(C\setminus A)\times D]\cup[C\times(D\setminus B)]$. Let $A\times B\in\mathscr{B}(S)\times\times\mathscr{B}(T)$ and $\varepsilon>0$; let

$$\delta_1=\min\bigl(\varepsilon\cdot[2(\mathscr{V}(\nu,B)+1)]^{-1},1\bigr),$$
$$\delta_2=\min\bigl(\varepsilon\cdot[2(\mathscr{V}(\mu,A)+1)]^{-1},1\bigr).$$

Then there exist $C,E\in\mathscr{B}(S)$ and $D,F\in\mathscr{B}(T)$ with $\bar{C}$ and $\bar{D}$ compact such that $C\subseteq\bar{C}\subseteq A\subseteq E^{\text{int.}}\subseteq E, D\subseteq\bar{D}\subseteq B\subseteq F^{\text{int.}}\subseteq F$ and $\mathscr{V}(\mu,E\setminus C)<\frac{\delta_1}{k}$, and $\mathscr{V}(\nu,F\setminus D)<\frac{\delta_2}{k}$. Then $C\times D\subseteq\overline{C\times D}\subseteq A\times B\subseteq(E\times F)^{\text{int.}}\subseteq E\times F$, $\overline{C\times D}$ is compact, and

$$\mathscr{V}(\mu\times\nu,(E\times F\setminus C\times D))\leqslant\mathscr{V}(\mu\times\nu,(E\setminus C)\times D)+\mathscr{V}(\mu\times\nu,E\times(F\setminus D))$$
$$\leqslant k\cdot\mathscr{V}(\mu,E\setminus C)\cdot\mathscr{V}(\nu,D)+k\mathscr{V}(\mu,E)\cdot\mathscr{V}(\nu,F\setminus D)<k\cdot\frac{\delta_1}{k}\mathscr{V}(\nu,D)+$$
$$+k\frac{\delta_2}{k}\mathscr{V}(\mu,E)\leqslant\delta_1\cdot(\mathscr{V}(\nu,B)+1)+\delta_2\cdot(\mathscr{V}(\mu,A)+1)<\varepsilon/2+\varepsilon/2=\varepsilon.$$

The preceding theorem shows that if μ and ν are "nice" functions on $\mathscr{B}(S)$ and $\mathscr{B}(T)$, then $\mu\times\nu$ is "nice" on $\mathscr{B}(S)\times\mathscr{B}(T)$. We now show that $\mu\times\nu$ has a unique "nice" extension on $\mathscr{B}(S\times T)$.

THEOREM II. 8. *Let μ: $\mathscr{B}(S)\to X$ and ν: $\mathscr{B}(T)\to Y$ be countably additive, regular, and of bounded variation; then there exists a unique λ: $(S\times T)\to Z$ such that λ is countably additive, regular, and of bounded variation and $\lambda(A\times B)=\mu(A)\cdot\nu(B)$ for all $A\in\mathscr{B}(S)$ and $B\in\mathscr{B}(T)$.*

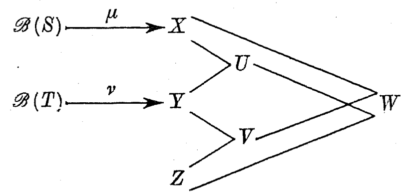
Proof. $\mu\times\nu$ is countably additive, regular, and of bounded variation on $\mathscr{B}(S)\times\mathscr{B}(T)$. Arsene and Stratila [1] have shown that there exists a unique countably additive, regular extension φ of bounded variation on the smallest σ-algebra containing $\mathscr{B}(S)\times\mathscr{B}(T)$; this σ-algebra contains the Baire subsets [5; 51.E]. Dinculeanu [3] has shown that there exists a unique countably additive, regular extension of φ of bounded variation on $\mathscr{B}(S\times T)$.

When we speak of "the extension of $\mu\times\nu$ to $\mathscr{B}(S\times T)$" we shall mean the unique extension described in Theorem II. 8; this extension will be denoted (somewhat ambiguously) by $\mu\times\nu$.

III. A FUBINI - TYPE THEOREM

Let X, Y, Z, U, V, and W be Banach spaces and suppose that there exist bilinear maps from $X\times Y$ into U, from $Y\times Z$ into V, from $X\times V$ into W, and from $U\times Z$ into W (all denoted by "$\cdot$" and such that $\|a\cdot b\|\leqslant k\cdot\|a\|\cdot\|b\|$ for some $k>0$) such that $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ for $x\in X$, $y\in Y$,

and $z\in Z$. As in the preceding chapter, let S and T be locally compact Hausdorff spaces with Borel σ-rings $\mathscr{B}(S)$ and $\mathscr{B}(T)$. Let μ: $\mathscr{B}(S)\to X$ and ν: $\mathscr{B}(T)\to Y$ be countably additive, regular, and of bounded variation.



1. Preliminaries. We shall consider a Bochner type integration of vector-valued function with respect to vector-valued set functions. We shall consider functions from T into Z and the set function ν from $\mathscr{B}(T)$ to Y but we note that any other σ-ring of any set and any bilinear system of Banach spaces would give the same results. The proofs for the unnumbered theorems listed below, as well as pertinant background information may be found in Dinculeanu [4] or Hille and Phillips [6].

DEFINITION. A function φ: $T\to Z$ is said to be a *simple function* if $\varphi=\sum_{i=1}^{n}\chi_{A_i}z_i$, where each $A_i\in\mathscr{B}(T)$, each $z_i\in Z$ and $A_i\cap A_j=\emptyset$ if $i\neq j$. The function φ: $T\to Z$ is *countably valued* if $\varphi=\sum_{i=1}^{\infty}\chi_{A_i}z_i$, where each $A_i\in\mathscr{B}(T)$, each $z_i\in Z$ and $A_i\cap A_j=\emptyset$ if $i\neq j$. A *simple* or countably valued function $\varphi=\sum\chi_{A_i}z_i$ is *ν-integrable* if $\sum\mathscr{V}(\nu,A_i)\|z_i\|$ is finite in which case we define

$$\int\varphi d\nu=\int\sum\chi_{A_i}z_i d\nu=\sum\nu(A_i)\cdot z_i.$$

DEFINITION. Let f: $T\to Z$; then f is

(a) *strongly measurable* if there exists a sequence $\{\varphi_n\}_1^\infty$ of countably valued functions on T to Z such that $\varphi_n\to f$ a.e. $[\mathscr{V}(\nu)]$;

(b) *weakly measurable* if for $\gamma\in Z^*$, $\gamma\circ f$ is a scalar valued measurable function.

A strongly measurable function f is ν-integrable if there is a sequence $\{\varphi_n\}$ of countably valued functions on T to Z such that $\varphi_n\to f$ a.e. $[\mathscr{V}(\nu)]$ and $\lim_{m,n\to\infty}\int\|\varphi_m-\varphi_n\|d\mathscr{V}(\nu)=0$. In this case $\{\int\varphi_n d\nu\}_1^\infty$ is a Cauchy sequence in Z and we define $\int f d\nu=\lim_{n\to\infty}\int\varphi_n d\nu$. We note that $\int(\cdot)d\nu$ is a linear function from the linear space of ν-integrable functions into V and $\|\int f d\nu\|\leqslant\int\|f\|d\mathscr{V}(\nu)$.

THEOREM. [6] *Let* $f\colon T\to Z$; *these are equivalent*:

(a) f *is strongly measurable*,

(b) f *is weakly measurable and there exists* $E\in\mathscr{B}(T)$ *such that* $\mathscr{V}(\nu,E)=0$ *and* $f(T\setminus E)$ *is separable* (f *is said be to almost separably valued*).

We also note that if f is strongly measurable, then there is a sequence of countably valued functions converging to f uniformly a.e. $[\mathscr{V}(\nu)]$.

THEOREM. *Let* $f\colon T\to Z$ *be strongly measurable, these are equivalent*:

(a) f *is* ν*-integrable.*

(b) $\|f\|$ *is* ν*-integrable.*

(c) f *is* $\mathscr{V}(\nu)$*-integrable.*

(d) $\|f\|$ *is* $\mathscr{V}(\nu)$*-integrable.*

DOMINATED CONVERGENCE THEOREM. *Let* $\{f_n\}$ *be a sequence of strongly measurable functions on* T *and* $f_n\to f$ *a.e.* $[\mathscr{V}(\nu)]$. *If there exist a non-negative,* $\mathscr{V}(\nu)$*-integrable function* g *on* T *such that* $\|f_n(t)\|\leqslant g(t)$ *a.e.* $[\mathscr{V}(\nu)]$ *for all* n, *then* f *and each* f_n *are integrable and* $\lim_{n\to\infty}\int f_n d\nu=\int f d\nu$.

We shall also use the following result which we are unable to find in the literature and whose proof is due to Fred B. Wright. This result allows us to apply the dominated convergence theorem in many cases.

THEOREM (WRIGHT). *Let* $f\colon T\to Z$ *be* ν*-integrable. Then there exists a sequence* $\{\varphi_n\}_1^\infty$ *of countably valued* ν*-integrable functions and a non-negative,* $\mathscr{V}(\nu)$*-integrable function* φ *and* T *such that* $\|\varphi_n(t)\|\leqslant\varphi(t)$ *a.e.* $[\mathscr{V}(\nu)]$ *for all* n *and* $\varphi_n\to f$ *uniformly a.e.* $[\mathscr{V}(\nu)]$.

Proof. Let $A=\{t\in T\colon \|f(t)\|>0\}$; then $A\in\mathscr{B}(T)$. For each pair j, k of positive integers, let $\varepsilon_{j,k}=\left[k\cdot 2^j\left(\mathscr{V}(\nu,A)+1\right)\right]^{-1}$; then for each k, $\sum_{j=1}^\infty \varepsilon_{j,k}\mathscr{V}(\nu,A)<1/k<\infty$ and we define $\psi_k(t)=\|f(t)\|+\sum_{j=1}^\infty\varepsilon_{j,k}\chi_A(t)$ and note that $\psi_1(t)\geqslant\psi_2(t)\geqslant\ldots$ and that each ψ_k is $\mathscr{V}(\nu)$-integrable. Since f is strongly measurable, there exists a sequence $\{h_n\}_1^\infty$ of countably valued functions on T such that $h_n\to f$ uniformly a.e. $[\mathscr{V}(\nu)]$. Then for each pair j, k there is an integer $N_{j,k}$ such that if $n\geqslant N_{j,k}$, then $\|h_n(t)-f(t)\|\leqslant\varepsilon_{j,k}$ a.e. $[\mathscr{V}(\nu)]$. For each k, define $\varphi_k(t)=\sum_{j=1}^\infty h_{N_{j,k}}(t)\chi_A(t)$. Each φ_k is countably valued, integrable, and $\|\varphi_k(t)-f(t)\|\leqslant 1/k$ a.e. $[\mathscr{V}(\nu)]$ so $\varphi_k\to f$ uniformly a.e. $[\mathscr{V}(\nu)]$. Let $\varphi(t)=\psi_1(t)$; then

$$\|\varphi_k(t)\|\leqslant\|f(t)\|+\sum_{j=1}^\infty\varepsilon_{j,k}\chi_A(t)=\psi_k(t)\leqslant\psi_1(t)=\varphi(t).$$

2. The Fubini theorem. For $E\in\mathscr{B}(S\times T)$ and $s\in S$, let $E_s=\{t\in T\colon (s,t)\in E\}$; for $t\in T$, let $E^t=\{s\in S\colon (s,t)\in E\}$; then $E_s\in\mathscr{B}(T)$ and $E^t\in\mathscr{B}(S)$. Define $\varphi_E\colon S\to Y$ by $\varphi_E(s)=\nu(E_s)$ and $\psi_E\colon T\to X$ by $\psi_E(t)=\mu(E^t)$.

THEOREM III. 1. *Let* $E\in\mathscr{B}(S\times T)$;

(a) *if* $\nu(\mathscr{B}(T))$ *is separable, then* $\varphi_E\colon S\to Y$ *is* μ*-integrable and* $\int\varphi_E d\mu=(\mu\times\nu)(E)$.

(b) *if* $\mu(\mathscr{B}(S))$ *is separable, then* $\psi_E\colon T\to X$ *is* ν*-integrable and* $\int\psi_E d\nu=(\mu\times\nu)(E)$.

Proof. We shall prove only (a) since (b) has a similar proof. $\varphi_E(S)\subseteq\nu(\mathscr{B}(T))$ which is separable so to show that φ_E is strongly measurable we need only show that is it weakly measurable. Let $\gamma\in Y^*$; then χ_E, from $S\times T$ to the reals, is measurable and $\int\chi_E d\left(\mathscr{V}(\mu)\times\mathscr{V}(\gamma\cdot\nu)\right)=[\mathscr{V}(\mu)\times\mathscr{V}(\gamma\cdot\nu)](E)<\infty$, so by the Fubini Theorem for real valued, countably additive set functions, $\int\chi_E(s,t)\,d(\gamma\circ\nu)(t)$ is measurable as a function in s. But $\int\chi_E(s,t)\,d(\gamma\cdot\nu)(t)=(\gamma\circ\nu_E)(S)$. Thus φ_E is strongly measurable,

$$\int\|\varphi_E(S)\|\,d\mathscr{V}(\mu)=\int\|\nu(E_s)\|\,d\mathscr{V}(\mu)\leqslant\int\mathscr{V}(\nu,E_s)\,d\mathscr{V}(\mu)$$
$$=[\mathscr{V}(\mu)\times\mathscr{V}(\nu)](E)<\infty.$$

If $E=A\times B$, where $A\in\mathscr{B}(S)$ and $B\in\mathscr{B}(T)$, then

$$\varphi_E(S)=\nu(E_s)=\chi_A(S)\cdot\nu(B)\ \text{so}\int\varphi_E d\mu=\int\chi_A d\mu\cdot\nu(B)=(\mu\times\nu)(E).$$

Now let $\varrho(E)=\int\varphi_E d\mu$ on $\mathscr{B}(S\times T)$; then ϱ is countably additive, regular, and of bounded variation on $\mathscr{B}(S\times T)$ and $\varrho=\mu\times\nu$ on $\mathscr{B}(S)\times\times\mathscr{B}(T)$. Thus $\varrho=\mu\times\nu$ on $\mathscr{B}(S\times T)$.

We note that the separability condition for the set function ν can be insured by any of the following assumptions:

(1) Y is separable.

(2) ν has relatively compact range.

(3) ν has a Radon–Nikodym derivative with respect to $\mathscr{V}(\nu)$, in the sense that there is a $\mathscr{V}(\nu)$-integrable function $f\colon T\to Y$ such that for $A\in\mathscr{B}(T)$, $\nu(A)=\int_A f d(\nu)$ (see Uhl [9]). For example, Y could be a reflexive space and any such ν would have a Radon–Nikodym derivative (see Phillips [7]).

(4) There is a countable subcollection in $\mathscr{B}(T)$, dense in $\mathscr{B}(T)$ with respect to the topology determined by the pseudo-metric

$$d(A,B)=\mathscr{V}(\nu,A\setminus B)+\mathscr{V}(\nu,B\setminus A).$$

COROLLARY III. 1. 1. *If* $\nu(\mathscr{B}(T))$ *is separable and* φ *is a simple function from* $S\times T$ *to* Z, *then*

$$\int\varphi d(\mu\times\nu)=\int\left(\int\varphi(s,t)\,d\nu(t)\right)d\mu(s).$$

Proof. Let $E\in\mathscr{B}(S\times T)$; then

$$\int\chi_E d(\mu\times\nu)=(\mu\times\nu)(E)=\int\nu(E_s)\,d\mu(s)=\int\left(\int\chi_{E_s}(t)\,d\nu(t)\right)d\mu(s)$$
$$=\int\left(\int\chi_E(s,t)\,d\nu(t)\right)d\mu(s).$$

The result for a simple function is a direct consequence of the linearity of the integral.

We are now ready for the Fubini-type theorem. The Fubini Theorem for scalar valued set functions involves two pairs of interated integrals; however, in our present context, only one of the pairs makes sense.

THEOREM III. 2 (FUBINI-TYPE THEOREM). *Let $\nu(\mathscr{B}(T))$ be separable and $f\colon S\times T\to Z$ be $\mathscr{V}(\mu)\times\mathscr{V}(\nu)$-integrable, then*

(a) $\int f(s,t)\,d\nu(t)$ *(as a function in s) is μ-integrable and*

(b) $\int f d(\mu\times\nu) = \int\big(\int f(s,t)\,d\nu(t)\big)\,d\mu(s)$.

Proof. (a) Since f is $\mathscr{V}(\mu)\times\mathscr{V}(\nu)$-integrable, then $\|f\|$ is $\mathscr{V}(\mu)\times\mathscr{V}(\nu)$-integrable so by the usual Fubini Theorem $\int\|f(s,t)\|\,d\mathscr{V}(\nu)(t)$ exists for almost all s, is measurable and $\mathscr{V}(\mu)$-integrable. Now without loss of generality f may be assumed to be separably valued. For $\gamma\in Z^*$, $(\gamma\cdot f)$ is measurable on $S\times T$ to $\mathscr{R}$ and thus $(\gamma\cdot f)(s,\cdot)$ is measurable for almost all s. Therefore, $f(s,\cdot)$ is weakly measurable and separably valued (since f is separably valued). From the first sentence of the proof, $f(s,\cdot)$ is ν-integrable so $\int f(s,t)\,d\nu(t)$ exists for almost all s and is μ-integrable.

(b) We prove the result first for countably valued integrable functions and use Wright's result and the dominated convergence theorem to get the result in general. Suppose $\varphi = \sum_1^\infty \chi_{A_i}\cdot z_i$ (where each $A_i\in\mathscr{B}(S\times T)$ and $A_i\cap A_j=\emptyset$ if $i\neq j$) and φ is $\mathscr{V}(\mu)\times\mathscr{V}(\nu)$ integrable. For all n, let $\varphi_n = \sum_1^n \chi_{A_i}\cdot z_i$ if $\psi(s,t)=\|\varphi(s,t)\|$, then ψ is $\mathscr{V}(\mu)\times\mathscr{V}(\nu)$-integrable, $\|\varphi_n(s,t)\|\leqslant\psi(s,t)$ for all n and $\varphi_n\to\varphi$ on $S\times T$. Thus by the dominated convergence theorem,

$$\int\varphi d(\mu\times\nu) = \lim_{n\to\infty}\int\varphi_n d(\mu\times\nu) = \lim_{n\to\infty}\Big[\int\Big(\int\varphi_n(s,t)\,d\nu(t)\Big)\,d\mu(s)\Big]$$

$$= \int\Big(\lim_{n\to\infty}\int\varphi_n(s,t)\,d\nu(t)\Big)\,d\mu(s)$$

(since $\|\int\varphi_n(s,t)\,d\nu(t)\|\leqslant\int|\psi(s,t)|\,d\mu(\nu)(t)$ which is μ-integrable)

$$= \int\Big(\int\lim_{n\to\infty}\varphi_n(s,t)\,d\nu(t)\Big)\,d\mu(s) = \int\Big(\int\varphi(s,t)\,d\nu(t)\Big)\,d\mu(s).$$

Now if $f\colon S\times T\to Z$ is $\mathscr{V}(\mu)\times\mathscr{V}(\nu)$-integrable, then there exists a sequence $\{\psi_n\}_1^\infty$ of countably valued, $\mathscr{V}(\mu)\times\mathscr{V}(\nu)$-integrable functions on $S\times T$ to Z and a non-negative, $\mathscr{V}(\mu)\times\mathscr{V}(\nu)$-integrable function ψ on $S\times T$ such that $\|\psi_n(s,t)\|\leqslant\psi(s,t)$ for all n and $\psi_n(s,t)\to f(s,t)$ a.e. Thus by use of the dominated convergence theorem as above, $\iint fd(\mu\times\nu) = \int\big(\int f(s,t)\,d\nu(t)\big)\,d\mu(s)$.

IV. CONVOLUTION OF VECTOR VALUED SET FUNCTIONS

Let X, Y, and Z be as in Chapter II and let G be a locally compact Hausdorff topological semigroup. Let $\mathscr{M}(X)$ be the collection of all countably additive, regular set functions of bounded variation from $\mathscr{B}(G)$ into X (similar definitions are made for $\mathscr{M}(Y)$ and $\mathscr{M}(Z)$). We recall that if $E\in\mathscr{B}(G)$ and $E_2=\{(g,h)\colon gh\in E\}$, then $E_2\in\mathscr{B}(G\times G)$.

DEFINITION IV. 1. Let $\mu\in\mathscr{M}(X)$ and $\nu\in\mathscr{M}(Y)$; then $\mu\times\nu\colon\mathscr{B}(G\times G)\to Z$. For each $E\in\mathscr{B}(G)$, let $(\mu*\nu)(E)=(\mu\times\nu)(E_2)$. Then $\mu*\nu$ is called the *convolution of μ and ν.*

THEOREM IV. 2. *If $\mu\in\mathscr{M}(X)$ and $\nu\in\mathscr{M}(Y)$, then $\mu*\nu\in\mathscr{M}(Z)$. Moreover, $\mathscr{V}(\mu*\nu)<k\mathscr{V}(\mu)*\mathscr{V}(\nu)$.*

Proof. Let λ be the extension of $\mu\times\nu$ to $\mathscr{B}(G\times G)$.

(a) $\mu*\nu$ is countably additive: Let $\{E_i\}_1^\infty$ be a pairwise disjoint sequence in $\mathscr{B}(G)$. Then

$$(\mu*\nu)\Big(\bigcup_1^\infty E_i\Big) = \lambda\Big(\bigcup_1^\infty E_i\Big)_2 = \lambda\Big(\bigcup_1^\infty (E_i)_2\Big) = \sum_1^\infty\lambda\big((E_i)_2\big) = \sum_1^\infty(\mu*\nu)(E_i).$$

(b) $\|\mu*\nu\|<\infty$ and $\mathscr{V}(\mu*\nu)\leqslant k\mathscr{V}(\mu)*\mathscr{V}(\nu)$:

First note that $\mathscr{V}(\mu)$ and $\mathscr{V}(\nu)$ are members of $\mathscr{M}(\boldsymbol{R})$ (where $\boldsymbol{R}$ denotes the real numbers).

$$\|(\mu*\nu)(E)\| = \|\lambda(E_2)\|\leqslant\mathscr{V}(\lambda,E_2)\leqslant k[\mathscr{V}(\mu)\times\mathscr{V}(\nu)](E_2)$$
$$= k[\mathscr{V}(\mu)*\mathscr{V}(\nu)](E).$$

Now

$$\mathscr{V}(\mu*\nu,A) = \sup\Big\{\sum\|\mu*\nu(E_i)\|\Big\}\leqslant\sup\Big\{\sum k[\mathscr{V}(u\mu)\times\mathscr{V}(\nu)]((E_i)_2)\Big\}$$
$$= k\sup\Big\{[\mathscr{V}(\mu)\times\mathscr{V}(\nu)]\Big(\bigcup_1^\infty (E_i)_2\Big)\Big\} = k[\mathscr{V}(\mu)\times\mathscr{V}(\nu)](A_2).$$

Thus $\|\mu*\nu\|<\infty$.

(c) $\mu*\nu$ is regular on $\mathscr{B}(G)$:

$\mu*\nu$ is regular if and only if $\mathscr{V}(\mu*\nu)$ is regular but $\mathscr{V}(\mu*\nu)\leqslant k\mathscr{V}(\mu)*{}*\mathscr{V}(\nu)$ which is regular so $\mathscr{V}(\mu*\nu)$ is also regular.

Suppose X, Y, Z, U, V, and W form an associative bilinear system as in Chapter III.

LEMMA. *If $\mu\in\mathscr{M}(X)$, $\nu\in\mathscr{M}(Y)$ and φ is a simple function from G to Z, then*

$$\int\varphi(g)\,d(\mu*\nu)(g) = \int\varphi(gh)\,d(\mu\times\nu)(g,h).$$

Proof. If $\varphi=\chi_A\cdot z$ with $A\in\mathscr{B}(G)$ and $z\in Z$, then $\chi_A(gh)=\chi_{A_2}(g,h)$ so

$$\int\varphi(gh)\,d(\mu\times\nu) = \int\chi_A(gh)\cdot z\,d(\mu\times\nu) = \int\chi_{A_2}(g,h)\,d(\mu\times\nu)\cdot z = (\mu\times\nu)(A_2)\cdot z$$
$$= (\mu*\nu)(A)\cdot z = \int\chi_A\cdot z\,d(\mu*\nu) = \int\varphi\,d(\mu*\nu).$$

The equality for simple functions follows from the linearity of the integrals.

THEOREM IV. 3. *Let* $\mu\in\mathcal{M}(X)$, $\nu\in\mathcal{M}(Y)$, $f\colon G\to Z$ *be* $\mathcal{V}(\mu)*\mathcal{V}(\nu)$ *integrable; then* $\int f d(\mu*\nu)=\int f(gh)d(\mu\times\nu)$.

Proof. We prove the result first for countably-valued, $\mathcal{V}(\mu)*\mathcal{V}(\nu)$-integrable functions. Suppose $\varphi=\sum_1^\infty \chi_{A_i}\cdot z_i$ (where each $A_i\in\mathcal{B}(G)$ and $A_i\cap A_j=\emptyset$ if $i\neq j$) and φ is $\mathcal{V}(\mu)*\mathcal{V}(\nu)$-integrable. For all n, let $\varphi_n=\sum_1^\infty \chi_{A_i}\cdot z_i$; if $\psi(g)=\|\varphi(g)\|$, then ψ is $\mathcal{V}(\mu)*\mathcal{V}(\nu)$-integrable and so $\psi(gh)$ is $\mathcal{V}(\mu)\times\mathcal{V}(\nu)$-integrable. Now $\|\varphi_n(g)\|\leqslant\psi(g)$, $\|g_n(\varphi h)\|\leqslant\psi(gh)$, and $\varphi_n\to\varphi$ on G. Thus

$$\int\varphi_n d(\mu*\nu)\to\int\varphi d(\mu*\nu),$$
$$\int\varphi_n(gh)d(\mu\times\nu)\to\int\varphi(gh)d(\mu\times\nu),$$

so

$$\int\varphi d(\mu*\nu)=\int\varphi(gh)d(\mu\times\nu).$$

Now let f be $\mathcal{V}(\mu)*\mathcal{V}(\nu)$-integrable. It is clear that $f(gh)$ is strongly measurable (as a function on $G\times G$). By Wright's result cited in Chapter IV, there is a sequence $\{\varphi_n\}_1^\infty$ of countably valued $\mathcal{V}(\mu)*\mathcal{V}(\nu)$-integrable functions on G to Z and a non-negative $\mathcal{V}(\mu)*\mathcal{V}(\nu)$-integrable function ψ on G such that $\|\varphi_n(g)\|\leqslant\psi(g)$ a.e. $[\mathcal{V}(\mu)*\mathcal{V}(\nu)]$ and $\varphi_n\to f$ uniformly a.e. $[\mathcal{V}(\mu)*\mathcal{V}(\nu)]$. Now if $\varphi_n\to f$ uniformly on $G\setminus E$ and $\big(\mathcal{V}(\mu)*\mathcal{V}(\nu)\big)(E)=0$, then $\varphi_n(gh)\to f(gh)$ uniformly on $(G\times G)\setminus E_2$ and $[\mathcal{V}(\mu)\times\mathcal{V}(\nu)](E_2)=\big(\mathcal{V}(\mu)*\mathcal{V}(\nu)\big)(E)=0$. Also $\|\varphi_n(gh)\|\leqslant\psi(gh)$ a.e. $[\mathcal{V}(\mu)\times\mathcal{V}(\nu)]$; thus

$$\int\varphi_n(gh)d(\mu\times\nu)\to\int f(gh)d(\mu\times\nu),$$
$$\int\varphi_n d(\mu*\nu)\to\int f d(\mu*\nu)$$

and thus

$$\int f(gh)d(\mu\times\nu)=\int f d(\mu*\nu).$$

LEMMA. *If* $E\in\mathcal{B}(G)$, *and* $g,h\in G$, *then* $\big(([E_2]_g)_2\big)_h=(E_2)_{gh}$.

Proof.

$$e\in\big([(E_2)_g]_2\big)_h\Leftrightarrow(h,e)\in[(E_2)_g]_2\Leftrightarrow he\in(E_2)_g\Leftrightarrow(g,he)\in E_2$$
$$\Leftrightarrow g(he)\in E\Leftrightarrow(gh)e\in E\Leftrightarrow(gh,e)\in E_2\Leftrightarrow e\in(E_2)_{gh}.$$

THEOREM IV. 4. *Let* $\mu\in\mathcal{M}(X)$, $\nu\in\mathcal{M}(Y)$ *and* $\lambda\in\mathcal{M}(Z)$ *and suppose* $\nu(\mathcal{B}(A))$ *and* $\lambda(\mathcal{B}(A))$ *are separable. Then* $(\mu*\nu)*\lambda=\mu*(\nu*\lambda)$.

Proof. Let $E\in\mathcal{B}(G)$. Then

$$\begin{aligned}(\mu*\nu)*\lambda(E)&=(\mu*\nu)\times\lambda(E_2)=\int(\mu*\nu)[(E_2)_g]d\lambda(g)\\&=\int\big[(\mu\times\nu)\big([(E_2)_g]_2\big)\big]d\lambda(g)=\int\Big[\int\mu\Big(\big([(E_2)_g]_2\big)_h\Big)d\nu(h)\Big]d\lambda(g)\\&=\int\Big[\int[(E_2)_{gh}]d\nu(h)\Big]d\lambda(g)\\&=\int\int\mu[(E_2)_{gh}]d(\mu\times\nu)=\int\mu[(E_2)_g]d(\nu*\lambda)\\&=[\mu\times(\nu*\lambda)](E_2)=[\mu*(\nu*\lambda)](E).\end{aligned}$$

References

[1] Gr. Arsene, and S. Stratila, *Prolongement des measures vectorielles*, Rev. Math. Pures Appl. 10 (1955), pp. 333–338.

[2] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fund. Math. 20 (1933), pp. 262–276.

[3] N. Dinculeanu, *On regular vector measures*, Acta Sci. Math. Szeged 24 (1963), pp. 236–243.

[4] — *Vector Measures*, Oxford 1967.

[5] P. R. Halmos, *Measure Theory*, Princeton 1950.

[6] E. Hille, and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Providence 1957.

[7] R. S. Phillips, *On weakly compact subsets of a Banach space*, Amer. J. Math. 65 (1943), pp. 108–136.

[8] W. Rudin, *Fourier analysis on groups*, New York 1962.

[9] J. J. Uhl Jr., *The range of a vector-valued measure*, Proc. Amer. Math. Soc. 23 (1969), pp. 158–163.

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