References

[1] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), pp. 273–299.
[2] R. A. Hunt, *On $L(p, q)$ spaces*, L'Enseignement Math. 12 (1966), pp. 249–276.
[3] P. Kree, *Interpolation d'espaces vectoriels qui ne sont ni normes, ni complets. Applications*, Ann. Inst. Fourier, Grenoble, 17 (1967), pp. 137–174.
[4] J. L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes Sci. Publ. Math. 19 (1964), pp. 5–68.
[5] J. Peetre, *Sur le nombre de parametres dans la definition de certains espaces d'interpolation*, Ricerche Mat. (1963), pp. 248–261.
[6] N. M. Rivière, *Interpolation theory in s-Banach spaces*, Thesis, University of Chicago, 1965, pp. 1–35.
[7] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci. Cl. III 5 (1957), pp. 471–473.
[8] E. M. Stein, *Interpolation of linear operators*, Trans. A. M. S. 83 (1956), pp. 482–492.
[9] — and G. Weiss, *An extension of a theorem of Marcinkiewicz and some of its applications*, J. Math. Mech. 8 (1959), pp. 263–284.
[10] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge 1968.

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The distribution of the values of a random function in the unit disk

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Abstract. Let $f(z)$ be defined in the unit disk by a power series whose coefficients are independent random variables and let $n(t, b)$ denote the number of zeros of $f(z)-b$ in $|z| \leqslant t$. It is shown that, for almost all functions of the family considered, $\inf_{|b|\leqslant K} \int_{1/2}^{r} \frac{n(t, b)}{t}\,dt$ has a well defined asymptotic behaviour. Furthermore $f(z)$ almost surely takes every finite value in every open sector of the unit disk. The paper contains some inequalities for $\int_E \log|X|\,d\mu$, where X is a random variable defined on a measure space $(\Omega, \mathcal{M}, \mu)$ and E belongs to $\mathcal{M}$ but is otherwise arbitrary.

§ 1. Introduction and principal results. This paper is concerned with the behaviour of functions

$$f(z) = \sum_{0}^{\infty} a_n z^n \tag{1.1}$$

defined in the unit disk for which the coefficients a_n are independent random variables. Our object is to show that the family (1.1) has certain properties almost surely. This implies some statistical basis and it becomes necessary to define this statistical basis precisely. Many years ago Littlewood and Offord [3] studied a similar problem for the family of entire functions

$$\sum_{0}^{\infty} \varepsilon_n a_n z^n \tag{1.2}$$

in which the coefficients a_n were given and the ε_n took the values ± 1 with equal probability. In 1964 one of the authors [4] returned to this problem and established the basic results of Littlewood and Offord under very general conditions on the distribution functions of the coefficients a_n. From this it followed that the behaviour of the family of entire functions was largely independent of the particular distribution functions chosen for the coefficients a_n. For this reason in the present investigation we confine ourselves to one distribution, the Steinhaus distribution which is defined as follows. We consider the family

$$f(z, \omega) = \sum_0^\infty e^{2\pi i \vartheta_n(\omega)} a_n z^n, \tag{1.3}$$

where the a_n are given real or complex numbers and are such that

$$\limsup_{n\to\infty} (|a_n|)^{1/n} = 1 \tag{1.4}$$

and

$$\sum_0^\infty |a_n|^2 = \infty. \tag{1.5}$$

The $\vartheta_n(\omega)$ are independent random variables uniformly distributed in $(0, 1)$. This particular distribution leaves the moduli of the coefficients unchanged and this is what commends it to us. However, our methods, although they do not cover the case of the family (1.2), do cover any distribution whose characteristic function satisfies a certain order condition at infinity. Also it is not necessary to assume that the expectations of the coefficients are zero; it is enough for them to be smaller than the standard deviation. The reader who is interested will find the necessary tools in [5].

There is a considerable literature on these problems (cf. 2) and we mention particularly the work of Zygmund and Kahane which inspired the present investigation. Zygmund [1, p. 157 and 2, p. 127] has shown that almost all functions of a family of type (1.2) map the unit disk onto a set which is everywhere dense in the complex plane. Later Kahane [2, p. 137] proved that if

$$\lim_{n\to\infty} \frac{a_n}{\sqrt{(n \log n)}} = \infty,$$

then a Gaussian–Taylor series almost surely takes every complex value. In a sense the results of the present paper can be regarded as completing those of Zygmund and Kahane because we shall show that, subject to (1.4) and (1.5), the function (1.3) almost surely takes every complex value infinitely often in every sector of the unit disk.

We proceed to state our theorems. We write $n(r, b, \omega)$ for the number of zeros of the function

$$\sum_0^\infty e^{2\pi i \vartheta_n(\omega)} a_n z^n - b$$

in the disk $|z| < r$. We write

$$N^*(r, K; \omega) = \sup_{|b|\leqslant K} \int_{1/2}^{r} \frac{n(t, b; \omega)}{t}\, dt \tag{1.6}$$

and

$$N_*(r, K; \omega) = \inf_{|b|\leqslant K} \int_{1/2}^{r} \frac{n(t, b; \omega)}{t}\, d\tau, \tag{1.7}$$

where K is any finite real number and b any complex number satisfying the condition. We also write

$$\mathfrak{M}(r) = \Big(\sum_0^\infty |a_n|^2 r^{2n}\Big)^{1/2}. \tag{1.8}$$

We prove

THEOREM 1. *If* (1.4) *and* (1.5) *hold and if K and δ are any positive real numbers, then, for almost all functions of the family* (1.3) *we have*

$$\lim_{r\to 1} \frac{N^*(r, K; \omega)}{\log \mathfrak{M}(r)} = 1, \tag{1.9}$$

$$\limsup_{r\to 1} \frac{N_*(r, K; \omega)}{\log \mathfrak{M}(r)} = 1, \tag{1.10}$$

and

$$\liminf_{r\to 1} \frac{N^*(r, K; \omega) - N_*(r, K; \omega)}{(\log \mathfrak{M}(r))^\delta} = 0. \tag{1.11}$$

An immediate consequence of this theorem is that the integral

$$\int_{1/2}^{1} \frac{n(t, b; \omega)}{t}\, dt$$

diverges almost surely whatever the value of b and so almost all functions of the family (1.3) take every finite value b an infinity of times within the unit disk, and so map the unit disk onto the finite complex plane. However, we shall prove more namely

THEOREM 2. *If* (1.4) *and* (1.5) *holds, then the function* (1.3) *almost surely takes every finite value infinitely often in every sector of the unit disk.*

It is natural to ask if the limit superior in (1.10) can be replaced by limit. If $\mathfrak{M}(r)$ is of regular growth, to be precise if

$$\mathfrak{M}\big(r + 4(\mathfrak{M}(r))^{-1/2}\big) \leqslant C\,\mathfrak{M}(r)$$

or if

$$\mathfrak{M}\big(\tfrac{1}{2}(1+r)\big) \leqslant C\,\mathfrak{M}(r)$$

for some numerical constant C and all r, then this will be true and indeed it will follow from our proofs. However, if $\mathfrak{M}(r)$ is of very irregular growth it may well be that no more than (1.10) can hold.

The proofs of our theorems are long. In § 2 we give the results needed from probability theory. These are of independent interest and are stated as Theorems 3 and 4. In § 3 we apply these results to the particular family (1.3). The proof of (1.9) is given in § 4 and that of (1.10) and (1.11) in §§ 5, 6, 7 and 8. The principal result from function theory needed is Jensen's theorem which asserts that if $f(z)$ is regular in $|z| < r$ and $f(0) \neq 0$, then

$$\int_0^r \frac{n(t,0)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(re^{i\theta})}{f(0)}\right| d\theta.$$

By taking the range of integration in the first member as $(\frac{1}{2}, r)$, we eliminate any trouble at $z = 0$. The proof of (1.9) is a straightforward deduction from Theorem 4. The proofs of (1.10) and (1.11) are more difficult and here in addition to Jensen's theorem we use Carathéodory's inequality (cf. [6], 139) which we state as Lemma 5.3. Inproving (1.10) and (1.11) it is necessary to treat separately three cases. The first when $\mathfrak{M}(r)$ increases rapidly we deal with in § 5. The second when $\mathfrak{M}(r)$ satisfies an order condition but its rate of growth is not too slow is treated in § 6. The third that of an $\mathfrak{M}(r)$ of extremely slow growth is dealt with in § 7 and in § 8 we collect these results together. The final section is devoted to the proof of Theorem 2.

§ 2. Some probability theorems. In this section we develop the main probability results required. Some of these results were given in the paper [5] which contains the special case of Theorem 3 when $\delta = 1$.

THEOREM 3. *Let X be a real random variable defined on the measure space $(\Omega, \mathcal{M}, \mu)$ and suppose* (i) $\mathscr{E}(X) = \alpha$, (ii) $\mathscr{E}(|x-\alpha|) = \beta$, (iii) *the distribution function $F(t)$ of X is such that for some number γ satisfying $\gamma \geqslant \sup(|\alpha|, \beta, 1)$ and for positive numbers δ and l we have for* $0 < t < 1$

$$t^{-\delta}\{F(\gamma t) - F(-\gamma t)\} \leqslant l. \tag{2.1}$$

Then for any set $E \in \mathcal{M}$ of measure not exceeding e^{-1} we have

$$\int_E \log|X|\,d\mu = \mu(E)\log\gamma - \eta\mu(E)\log\mu(E), \tag{2.2}$$

where

$$-\delta^{-1}(2l+1) \leqslant \eta \leqslant 5.$$

It is very important for our applications that E should be quite arbitrary, apart from the requirement that it should be a member of $\mathcal{M}$.

Theorem 3 is a consequence of the following two lemmas. We observe that Lemma 2.2 does not require hypothesis (iii) of Theorem 3.

LEMMA 2.1. *Under the hypotheses of Theorem* 1

$$\int_E \log|X|\,d\mu \geqslant \mu(E)\log\gamma + \delta^{-1}(2l+1)\mu(E)\log\mu(E). \tag{2.3}$$

LEMMA 2.2. *If X is a real random variable defined on the space $(\Omega, \mathcal{M}, \mu)$ such that $\mathscr{E}(X) = \alpha$ and $\mathscr{E}(|X-\alpha|) = \beta$ and if $\gamma \geqslant \sup(|\alpha|, \beta)$, then*

$$\int_E \log^+|X|\,d\mu \leqslant \mu(E)\log^+\gamma - 2\mu(E)\log\mu(E) + 3\mu(E). \tag{2.4}$$

The proof of Lemma 2.2 is given in [5, p. 176]. We give only the proof of Lemma 2.1.

Proof of Lemma 2.1. We have

$$\int_E \log|X|\,d\mu = \mu(E)\log\gamma + \int_E \log\frac{|X|}{\gamma}\,d\mu.$$

Writing $X' = \inf(|X|\gamma^{-1}, 1)$ the integral in the second member is not less than

$$\int\limits_{\substack{E\\ X' < (\mu(E))^{1/\delta}}} \log X'\,d\mu + \int\limits_{\substack{E\\ X' \geqslant (\mu(E))^{1/\delta}}} \log X'\,d\mu = I_1 + I_2$$

and

$$I_1 \geqslant \int_0^{(\mu(E))^{1/\delta}} \log u\,dG(u),$$

where $G(u)$ is the distribution function of X'. But by hypothesis

$$G(u)|u|^{-\delta} \leqslant l \quad \text{for} \quad |u| \leqslant 1$$

and so, integrating by parts

$$\begin{aligned} I_1 &\geqslant \frac{1}{\delta}\log\mu(E)G\big((\mu(E))^{1/\delta}\big) - \int_0^{(\mu(E))^{1/\delta}} \frac{G(u)}{u}\,du \\ &\geqslant \frac{l}{\delta}\mu(E)\log\mu(E) - l\int_0^{(\mu(E))^{1/\delta}} \frac{du}{u^{1-\delta}} \\ &\geqslant 2l\delta^{-1}\mu(E)\log\mu(E). \end{aligned}$$

While

$$I_2 \geqslant \delta^{-1}\mu(E)\log\mu(E).$$

The extension to a complex valued random variable Z is as in [5, 178]. We write $Z = X + iY$, where X and Y need not be independent

$$\mathscr{E}(Z) = \mathscr{E}(X) + i\mathscr{E}(Y) = \alpha_1 + i\alpha_2 = \alpha,$$

and

$$\mathscr{E}(|Z-\alpha|) = \beta.$$

Then

$$Ze^{-i\psi} = X\cos\psi + Y\sin\psi + i(X\cos\psi - Y\sin\psi).$$

So

$$|\mathscr{E}(X\cos\psi + Y\sin\psi)| \leqslant |\alpha|$$

for all ψ, and

$$\mathscr{E}(|X\cos\psi + Y\sin\psi - a_1\cos\psi - a_2\sin\psi|) \leqslant \beta .$$

Write $P(t, \psi)$ for the probability that $-t < X\cos\psi + Y\sin\psi < t$, then the hypotheses of Theorem 1 will be satisfied for some ψ if $\gamma \geqslant \sup(|a|, \beta, 1)$ and

$$t^{-\delta}P(\gamma t, \psi) \leqslant l(\psi)$$

for $0 < t < 1$. Or if

(2.5) $$t^{-\delta}\gamma^{\delta}P(t, \psi) \leqslant l(\psi)$$

for $0 < t < 1/\gamma$. Here ψ is at our choice and we may choose it to give $l(\psi)$ the smallest value possible. We have

LEMMA 2.3. *For any measurable set E, where $\mu(E) \leqslant e^{-1}$ and* $\gamma \geqslant \sup(|a|, \beta)$

$$\int_E \log^+|Z|\,d\mu \leqslant \mu(E)\log^+\gamma - 6\mu(E)\log\mu(E).$$

The proof of this lemma follows from Lemma 2.2 as in [5]. The result corresponding to Theorem 3 for a complex valued random variable is

THEOREM 4. *If E is any set of measure not less than e^{-1}, $\gamma \geqslant \sup(|a|, \beta, 1)$, $\delta > 0$ and*

(2.6) $$l = \inf_{\psi}\sup_{0<t\leqslant 1/\gamma} t^{-\delta}\gamma^{\delta}P(t, \psi),$$

then

$$\int_E \log|Z|\,d\mu = \mu(E)\log\gamma - \eta\mu(E)\log\mu(E),$$

where

$$-\delta^{-1}(2l+1) \leqslant \eta \leqslant 6.$$

The second inequality follows from Lemma 2.3 and does not make use of (2.6).

§ 3. Lemmas for the family (1.3). In this section we apply the results of the previous section to the family of random functions (1.3). The independent random variables $e^{2\pi i\vartheta_n(\omega)}$ have expectation zero, variance one and characteristic function

(3.1) $$J_0(\varrho) = \int_0^1 \exp(i\xi\cos 2\pi u + i\eta\sin 2\pi u)\,du,$$

where $\varrho = (\xi^2+\eta^2)^{1/2}$ and $J_0(\varrho)$ is a Bessel function. So that the characteristic function depends only on the modulus of the terms and that of

$$e^{2\pi i\vartheta_n(\omega)}a_n z^n,$$

is therefore

$$J_0(\varrho|a_n|r^n),$$

and that of $f(z, \omega)$

$$\prod_0^{\infty} J_0(\varrho|a_n|r^n).$$

A simply calculation shows that

(3.2) $$P(t, \psi) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin t\varrho}{\varrho}\prod_0^{\infty}J_0(\varrho|a_n|r^n)\,d\varrho,$$

so that P is independent of ψ. With the notation (2.6) we shall show that

(3.3) $$|P(t)| \leqslant Ct^{1/3}\big(\mathfrak{M}(r)\big)^{-1/3},$$

where C is a numerical constant. However, unless the series for $f(z)$ is dominated by one or two terms we can prove much more namely

(3.4) $$|P(t)| \leqslant Ct\big(\mathfrak{M}(r)\big)^{-1}.$$

But while it may be possible to improve on (3.3) in the particular case of the family (1.3) the method would not be general so we shall content ourselves with (3.3) which is sufficient for our purposes. We require

LEMMA 3.1. *If $S = \sum_0^{\infty} c_n$ is a convergent series of non-increasing non-negative terms, then we can find an integer k such that*

(i) $\sum_0^{k} c_n \geqslant \frac{1}{4}S$,

(ii) $\sum_k^{\infty} c_n \geqslant \frac{1}{2}S$,

(iii) $c_k \geqslant \frac{1}{4}(k+1)^{-2}S$.

Remark. The hypothesis that the c_n are non-increasing is made only for convenience of statement and the lemma can be applied to any convergent series of non-negative terms since such a series can be re-ordered.

Proof. If $c_0 \geqslant \frac{1}{4}S$ there is nothing to prove since the theorem will hold with $k = 0$. Hence we may assume $k \geqslant 1$. Then we can find $m \geqslant 1$ such that

$$\sum_0^{m-1} c_n < \tfrac{1}{2}S \leqslant \sum_0^{m} c_n .$$

If $c_m \geqslant \frac{1}{4}(m+2)^{-2}S$, then the conclusions of the lemma hold since (i) and (iii) are satisfied by definition and

$$\sum_m^{\infty} c_n = S - \sum_0^{m-1} c_n \geqslant \tfrac{1}{2}S .$$

If $c_m < \frac{1}{4}(m+1)^{-2}S$ we can find ν satisfying $1 \leqslant \nu \leqslant m$ such that for $\nu \leqslant n \leqslant m$ we have $c_n < \frac{1}{4}(n+1)^{-2}S$ while

$$c_{\nu-1} \geqslant \tfrac{1}{4}\nu^{-2}S.$$

Such a $\nu \geqslant 1$ must always exist, because if not

$$\sum_0^m c_n \leqslant \frac{1}{4}S\left(1+\frac{1}{2^2}+\ldots+\frac{1}{(m+1)^2}\right) < \frac{1}{2}S$$

contrary to hypothesis. But then

$$\sum_0^{\nu-1} c_n = \sum_0^m c_n - \sum_\nu^m c_n \geqslant \frac{1}{2}S - \frac{1}{4\nu}S \geqslant \frac{1}{4}S$$

and so taking $k = \nu-1$ (i) and (iii) are satisfied while

$$\sum_{n=k}^\infty c_n \geqslant \sum_{n=m}^\infty c_n = S - \sum_0^{m-1} c_n \geqslant \tfrac{1}{2}S.$$

We now have

LEMMA 3.2. *Under the conditions of Lemma 3.1 and if in addition $k \geqslant 2$, then*

$$\int_0^\infty \Big|\prod_0^\infty J_0(\varrho\sqrt{c_n})\Big| d\varrho \leqslant C\Big(\sum_0^\infty c_n\Big)^{-1/2},$$

where C is a numerical constant.

Proof. We use the following properties of the Bessel function:

(i) $|J_0(\varrho)| \leqslant 1$ for all ϱ,

(ii) $|J_0(\varrho)| \leqslant 1-\frac{1}{5}\varrho^2$ for $0 \leqslant \varrho \leqslant 1$,

(iii) $|J_0(\varrho)| \leqslant \frac{4}{5}$ for $\varrho \geqslant 1$,

(iv) $|J_0(\varrho)| \leqslant 3\varrho^{-1/2}$ for $\varrho \geqslant 1$.

Without loss of generality we may arrange the c_n's in decreasing order. With the notation of Lemma 3.1

$$I_1 = \int_0^{1/\sqrt{c_k}} \Big|\prod_0^\infty J_0(\varrho\sqrt{c_n})\Big| d\varrho \leqslant \int_0^{1/\sqrt{c_k}} \prod_k^\infty (1-\tfrac{1}{5}\varrho^2 c_n)\, d\varrho$$

$$\leqslant \int_0^\infty \exp\Big(-\tfrac{1}{5}\varrho^2\sum_k^\infty c_n\Big) d\varrho < 3S^{-1/2}.$$

Again by (iv) and the hypothesis $k \geqslant 2$

$$I_2 = \int_{16/\sqrt{c_k}}^\infty \Big|\prod_0^\infty J_0(\varrho\sqrt{c_n})\Big| d\varrho \leqslant 3^{k+1}\prod_0^k\left(\frac{1}{\sqrt[4]{c_n}}\right)\int_{16/\sqrt{c_n}}^\infty \frac{d\varrho}{\varrho^{(k+1)/2}}$$

$$\leqslant \left(\frac{3}{4}\right)^{k-1}\frac{18}{k-1}\,\frac{1}{\sqrt[4]{(c_0c_1)}} \leqslant 81\,S^{-1/2}.$$

Finally using (iii)

$$I_3 = \int_{1/\sqrt{c_k}}^{16/\sqrt{c_k}} \Big|\prod_0^\infty J_0(\varrho\sqrt{c_n})\Big| d\varrho$$

$$\leqslant \left(\frac{4}{5}\right)^{k+1}\frac{15}{\sqrt{c_k}} \leqslant 30\left(\frac{4}{5}\right)^{k+1}(k+1)\,S^{-1/2} < 60\,S^{-1/2}.$$

This proof does not cover the case, when in Lemma 3.1, k is 0 or 1.

In this case either $c_0 = \frac{1}{4}S$ or $c_1 \geqslant \frac{1}{16}S$ but in the latter case we have $c_0 \geqslant c_1 \geqslant \frac{1}{16}S$ and this we shall assume. We have

LEMMA 3.3.

$$\left|\int_0^\infty \frac{\sin t\varrho}{t^{1/3}\varrho}\prod_0^\infty J_0(\varrho\sqrt{c_n})\,d\varrho\right| \leqslant Cc_0^{-1/6},$$

where C is a numerical constant.

Proof. By (i) and (iv) of Lemma 3.2 the above integral does not exceed

$$\int_0^\infty \varrho^{-2/3}|J_0(\varrho\sqrt{c_0})|\,d\varrho \leqslant c_0^{-1/6}\Big(\int_0^1 \varrho^{-2/3}d\varrho + \int_0^\infty \varrho^{-1/6}d\varrho\Big)$$

and the desired result follows. Combining these results we have

LEMMA 3.4. *If $P(t)$ is defined by (3.2), then, for all $t \geqslant 0$,*

$$\Pr\big(|f(z)| \leqslant t\big) \leqslant P(t) \leqslant C\max\left(\frac{t}{\mathfrak{M}(r)},\left(\frac{t}{\mathfrak{M}(r)}\right)^{1/3}\right),$$

where C is a numerical constant.

Hence by Theorem 4 and Lemma 3.4, we have Lemma 3.5. For the family (1.3) and for any set E of measure not exceeding e^{-1}

$$\int_E \log|f(z,\omega)|\,d\mu = \mu(E)\log\mathfrak{M}(r) - \eta\mu(E)\log\mu(E)$$

where, for some numerical constant C,

$$-C \leqslant \eta \leqslant 6.$$

§ 4. Proof of (1.9). As stated in the introduction the purpose of this section is to establish the equality (1.9) which is given as Lemma 4.2. We require the following lemma.

LEMMA 4.1. *If a_k is the first non-vanishing coefficient of $a_1, a_2, \ldots$, then there exists a number δ satisfying $0<\delta\leqslant\frac{1}{2}$ depending only on the coefficients $a_k, a_{k+1}, \ldots$ such that for all b, ϑ and ω*

$$\max\left(|f(\delta e^{i\theta})-b|,\ |f(\tfrac{1}{4}\delta e^{i\theta})-b|\right)\geqslant\tfrac{1}{16}|a_k|\,\delta^k \tag{4.1}$$

and

$$\inf_b\frac{1}{2\pi}\int_0^{2\pi}\log|f(\delta e^{i\theta})-b|\,d\theta\geqslant k\log\delta+\log|a_k|-3. \tag{4.2}$$

Proof. We choose δ_1 so that

$$\sum_{n=k+1}^{\infty}|a_n|\,\delta_1^{n-k}=\tfrac{1}{3}|a_k|$$

and define $\delta=\min(\delta_1,\frac{1}{2})$. We distinguish two cases (i) when

$$|a_0-b|\leqslant\tfrac{1}{3}\delta^k|a_k|$$

and (ii) when

$$|a_0-b|>\tfrac{1}{3}\delta^k|a_k|.$$

In case (i) we have

$$|f(\delta e^{i\theta})-b|\geqslant|a_k|\,\delta^k-|a_0-b|-\sum_{k+1}^{\infty}|a_n|\,\delta^n\geqslant\tfrac{1}{3}|a_k|\,\delta^k.$$

In case (ii) we have

$$|f(\tfrac{1}{4}\delta e^{i\theta})-b|\geqslant|a_0-b|-|a_k|\,(\tfrac{1}{4}\delta)^k-\tfrac{1}{4}(\tfrac{1}{4}\delta)^k\sum_{k+1}^{\infty}|a_n|\,\delta^{n-k}\geqslant\tfrac{1}{16}\delta^k|a_k|,$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(\delta e^{i\theta})-b|\,d\theta\geqslant\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{4}\delta e^{i\theta}\right)-b\right|d\theta$$
$$\geqslant k\log\delta+\log|a_k|-\log 16$$

and from these four inequalities the result follows.

LEMMA 4.2. *Under the hypotheses of Theorem 1*

$$\lim_{r\to 1}\frac{N^*(r,K;\omega)}{\log\mathfrak{M}(r)}=1$$

almost surely.

Proof. By Jensen's theorem

$$N^*(r,K;\omega)=\sup_{|b|\leqslant K}\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f(re^{i\theta})-b}{f(\frac{1}{2}e^{i\theta})-b}\right|d\theta. \tag{4.3}$$

Define the sequence $\{r_\nu\}$ by

$$\log\mathfrak{M}(r_\nu)=\nu^2. \tag{4.4}$$

It follows by Lemma 4.1 that

$$\limsup_{\nu\to\infty}\frac{N^*(r_\nu,K;\omega)}{\log\mathfrak{M}(r_\nu)}\leqslant\frac{1}{2\pi}\limsup_{\nu\to\infty}\frac{\int_0^{2\pi}\log^+|f(r_\nu e^{i\theta})|\,d\theta}{\log\mathfrak{M}(r_\nu)}. \tag{4.5}$$

But by Lemmas 2.3 and 3.5 for any measurable set E

$$\frac{1}{2\pi}\int_E d\mu\int_0^{2\pi}\log^+|f(r_\nu e^{i\theta})|\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_E\log^+|f(r_\nu e^{i\theta})|\,d\mu$$
$$\leqslant\mu(E)\log\mathfrak{M}(r_\nu)-6\mu(E)\log\mu(E).$$

Hence if E_ν is the set for which

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(r_\nu e^{i\theta})|\,d\theta\geqslant\log\mathfrak{M}(r_\nu)+\left(\log\mathfrak{M}(r_\nu)\right)^\delta \tag{4.6}$$

we have

$$\{\log\mathfrak{M}(r_\nu)+\left(\log\mathfrak{M}(r_\nu)\right)^\delta\}\mu(E_\nu)\leqslant\mu(E_\nu)\log\mathfrak{M}(r_\nu)-6\mu(E_\nu)\log\mu(E_\nu)$$

and so

$$\mu(E_\nu)\leqslant\exp\{-\tfrac{1}{6}\left(\log\mathfrak{M}(r_\nu)\right)^\delta\}\leqslant\frac{1}{\nu^2}$$

for ν large enough. Hence

$$\mu\Big(\bigcup_{\nu\geqslant\nu_0}E_\nu\Big)\leqslant\sum_{\nu\geqslant\nu_0}\mu(E_\nu)\leqslant(\nu_0-1)^{-1}.$$

But outside $\bigcup_{\nu\geqslant\nu_0}E_\nu$, we have, by (4.5) and (4.6),

$$\limsup_{\nu\to\infty}\frac{N^*(r_\nu,K;\omega)}{\log\mathfrak{M}(r_\nu)}\leqslant 1 \tag{4.7}$$

and since ν_0 may be taken as large as we please we conclude that (4.7) holds almost surely.

On the other hand from (4.3) it follows that

$$N^*(r_\nu,K;\omega)\geqslant\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f(re^{i\theta})}{f(\frac{1}{2}e^{i\theta})}\right|d\theta,$$

and hence

$$\liminf_{\nu\to\infty}\frac{N^*(r_\nu, K;\omega)}{\log\mathfrak{M}(r_\nu)} \geqslant \frac{1}{2\pi}\liminf_{\nu\to\infty}\frac{\int_0^{2\pi}\log|f(r_\nu e^{i\theta})|\,d\theta}{\log\mathfrak{M}(r_\nu)}.$$

But by Lemma 3.5 for any measurable set E_ν of measure not exceeding e^{-1} we have as before

$$\frac{1}{2\pi}\int_{E_\nu}d\mu\int_0^{2\pi}\log|f(r_\nu e^{i\theta})|\,d\theta \geqslant \mu(E_\nu)\log\mathfrak{M}(r_\nu)+C\mu(E_\nu)\log\mu(E_\nu)$$

and if therefore E_ν is the set for which

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(r_\nu e^{i\theta})|\,d\theta \leqslant \log\mathfrak{M}(r_\nu)-\big(\log\mathfrak{M}(r_\nu)\big)^\delta$$

we have

$$\{\log\mathfrak{M}(r_\nu)-\big(\log\mathfrak{M}(r_\nu)\big)^\delta\}\mu(E_\nu) \geqslant \mu(E_\nu)\log\mathfrak{M}(r_\nu)+C\mu(E_\nu)\log\mu(E_\nu)$$

and we conclude as before that for ν large enough

$$\mu(E_\nu)\leqslant \nu^{-2}$$

and hence

$$\liminf_{\nu\to\infty}\frac{N^*(r_\nu, K;\omega)}{\log\mathfrak{M}(r_\nu)} \geqslant 1$$

almost surely, and this together with (4.7) implies that

$$\lim_{\nu\to\infty}\frac{N^*(r_\nu, K;\omega)}{\log\mathfrak{M}(r_\nu)} = 1$$

almost surely. But $N^*(r, K;\omega)$ is an increasing function of r and for $r_\nu\leqslant r\leqslant r_{\nu+1}$

$$\frac{N^*(r_\nu, K;\omega)}{\log\mathfrak{M}(r_{\nu+1})} \leqslant \frac{N^*(r, K;\omega)}{\log\mathfrak{M}(r)} \leqslant \frac{N^*(r_{\nu+1}, K;\omega)}{\log\mathfrak{M}(r_\nu)}.$$

From this and (4.4), (4.3) follows.

§ 5. Lemmas for (1.10) first case. We now give the essential part of Theorem 1 for the case when $\mathfrak{M}(r)$ increases rapidly. We need the following lemmas.

LEMMA 5.1. *If* $\mathfrak{M}(r)$ *satisfies*

(5.1) $$\limsup_{r\to 1}(1-r)^2\mathfrak{M}(r)\geqslant 260$$

and if

(5.2) $$\tau(r) = 2\big(\mathfrak{M}(r)\big)^{-1/2},$$

then we can find $\{r_\nu\}$, $0<r_\nu<1$, $r_\nu\to 1$, *such that*

(5.3) $$\mathfrak{M}\big(r_\nu+2\tau(r_\nu)\big)\leqslant 4\,\mathfrak{M}(r_\nu)$$

and

(5.4) $$(1-r_\nu)^2\mathfrak{M}(r_\nu)\geqslant 16.$$

Proof. Choose a sequence $\{\varrho_\nu\}$, $\varrho_\nu\to 1$ such that

(5.5) $$(1-\varrho_\nu)^2\mathfrak{M}(\varrho_\nu)\geqslant 256$$

and define

$$\varrho_\nu^{(0)} = \varrho_\nu,\qquad \varrho_\nu^{(k)} = \varrho_\nu^{(k-1)}+2\tau(\varrho_\nu^{(k-1)}),\qquad k\geqslant 1.$$

Suppose that for $k = 1, 2, \ldots, p$

(5.6) $$\mathfrak{M}(\varrho_\nu^{(k)}) > 4\mathfrak{M}(\varrho_\nu^{(k-1)}).$$

We shall show that if p is infinite $\mathfrak{M}(\varrho_\nu^{(k)})$ will tend to infinity while $\varrho_\nu^{(k)}<\frac{1}{2}(1+\varrho_\nu)$. Hence we shall be able to conclude that p is finite and that

(5.7) $$\mathfrak{M}(\varrho_\nu^{(p+1)})\leqslant 4\mathfrak{M}(\varrho_\nu^{(p)}).$$

We have

$$\varrho_\nu^{(k)}-\varrho_\nu^{(k-1)} = 2\tau(\varrho_\nu^{(k-1)})\leqslant 2^{-k+2}\tau(\varrho_\nu).$$

Hence

$$\varrho_\nu^{(k)}-\varrho_\nu\leqslant 4\tau(\varrho_\nu)\leqslant \tfrac{1}{2}(1-\varrho_\nu)$$

by (5.4); so that $\varrho_\nu^{(k)}\leqslant\frac{1}{2}(1+\varrho_\nu)$ for all k. But

$$\mathfrak{M}(\varrho_\nu^{(k)}) > 4^k\,\mathfrak{M}(\varrho_\nu)$$

and tends to infinity if k tends to infinity. We conclude that p is finite and that (5.7) holds. We write $r_\nu = \varrho_\nu^{(p)}$ and we have

$$\mathfrak{M}\big(r_\nu+2\tau(r_\nu)\big)\leqslant 4\mathfrak{M}(r_\nu).$$

But

$$\mathfrak{M}(r_\nu)\geqslant\mathfrak{M}(\varrho_\nu)\geqslant 64(1-\varrho_\nu)^{-2}$$

and $r_\nu\leqslant\frac{1}{2}(1+\varrho_\nu)$ so that (5.4) follows.

LEMMA 5.2. *If* $\{r_\nu\}$ *satisfies the hypothesis of Lemma* 5.1 *and if* $\tau_\nu = \tau(r_\nu)$, *then*

$$\sum_0^\infty|a_n|(r_\nu+\tau_\nu)^n\leqslant 4\tau_\nu^{-1/2}\,\mathfrak{M}(r_\nu)$$

and

$$\sum_1^\infty n|a_n|(r_\nu+\tau_\nu)^n\leqslant 4\tau_\nu^{-3/2}\,\mathfrak{M}(r_\nu).$$

Proof. We have

$$\Big(\sum_0^\infty |a_n| r^n\Big)^2 \leqslant \sum_0^\infty |a_n|^2 (r+\delta)^{2n} \sum_0^\infty \Big(\frac{r}{r+\delta}\Big)^{2n} \leqslant \delta^{-1}\big(\mathfrak{M}(r+\delta)\big)^2.$$

Writing $r = r_\nu + \tau_\nu$ and $\delta = \tau_\nu$ and using (5.2) we get the desired result. The proof of the second inequality is similar.

In proving (1.10) of Theorem 1 and Theorem 2 we shall make considerable use of Carathéodory's inequality (cf. [6], Aufgaben 284). It is therefore convenient to state this as a lemma with various parameters to which we can give appropriate values for each application.

Lemma 5.3. *If $\varphi(z)$ is regular and δ has no zeros in a domain D which includes the disk $|z-z_0| < A$ and if further $|\varphi(z)| \leqslant M$ in this disk, then for $|z-z_0| < B < A$ we have*

$$\log|\varphi(z)| \geqslant -\frac{2B}{A-B}\log M + \frac{A+B}{A-B}\log|\varphi(z_0)|. \tag{5.8}$$

If further $B \leqslant \frac{1}{2}A$, then

$$\log|\varphi(z)| \geqslant -2\log M + 3\log|\varphi(z_0)|. \tag{5.9}$$

Proof. Since $\varphi(z)$ has no zeros in D, $\log\varphi(z)$ has a branch which is regular in D and

$$\mathrm{Re}\log\varphi(z) \leqslant \log M.$$

Hence by Carathéodory's inequality applied to the disks $|z-z_0| < A$ and $|z-z_0| < B$

$$\left|\log\frac{\varphi(z)}{\varphi(z_0)}\right| \leqslant \frac{2B}{A-B}\log\frac{M}{|\varphi(z_0)|}$$

whence

$$\log|\varphi(z)| \geqslant -|\log\varphi(z)| \geqslant -\frac{2B}{A-B}\log\frac{M}{|\varphi(z_0)|} + \log|\varphi(z_0)|$$

and (5.6) and (5.7) follow.

The next lemma is probably the most important lemma of the paper, because the argument used in the proof will be repeated with variations many times. Whenever these variations are not significant we shall refer the reader back to the proof of this lemma. On other occasions we shall just indicate the variations required. An important feature of this lemma is the arbitrary character of the set E and the function $b(\omega)$. The function $b(\omega)$ is any measurable function which may, and probably will, depend on $f(z,\omega)$, but it is independent of θ. It is this arbitrary character of $b(\omega)$ which gives rise to some delicate arguments.

Lemma 5.4. *If the conditions of Lemma 5.1 are satisfied with $\{r_\nu\}$ as defined in that lemma and if further E is any set of measure not exceeding e^{-1}, $b(\omega)$ any measurable complex valued function of ω satisfying $|b(\omega)| \leqslant \log\mathfrak{M}(r_\nu)$, then*

$$\frac{1}{2\pi}\int_E d\mu \int_0^{2\pi} \log\big|f\big((r_\nu+\tau_\nu)e^{i\theta},\omega\big) - b(\omega)\big|\, d\theta \tag{5.10}$$

$$\geqslant \mu(E)\log\mathfrak{M}(r_\nu+\tau_\nu) + C\mu(E)\log\mu(E) - C\big(\mathfrak{M}(r_\nu+\tau_\nu)\big)^{-1/8}.$$

Proof. It is necessary to remove the zeros of $f(z)-b$ in the disk $|z| < r_\nu+\tau_\nu$ and to do this we write

$$g(z,b;\omega) = \frac{f(z,\omega)-b(\omega)}{\pi(z,b,r_\nu;\omega)}, \tag{5.11}$$

where π is the Blaschke product of the zeros of $f-b$ in the disk $|z| < r_\nu+\tau_\nu$. We shall then prove that

$$\frac{1}{2\pi}\int_E d\mu \int_0^{2\pi} \log|g(r_\nu e^{i\theta}, b;\omega)|\, d\theta \tag{5.12}$$

satisfies the above inequality. But the integral of $g(re^{i\theta})$ with respect to θ in this expression is an increasing function of r and for $r = r_\nu+\tau_\nu$

$$\big|g\big((r_\nu+\tau_\nu)e^{i\theta}\big)\big| = \big|f\big((r_\nu+\tau_\nu)e^{i\theta},\omega\big) - b(\omega)\big|$$

and so (5.10) will follow.

Throughout the proof we write τ for τ_ν since there will not be any ambiguity. We divide the interval $(0, 2\pi)$ into N equal disjoint subintervals $\theta_j-\delta, \theta_j+\delta$, where $N = [\tau^{-3}]$ and put (5.12) in the form

$$\frac{1}{2\pi}\sum_{j=1}^N \int_E d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g|\, d\theta.$$

We now express E as the union of three disjoint sets

$$E = E_{1,j}\cup E_{2,j}\cup E_{3,j},$$

where, writing z_j for $r_\nu e^{i\theta_j}$ and r for r_ν,

$$\begin{aligned} E_{1,j} &= \{\omega \mid \omega\in E,\ |f(z_j,\omega)| \geqslant (\mathfrak{M}(r))^{1/2}\},\\ E_{2,j} &= \{\omega \mid \omega\in E\setminus E_{1,j}\ \sup_{|z-z_j|\leqslant\delta} |f(z,\omega)-b(\omega)| \geqslant \delta(\mathfrak{M}(r))^{-3}\}. \end{aligned} \tag{5.13}$$

Write

$$\sum\nolimits_1 = \frac{1}{2\pi}\sum_{j=1}^N \int_{E_{1,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g(z,\omega)|\, d\theta,$$

where $z = r_\nu e^{i\theta}$. In $E_{1,j}$

$$|f(z)-b(\omega)| \geqslant |f(z_j)| - |f(z)-f(z_j)| - |b(\omega)|.$$

But

$$|f(z)-f(z_j)| \leqslant |z-z_j| \sum_1^\infty n|a_n| r^n \leqslant 4\tau^{3/2}\mathfrak{M}(r) = C(\mathfrak{M}(r))^{1/4}$$

by Lemma 5.2. Hence in $E_{1,j}$ for $\theta_j - \delta \leqslant \theta \leqslant \theta_j + \delta$

$$|f(z)-f(\omega)| \geqslant \frac{1}{2}|f(z_j)|.$$

Now for $|z| = r < r_\nu + \tau_\nu$

$$|g(z)| \geqslant |f(z)-b|$$

and so

$$\sum\nolimits_1 \geqslant 2\delta \sum_{j=1}^N \int_{E_{1,j}} \log|f(z_j)|\,d\mu - \mu(E)\log 2.$$

But, by Lemma 3.5,

$$\int_{E_{1,j}} \log|f(z_j)|\,d\mu \geqslant \mu(E_{1,j})\log\mathfrak{M}(r) + C\mu(E_{1,j})\log\mu(E_{1,j}).$$

Now $|x\log x|$ is an increasing function of x so

$$\mu(E_{1,j})\log\mu(E_{1,j}) \geqslant \mu(E)\log\mu(E).$$

Also by Lemma 3.4

$$\mu(E_{1,j}^c) \leqslant C(\mathfrak{M}(r))^{-1/6}.$$

Therefore

$$\sum\nolimits_1 \geqslant \mu(E)\log\mathfrak{M}(r) + C\mu(E)\log\mu(E) - C(\mathfrak{M}(r))^{-1/6}\log\mathfrak{M}(r).$$

We have defined $E_{2,j}$ by (5.13). Denote by $\zeta_j = \zeta_j(\omega)$ the point on $|z-z_j| = \delta$, where $|f(z,\omega)-b(\omega)|$ attains its maximum. On $|z| = r_\nu + \tau$

$$|g(z)| = |f(z,\omega)-b(\omega)| \leqslant \sum_0^\infty |a_n|(r_\nu+\tau)^n + \log\mathfrak{M}(r_\nu) \leqslant 5(\mathfrak{M}(r_\nu))^{5/4}$$

by Lemma 5.2. This inequality true for $|z| = r_\nu + \tau$ holds for $|z| < r_\nu + \tau$ by the maximum modulus principle. Now apply Lemma 5.3 with $z_0 = \zeta_j$, $A = \tau - \delta$, $B = 2\delta$. We deduce that

$$\log|g(z)| \geqslant -C\log\mathfrak{M}(r_\nu) - \log\left(\delta^{-1}(\mathfrak{M}(r_\nu))^3\right) \geqslant -C\log\mathfrak{M}(r_\nu)$$

for $|z-\zeta_j| \leqslant 2\delta$ and so for $|z-z_j| \leqslant \delta$. Consequently

$$\sum\nolimits_2 = \sum_1^N \int_{E_{2,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g(r_\nu e^{i\theta})|\,d\theta \geqslant -CN\delta\log\mathfrak{M}(r_\nu)\sup_j \mu(E_{2,j}).$$

Now

$$\mu(E_{2,j}) \leqslant \mu(E_{1,j}^c) \leqslant (\mathfrak{M}(r_\nu))^{-1/6}$$

and so

$$\sum\nolimits_2 \geqslant -C(\mathfrak{M}(r_\nu))^{-1/6}\log\mathfrak{M}(r_\nu).$$

If $\omega \in E_{3,j}$, then, integrating round $|z-z_j| = \delta$,

$$|f'(z_j,\omega)| \leqslant \frac{1}{2\pi}\left|\int \frac{f(z)-b}{(z-z_j)^2}\,dz\right| \leqslant (\mathfrak{M}(r_\nu))^{-3}.$$

Hence by Lemma 3.4

$$\mu(E_{3,j}) \leqslant C(\mathfrak{M}(r_\nu))^{-1}\Big(\sum_1^\infty n^2|a_n|^2 r_\nu^{2n}\Big)^{-1/6} \leqslant C(\mathfrak{M}(r_\nu))^{-1}. \tag{5.14}$$

Write $z_1 = \frac{3}{4}e^{i\theta_j}$ and consider $|f(z)-b|$ in the disk $|z-z_1| \leqslant \tau^2$. This function attains its maximum at a point ζ_1 on the circumference and

$$|f'(z_1)| \leqslant \tau^{-2}|f(\zeta_1,\omega)-b(\omega)| \leqslant \tau^{-2}|g(\zeta_1)|.$$

Now apply Lemma 5.3 to the function $g(z)$ with $z_0 = \zeta_1$ and the numbers A and B chosen so that both disks include the arc joining $(r_\nu+\frac{1}{2}\tau)e^{\theta_j-\delta}$, $(r_\nu+\frac{1}{2}\tau)e^{\theta_j+\delta}$, lie entirely within $|z| < r_\nu+\tau$ and are such that $A-B \geqslant \frac{1}{4}\tau$. This is possible because $\delta = O(\tau^3)$. We deduce that

$$\log|g(z)| \geqslant -\frac{C}{\tau}\log\mathfrak{M}(r_\nu) - \frac{C}{\tau}\log|f'(z_1)| + \log|g(I_1)|$$

$$\geqslant -\frac{C}{\tau}\log\mathfrak{M}(r_\nu) - \frac{C}{\tau}\log|f'(z_1)|.$$

Hence

$$\int_{E_{3,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g|\,d\theta \geqslant -C\frac{\delta}{\tau}\log\mathfrak{M}(r_\nu)\mu(E_{3,j}) - C\frac{\delta}{\tau}\int_{E_{3,j}} \log|f'(z_1)|\,d\mu.$$

But by Lemma 3.5 the last term is not less than

$$-\tfrac{1}{2}\log\Big(\sum_1^\infty n^2|a_n|^2(\tfrac{3}{4})^{2n}\Big)\mu(E_{3,j}) + C\mu(E_{3,j})\log\mu(E_{3,j})$$

therefore by (5.14)

$$\sum\nolimits_3 = \sum_1^N \int_{E_{3,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g|\,d\theta \geqslant -C(\mathfrak{M}(r_\nu))^{-1/6}\log\mathfrak{M}(r_\nu).$$

Combining the inequalities for $\sum_1$, $\sum_2$ and $\sum_3$ we deduce that

$$\frac{1}{2\pi}\int_E d\mu \int_0^{2\pi} \log|g(r_\nu e^{i\theta})|\,d\theta \tag{5.15}$$

$$\geqslant \mu(E)\log\mathfrak{M}(r_\nu) + C\mu(E)\log\mu(E) - C(\mathfrak{M}(r_\nu))^{-1/6}\log\mathfrak{M}(r_\nu).$$

But the first number is an increasing function of r and on $|z| = r_\nu + \tau$ we have

$$|g((r_\nu+\tau)e^{i\theta})| = |f((r_\nu+\tau)e^{i\theta} - b(\omega)|$$

and so (5.15) and (5.3) yield the desired result.

§ 6. Lemmas for (1.10) second case. Our next task is to consider the case when $(1-r)^2\mathfrak{M}(r)$ is bounded. The argument of Lemma 5.1 cannot now be applied because, in the notation of the lemma, $r_\nu+\tau_\nu$ may lie outside the unit circle. But although we cannot use this lemma we now have the advantage of an order condition on $\mathfrak{M}(r)$. We do, however, get into trouble if $\mathfrak{M}(r)$ increases very slowly and this case we reserve for the next section. The analogue of Lemma 5.1 is

LEMMA 6.1. *If, for some numerical constant C*

(6.1) $$\mathfrak{M}(r) \leqslant C(1-r)^{-2}$$

for all r, and

(6.2) $$\limsup_{r\to 1}\left(\log\left(\frac{1}{1-r}\right)\right)^{-5}\mathfrak{M}(r) = \infty,$$

then we can find $\{r_\nu\}$ so that $r_\nu \to 1$ and for all ν

(6.3) $$\mathfrak{M}\left(\frac{1+r_\nu}{2}\right) \leqslant 8\mathfrak{M}(r_\nu),$$

and

(6.4) $$\lim_{\nu\to\infty}\left(\log\left(\frac{1}{1-r_\nu}\right)\right)^{-5}\mathfrak{M}(r_\nu) = \infty.$$

Proof. Choose ϱ_ν so that

$$\mathfrak{M}(\varrho_\nu)\left(\log\left(\frac{1}{1-\varrho_\nu}\right)\right)^{-5} \to \infty.$$

Write

$$\varrho_\nu^{(0)} = \varrho_\nu, \quad \varrho_\nu^{(k)} = \tfrac{1}{2}(1+\varrho_\nu^{(k-1)})$$

so that

$$1-\varrho_\nu^{(k)} = 2^{-k}(1-\varrho_\nu)$$

and suppose if possible that

$$\mathfrak{M}(\varrho_\nu^{(k)}) \geqslant 8\mathfrak{M}(\varrho_\nu^{(k-1)})$$

for $k = 1, 2, \ldots, p$. Then

$$\mathfrak{M}(\varrho_\nu^{(k)}) \geqslant 8^k\mathfrak{M}(\varrho_\nu)$$

so that

$$(1-\varrho_\nu^{(k)})^2\mathfrak{M}(\varrho_\nu^{(k)}) \geqslant 2^k(1-\varrho_\nu)^2\mathfrak{M}(\varrho_\nu).$$

But by hypothesis the first member is bounded and hence p is bounded and indeed

(6.5) $$2^p \leqslant C(1-\varrho_\nu)^{-2}(\mathfrak{M}(\varrho_\nu))^{-1}.$$

We have therefore found $\varrho_\nu^{(p)}$ such that

$$\mathfrak{M}(\varrho_\nu^{(p+1)}) \leqslant 8\mathfrak{M}(\varrho_\nu^{(p)}),$$

writing $r_\nu = \varrho_\nu^{(p)}$ we obtain (6.3). Further from (6.5)

$$\frac{1}{1-r_\nu} = \frac{2^p}{1-\varrho_\nu} \leqslant \frac{C}{(1-\varrho_\nu)^3\mathfrak{M}(\varrho_\nu)} \leqslant \frac{1}{(1-\varrho_\nu)^3}$$

if ϱ_ν is near enough to unity. Hence

$$\left(\log\left(\frac{1}{1-r_\nu}\right)\right)^{-5}\mathfrak{M}(r_\nu) \geqslant \left(3\log\left(\frac{1}{1-\varrho_\nu}\right)\right)^{-5}\mathfrak{M}(\varrho_\nu) \to \infty$$

as desired.

LEMMA 6.2. *If* (6.1) *is satisfied, then for all r*

$$\sum_0^\infty |a_n|\left(\frac{1+r}{2}\right)^n \leqslant C(1-r)^{-5/2},$$

and

$$\sum_1^\infty n|a_n|\left(\frac{1+r}{2}\right)^n \leqslant C(1-r)^{-7/2}.$$

Proof. We have

$$\sum_0^\infty |a_n|\left(\frac{1+r}{2}\right)^n \leqslant \mathfrak{M}\left(\frac{3+r}{4}\right)\left(\sum_0^\infty\left(\frac{2+2r}{3+r}\right)^{2n}\right)^{1/2} \leqslant C(1-r)^{-5/2},$$

and similarly for the second inequality.

The following lemma is analogous to Lemma 5.4 and its proof is similar.

LEMMA 6.3. *If the hypotheses of Lemma 6.1 are satisfied with $\{r_\nu\}$ as defined in that lemma and if further E is any set of measure less than e^{-1} and $b(\omega)$ any complex valued measurable function satisfying $|b(\omega)| \leqslant \log\mathfrak{M}(r_\nu)$, then for ν large enough*

$$\frac{1}{2\pi}\int_E d\mu\int_0^{2\pi}\log\left|f\left(\frac{1+r_\nu}{2}e^{i\theta}, \omega\right) - b(\omega)\right| d\theta$$
$$\geqslant \mu(E)\log\mathfrak{M}\left(\frac{1+r_\nu}{2}\right) + C\mu(E)\log\mu(E) - \left(\mathfrak{M}\left(\frac{1+r_\nu}{2}\right)\right)^{-1/8}.$$

Proof. We denote by $\pi(z, b, r_\nu; \omega)$ the Blaschke product of the zeros of $f(z,\omega)-b(\omega)$ in the disk $|z| < \frac{1}{2}(1+r_\nu)$ and as in the proof of Lemma 5.4 we write

$$g(z,\omega) = \frac{f(z,\omega)-b(\omega)}{\pi(z)}.$$

Then as in the case of Lemma 5.4 we consider

$$I = \frac{1}{2\pi}\int_E d\mu \int_0^{2\pi} \log|g(r_\nu e^{i\theta})|\,d\theta.$$

We divide the range $(0, 2\pi)$ into N equal disjoint intervals $(\theta_j-\delta,\ \theta_j+\delta)$ with $N\delta = \pi$ and $N = [\pi(1-r)^{-1/2}]$. Then if $z_j = r_\nu e^{i\theta_j}$, $|z-z_j| \leqslant \delta$ and $|b(\omega)| \leqslant \log \mathfrak{M}(r_\nu)$

$$|f(z,\omega)-b(\omega)| \geqslant |f(z_j)| - \log\mathfrak{M}(r_\nu) - C.$$

We define

$$E_{1,j} = \{\omega \mid \omega\in E,\ |f(z_j,\omega)| \geqslant 3\log\mathfrak{M}(r_\nu)\},$$

$$E_{2,j} = \{\omega \mid \omega \in E\setminus E_{1,j} \sup_{|z-z_j|\leqslant(1-r_\nu)/16} |f(z,\omega)-b(\omega)| \geqslant (1-r_\nu)^7\},$$

$$E_{3,j} = E\setminus E_{1,j}\cup E_{2,j}.$$

Then in $E_{1,j}$ for $|z| = r_\nu$ $|\theta-\theta_j| \leqslant \delta$

$$|f(z,\omega)-b(\omega)| \geqslant \frac{1}{2}|f(z_j)|.$$

Hence as in the proof of Lemma 5.4

$$\sum\nolimits_1 = \frac{1}{2\pi}\sum_1^N \int_{E_{1,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g|\,d\theta$$

$$\geqslant \mu(E)\log\mathfrak{M}(r_\nu) + C\mu(E)\log\mu(E) - \mu(E)\log 2.$$

If $\omega\in E_{2,j}$ let ζ_j be the point on $|z-z_j| = (1-r_\nu)/16$, where $|f(z,\omega)-b(\omega)|$ attains its maximum. Then by Lemma 5.3 with $z_0 = \zeta_j$, $A = \frac{1}{4}(1-r_\nu)$, $B = \frac{1}{2}A$ we have, for $|z-z_j| < (1-r_\nu)/16$

$$\log|g(z)| \geqslant C\log(1-r_\nu)$$

whence

$$\sum\nolimits_2 = \sum_1^N \int_{E_{2,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g|\,d\theta \geqslant 2C\delta\log(1-r_\nu)\sum_1^N \mu(E_{2,j}).$$

But $E_{2,j} \subset E_{1,j}^c$ and so

$$\mu(E_{2,j}) \leqslant C\big(\mathfrak{M}(r_\nu)\big)^{-1/3}\log\mathfrak{M}(r_\nu)$$

whence

$$\sum\nolimits_2 \geqslant C\log(1-r_\nu)\cdot\big(\mathfrak{M}(r_\nu)\big)^{-1/3}\log\mathfrak{M}(r_\nu) \geqslant -\big(\mathfrak{M}(r_\nu)\big)^{-1/8}$$

for large enough ν by condition (6.2). In $E_{3,j}$ we have, by the argument used in Lemma 5.4

$$|f(z_j)| \leqslant C(1-r_\nu)^6.$$

Whence

$$\mu(E_{3,j}) \leqslant C(1-r_\nu)^2\Big(\sum_1^\infty n^2|a_n|^2 r^{2n}\Big)^{-1/6} = o\big((1-r_\nu)^2\big). \tag{6.6}$$

Reasoning similarly to § 5 we apply Carathéodory's inequality to disks centre a point in the neighbourhood of $\frac{3}{4}e^{i\theta_j}$ and of radii to include the arc $r_\nu e^{i(\theta_j-\delta)}$, $r_\nu e^{i(\theta_j+\delta)}$ we deduce that

$$\int_{E_{3,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g(r_\nu e^{i\theta},\omega)|\,d\theta \geqslant C\frac{\delta}{1-r_\nu}\log(1-r_\nu)\,\mu(E_{3,j})$$

whence in view of (6.6)

$$\sum\nolimits_3 = \sum_1^N \int_{E_{3,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g|\,d\theta \geqslant C(1-r_\nu)\log(1-r_\nu) \geqslant -\big(\mathfrak{M}(r_\nu)\big)^{-1/4}$$

for ν large enough. Collecting together the three terms $\sum_1, \sum_2, \sum_3$ remembering that the integral of $g(re^{i\theta})$ with respect to θ in $(0, 2\pi)$ is an increasing function of r and using inequality (6.3) we get the desired result.

§ 7. Lemmas for (1.10) third case. We have finally to deal with the case when $\mathfrak{M}(r)$ increases very slowly. The analogues of Lemmas 6.1 and 6.2 are

LEMMA 7.1. *If for some numerical constants C and k*

$$\mathfrak{M}(r) \leqslant C\Big(\log\frac{1}{1-r}\Big)^k, \tag{7.1}$$

then we can find a sequence $\{r_\nu\}$, $r_\nu \to 1$ such that

$$\mathfrak{M}\Big(\frac{1+r_\nu}{2}\Big) \leqslant 8\,\mathfrak{M}(r_\nu). \tag{7.2}$$

LEMMA 7.2. *If $\mathfrak{M}(r)$ satisfies (7.1), then*

$$\sum_0^\infty |a_n| r^n \leqslant C(1-r)^{-1/2}\Big(\log\frac{1}{1-r}\Big)^k,$$

and

$$\sum_1^\infty n|a_n| r^n \leqslant C(1-r)^{-3/2}\Big(\log\frac{1}{1-r}\Big)^k.$$

The proofs of these lemmas are similar to those of Lemmas 6.1 and 6.2. In this case there is no result corresponding to (6.4) and condition (6.2) is not needed.

LEMMA 7.3. *If $\mathfrak{M}(r)$ satisfies (7.1) and if $\{r_\nu\}$ is a sequence satisfying (7.2), then for any set E of measure less than e^{-1} and for any complex valued measurable function $b(\omega)$ satisfying $|b(\omega)| \leqslant \log \mathfrak{M}(r_\nu)$*

$$\frac{1}{2\pi}\int_E d\mu \int_0^{2\pi} \log\left| f\left(\frac{1+r_\nu}{2}e^{i\theta}, \omega\right) - b(\omega)\right| d\theta$$

$$\geqslant \mu(E)\log \mathfrak{M}\left(\frac{1+r_\nu}{2}\right) + C\mu(E)\log\mu(E) - K\left(\mathfrak{M}\left(\frac{1+r_\nu}{2}\right)\right)^{-1/8}$$

provided ν is large enough. Here C is a numerical constant, K depends on $a_1, a_2, \ldots$ but is independent of r.

Proof. As in Lemma 6.3 we write $\pi(z,\omega)$ for the Blaschke product of the zeros of $f(z)-b(\omega)$ in the disk $|z| < \frac{1}{2}(1+r_\nu)$ and define

$$g(z,\omega) = \frac{f(z,\omega)-b(\omega)}{\pi(z,\omega)}.$$

We consider on $|z| = r_\nu$ the expression

$$I = \frac{1}{2\pi}\int_E d\mu \int_0^{2\pi} \log|g(z,\omega)|\,d\theta.$$

We divide $(0, 2\pi)$ into N equal disjoint intervals $(\theta_j-\delta, \theta_j+\delta)$, where $N\delta = \pi$ and $N = [(1-r_\nu)^{-2}]$. We write $z_j = r_\nu e^{i\theta_j}$ and $F = F(z_j, \delta; \omega) = \sup\limits_{|z-z_j|\leqslant\delta} |f(z,\omega)-b(\omega)|$,

$$E_{1,j} = \{\omega \mid \omega\in E, |f(z_j)| \geqslant 3\log\mathfrak{M}(r_\nu)\},$$
$$E_{2,j} = \left\{\omega \mid \omega\in E\setminus E_{1,j}\ F \geqslant \exp\left(-(\mathfrak{M}(r_\nu))^{1/5}\right)\right\},$$
$$E_{3,j} = \{\omega \mid \omega\in E\setminus E_{2,j}\ F\leqslant (1-r_\nu)^7\},$$
$$E_{4,j} = E\setminus E_{1,j}\cup E_{2,j}\cup E_{3,j}.$$

Then

$$I = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4,$$

where

$$\sum\nolimits_k = \frac{1}{2\pi}\sum_{j=1}^{N}\int_{E_{k,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g(z,\omega)|\,d\theta$$

for $k = 1, 2, 3, 4$. The treatment of $\sum_1$, $\sum_2$ and $\sum_3$ is similar to that in Lemma 6.3. The only change in the treatment of $\sum_1$ is that the small order of $\mathfrak{M}(r)$ enables us to have a larger δ. For $\sum_2$, let ζ_j be the point of $|z-z_j| = \delta$, where $|f(z)-b(\omega)|$ attains its maximum; then by (5.8) of Lemma 5.3 with $A = \frac{1}{4}(1-r_\nu)$, $B = 2\delta$ we get

$$\log|g(z)| \geqslant -(\mathfrak{M}(r_\nu))^{1/5} - o\left((1-r_\nu)^{1/2}\right)$$

in $|z-z_j|\leqslant\delta$, and remembering that

$$\mu(E_{2,j}) \leqslant C(\mathfrak{M}(r_\nu))^{-1/3}\log\mathfrak{M}(r_\nu)$$

we deduce that, for ν large enough

$$\sum\nolimits_2 \geqslant -C(\mathfrak{M}(r_\nu))^{-1/8}.$$

For $\sum_3$ we use

$$|f'(z_j)| \leqslant \delta^{-1}F \leqslant (1-r_\nu)^5$$

whence as before

$$\mu(E_{3,j}) \leqslant (1-r_\nu)^{5/3}(\mathfrak{M}(r_\nu))^{-1/3}.$$

Then, by using Lemma 5.3 as in Lemma 6.3, we deduce that in $|z-z_j|\leqslant\delta$

$$\log|g(z)| \leqslant C(1-r_\nu)^{-1}\log(1-r_\nu)$$

whence

$$\sum\nolimits_3 \geqslant -C(1-r_\nu)^{1/2}.$$

This leaves only the sum $\sum_4$ which has no analogue in Lemmas 5.4 and 6.3. It is necessary to make a further division of the sets $E_{4,j}$. We write

$$E_{4,j} = \bigcup_{p=5}^{P} E_{p,j},$$

where the sets $E_{p,j}$ are disjoint and in $E_{p,j}$

$$\Lambda_p^{-1} \leqslant \sup_{|z-z_j|\leqslant\delta} |f(z,\omega)-b(\omega)| \leqslant \Lambda_{p-1}^{-1}$$

and Λ_p is defined by

$$\Lambda_4 = \exp\left((\mathfrak{M}(r_\nu))^{1/5}\right), \quad \Lambda_p = 2^{p-4}\Lambda_4, \quad p\geqslant 4,$$
$$\Lambda_{P-1} < (1-r_\nu)^{-7} \leqslant \Lambda_P$$

and so for all p

$$\log\Lambda_p \leqslant 8\log\left(\frac{1}{1-r_\nu}\right).$$

Let $\zeta_j = \zeta_j(\omega)$ be the point, where $|f(z,\omega)-b(\omega)|$ attains its maximum on $|z-z_j| = \delta$. By Lemma 5.3 with $z_0 = \zeta_j$, $A = \frac{1}{4}(1-r_\nu)$, $B = 2\delta$, we get, for $\omega\in E_{p,j}$

$$\log|g(z)| \geqslant -C(1-r_\nu)^{1/2} - \log\Lambda_p \geqslant -2\log\Lambda_p$$

if ν is large enough. Therefore

(7.3) $$\sum_p = \sum_{j=1}^{N} \int_{E_{p,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} \log|g(z)|\,d\theta \geqslant -4\pi \log \Lambda_p \frac{1}{N}\sum_{j=1}^{N}\mu(E_{p,j})$$

and we have to estimate the sum $\frac{1}{N}\sum_{1}^{N}\mu(E_{p,j})$. Let $\chi_j(\omega)=\chi_{p,j}(\omega)$ be the characteristic function of the set $E_{p,j}$. Then

$$\sum_{1}^{N}\mu(E_{p,j}) = \sum_{1}^{N}\int_{\Omega}\chi_j(\omega)\,d\mu \leqslant \Big(\int_{\Omega}\Big(\sum_{1}^{N}\chi_j(\omega)\Big)^2 d\mu\Big)^{1/2},$$

and

$$\int_{\Omega}\Big(\sum_{1}^{N}\chi_j(\omega)\Big)^2 d\mu = \sum_{1}^{N}\int_{\Omega}\chi_j^2(\omega)\,d\mu + \sum\sum_{j\neq j'}\int_{\Omega}\chi_j(\omega)\chi_{j'}(\omega)\,d\mu$$
$$= \sum_{1}^{N}\mu(E_{p,j}) + \sum\sum_{j\neq j'}\mu(E_{p,j}\cap E_{p,j'}).$$

Now if $\sum\mu(E_{p,j})\leqslant 2$, then the second member of (7.3) is small and so there is nothing to prove. If this is not so, then

(7.4) $$\Big(\sum_{1}^{N}\mu(E_{p,j})\Big)^2 \leqslant 2\sum\sum_{j\neq j'}\mu(E_{p,j}\cap E_{p,j'}).$$

But if $\omega\in E_{p,j}\cap E_{p,j'}$, then

$$|f(z_j)-b(\omega)|\leqslant \Lambda_{p-1}^{-1}, \qquad |f(z_{j'})-b(\omega)|\leqslant \Lambda_{p-1}^{-1}$$

and so

$$|f(z_j)-f(z_{j'})|\leqslant 2\Lambda_{p-1}^{-1}.$$

But if $j\neq j'$

$$f(z_j)-f(z_{j'}) = 2i\sum_{1}^{\infty} e^{2\pi i\theta_n(\omega)}\, a_n r^n e^{\frac{1}{2}in(\theta_j+\theta_{j'})}\sin n\Big(\frac{\theta_j-\theta_{j'}}{2}\Big).$$

So by Lemma 3.4

(7.5) $$\mu(E_{p,j}\cap E_{p,j'}) \leqslant Pr\big(|f(z_j)-f(z_{j'})|\leqslant 2\Lambda_{p-1}^{-1}\big)$$
$$\leqslant C\Lambda_{p-1}^{-1/3}\Big(\sum_{1}^{\infty}|a_n|^2 r^{2n}\sin^2 n\Big(\frac{\theta_j-\theta_{j'}}{2}\Big)\Big)^{-1/6}.$$

We now distinguish two cases, case (i) when

$$\sum_{1}^{\infty}|a_n|^2 r^{2n}\sin^2 n\Big(\frac{\theta_j-\theta_{j'}}{2}\Big)\geqslant \Lambda_{p-1}^{-1}$$

and case (ii) when the opposite inequality is satisfied. In case (i) we have

$$\mu(E_{p,j}\cap E_{p,j'})\leqslant C\Lambda_{p-1}^{-1/6}$$

so that if $\sum^{(1)}$ denotes the sum over those terms for which case (i) holds

(7.6) $$\sum{}^{(1)} = \sum{}^{(1)}\mu(E_{p,j}\cap E_{p,j'})\leqslant CN^2\Lambda_{p-1}^{-1/6}.$$

Let a_k be the first non-vanishing term in the sequence $a_1, a_2, \ldots$ Then

$$\sum_{1}^{\infty}|a_n|^2 r^{2n}\sin^2\frac{n}{2}(\theta_j-\theta_{j'})\geqslant |a_k|^2 r^{2k}\sin^2\frac{k}{2}(\theta_j-\theta_{j'})$$

so that in case (ii)

$$\Big|\sin\frac{k}{2}(\theta_j-\theta_{j'})\Big|\leqslant (\Lambda_{p-1}^{1/2}|a_k|r^k)^{-1}.$$

The number of terms satisfying this condition is clearly

$$CN^2\Lambda_{p-1}^{-1/2}/|a_k|r^k\leqslant KN^2\Lambda_{p-1}^{-1/2},$$

where K depends on $\mathfrak{M}(r)$ but is independent of p.

So from (7.3) and (7.4) and (7.6)

$$\sum_p \geqslant -K\log\Lambda_p\{\Lambda_{p-1}^{-1/12}+\Lambda_{p-1}^{-1/4}\}.$$

Putting in the value of Λ_p we get

$$\sum_4 = \frac{1}{2\pi}\sum_{p=4}^{P}\sum_p \geqslant -K\frac{\log\Lambda_4}{\Lambda_4^{1/12}}\geqslant -\frac{K}{\mathfrak{M}(r_\nu)}$$

for sufficiently large ν. Collecting the inequalities for $\sum_1, \sum_2, \sum_3$ and $\sum_4$ we get the desired result.

§ 8. Proof of Theorem 1. We are now in a position to prove Theorem 1. We begin with the following lemma

LEMMA 8.1. *If (1.4) and (1.5) are satisfied there exists a sequence $\{\varrho_\nu\}$ $\varrho_\nu\to 1$ determined by the function $\mathfrak{M}(r)$ and to each ν a set E_ν of measure at most $(\log\mathfrak{M}(\varrho_\nu))^{-1}$ such that if K and δ are any positive constants*

(8.1) $$\inf_{|b|\leqslant K}\frac{1}{2\pi}\int_{0}^{2\pi}\log|f(\varrho_\nu e^{i\theta},\omega)-b|\,d\theta\geqslant \log\mathfrak{M}(\varrho_\nu)-\big(\log\mathfrak{M}(\varrho_\nu)\big)^{\delta}$$

for $\nu\geqslant\nu_0(K,\delta)$ and all ω in the complement of E_ν.

Proof. The sequence $\{r_\nu\}$ is as defined in Lemmas 5.1 and 6.1 and 7.1 and $\varrho_\nu = r_\nu+\tau_\nu$ when the hypotheses of Lemma 5.1 are satisfied and

$\varrho_\nu = \frac{1}{2}(1+r_\nu)$ when those of Lemmas 6.1 or 7.1 are satisfied. Since by hypothesis $\mathfrak{M}(r)$ tends to infinity we may suppose that ν_0 is such that for $\nu \geqslant \nu_0$, $K \leqslant \log \mathfrak{M}(\varrho_\nu)$. To each ϱ_ν there exists a measurable function $b(\omega)$ such that

$$I(\omega) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\varrho_\nu e^{i\theta}) - b(\omega)|\,d\theta \leqslant \inf_{|b|\leqslant K} \frac{1}{2\pi}\int_0^{2\pi} \log|f(\varrho_\nu e^{i\theta}) - b|\,d\theta + 1.$$

However, we have shown in Lemmas 5.4, 6.3 and 7.3 that for any measurable function $b(\omega)$ satisfying these conditions and for any set E of measure at most e^{-1}

$$\int_E I(\omega)\,d\mu \geqslant \mu(E)\log\mathfrak{M}(\varrho_\nu) + C\mu(E)\log\mu(E) - C\big(\mathfrak{M}(\varrho_\nu)\big)^{1/8}. \tag{8.2}$$

We are free to choose for E any measurable set and we now exercise this choice by taking for E the ω-set E_ν for which

$$I(\omega) \leqslant \mathfrak{M}(\varrho_\nu) - \big(\log\mathfrak{M}(\varrho_\nu)\big)^\delta + 1.$$

Then

$$\mu(E_\nu)\{\log\mathfrak{M}(\varrho_\nu) - \big(\log\mathfrak{M}(\varrho_\nu)\big)^\delta + 1\}$$
$$\geqslant \mu(E_\nu)\log\mathfrak{M}(\varrho_\nu) + C\mu(E_\nu)\log\mu(E_\nu) - \big(\mathfrak{M}(\varrho_\nu)\big)^{-1/8}$$

or

$$\big(\log\mathfrak{M}(\varrho_\nu)\big)^\delta \leqslant C\log\frac{1}{\mu(E_\nu)} + \frac{2}{\mu(E_\nu)}.$$

But if $\mu(E_\nu) \geqslant \big(\log\mathfrak{M}(\varrho_\nu)\big)^{-1}$ this inequality will not be satisfied for ϱ_ν near enough to unity. We conclude that $\mu(E_\nu) \leqslant \big(\log\mathfrak{M}(\varrho_\nu)\big)^{-1}$ whenever (8.2) is satisfied. Consequently outside E_ν we must have for all $b(\omega)$ satisfying $|b(\omega)| \leqslant K$

$$I(\omega) \geqslant \log\mathfrak{M}(\varrho_\nu) - \big(\log\mathfrak{M}(\varrho_\nu)\big)^\delta,$$

and so (8.1) can hold in a set E_ν of measure $\big(\log\mathfrak{M}(\varrho_\nu)\big)^{-1}$ at most.

By Jensen's theorem we have from (1.7)

$$N_*(r, K; \omega) = \inf_{|b|\leqslant K}\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f(re^{i\theta})-b}{f(\frac{1}{2}e^{i\theta})-b}\right|d\theta$$
$$\geqslant \inf_{|b|\leqslant K}\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})-b|\,d\theta - \sup_{|b|\leqslant K}\int_0^{2\pi}\log|f(\tfrac{1}{2}e^{i\theta})-b|\,d\theta.$$

But the second integral is trivially bounded. Hence by Lemma 8.1 we have

$$N_*(r, K; \omega) \geqslant \log\mathfrak{M}(\varrho_\nu) - \big(\log\mathfrak{M}(\varrho_\nu)\big)^\delta + O(1) \tag{8.3}$$

outside a set E_ν of measure at most $\big(\log\mathfrak{M}(\varrho_\nu)\big)^{-1}$. By taking a subsequence of $\{\varrho_\nu\}$ if necessary we can arrange that

$$\sum_1^\infty \big(\log\mathfrak{M}(\varrho_\nu)\big)^{-1} < \infty. \tag{8.4}$$

Making this refinement, it follows that outside a set $\bigcup_{\nu\geqslant\nu_0} E_\nu$ whose measure

$$\sum_{\nu=\nu_0}^\infty \big(\log\mathfrak{M}(\varrho_\nu)\big)^{-1}$$

tends to zero as ν_0 tends to infinity (8.3), holds for all $\nu \geqslant \nu_0$. Hence

$$\liminf_{\varrho_\nu\to 1}\frac{N_*(\varrho_\nu, K; \omega)}{\log\mathfrak{M}(\varrho_\nu)} \geqslant 1$$

almost surely, and this with (1.9) yields (1.10).

To deduce (1.10) we observe that the argument used in the proof of Lemma 4.2 establishes (4.6) for any sequence $\{\varrho_\nu\}$ satisfying (8.4). Then from (4.6) and (8.3) outside the exceptional set

$$N^*(\varrho_\nu, K; \omega) - N_*(\varrho_\nu, K; \omega) \leqslant \big(\log\mathfrak{M}(\varrho_\nu)\big)^\delta + O(1)$$

whence as before for $\delta > 0$

$$\frac{N^*(\varrho_\nu, K; \omega) - N_*(\varrho_\nu, K; \omega)}{\big(\log\mathfrak{M}(\varrho_\nu)\big)^\delta} \to 0$$

almost surely and (1.11) follows.

§ 9. Proof of Theorem 2. The method we used to prove Theorem 1 enables us also to establish Theorem 2, but this time, since we say nothing about the order of the number of zeros, we can make some preliminary reductions. In the first place it is sufficient to show that almost all functions take every finite value an infinity of times in a particular sector; because, since the union of a countable number of sets of measure zero is of measure zero, we can then extend this to every rational sector and so to every sector.

Now suppose that there exists an ω-set E such that to each $\omega \in E$ there exists a number $b(\omega)$ such that $f(z, \omega) - b(\omega)$ has at most a finite number of zeros in the sector $0 < |z| < 1$, $|\arg z| < \alpha$. We have to show that E is of zero measure. Suppose on the contrary that E has positive measure and denote by $\{S_n\}$, $n = 1, 2, \ldots$, the set of all rational sectors in this given sector. Let E_n be the subset of E such that if $\omega \in E_n$, $f(z, \omega)$ takes every value at least once in S_n. Now if for all n, $\mu(E_n) = \mu(E)$, then $\mu(E\setminus E_n) = 0$ and so $\mu\big(\bigcup_1^\infty (E\setminus E_n)\big) = 0$ or $\mu\big(\bigcap_1^\infty E_n\big) = \mu(E)$.

But if $\omega \in \bigcap_1^\infty E_n$, then $f(z, \omega)$ takes every value at least one in every sector S_n and so $f(z, \omega)$ takes every value an infinity of times in $0 < |z| < 1$, $|\arg z| < \alpha$ contrary to hypothesis. Hence there must exist E_n such that $\mu(E_n) < \mu(E)$ or $\mu(E \setminus E_n) > 0$. That is we have found a set $E \setminus E_n$ of positive measure and a sector S_n such that to each $\omega \in E \setminus E_n$ there exists $b(\omega)$ such that $f(z, \omega) - b(\omega)$ has no zeros in S_n. We shall show that this is impossible and from this contradiction it will follow that E has zero measure. We can make yet two further reductions. Let $\{C_j\}$ denote the complex integers and

$$F_j = \{\omega \,|\, \omega \in E \setminus E_n, \ |b(\omega) - c_j| \leqslant 1\};$$

then

$$E \setminus E_n = \bigcup_1^\infty F_j$$

so if we can show that $\mu(F_j) = 0$ for all j the result will follow. However, since altering the constant term in the expansion of $f(z)$ makes no difference it will be enough to show that $\mu(F_1) = 0$. Again if $f(z) - b$ has no zeros in the sector, then $|f(z, \omega) - b(\omega)|$ will be bounded below for $\delta \leqslant |z| \leqslant 1 - \delta$ and a sector $S_n' \subset S_n$. Denote this lower bound by $B(\omega)$ and let E_ν be the subset of $E \setminus E_n$ for which $B(\omega) \geqslant 1/\nu$. Then $E \setminus E_n = \bigcup E_\nu$ so $\mu(E \setminus E_n) = \lim_{\nu \to \infty} \mu(E_\nu)$ and this implies that, for some ν, E_ν will be of positive measure. So in what follows we may assume without loss of generality that for a set of positive measure and a positive number δ, $|f(z, \omega) - b(\omega)| \geqslant B > 0$, where B is independent of z and ω for $\delta \leqslant |z| \leqslant 1 - \delta$, $S_n' \subset S_n$ and $\omega \in E_n$.

We can now restate the problem with revised notation in the following form. If to each $\omega \in E$ there exists a complex number $b(\omega)$ satisfying $|b(\omega)| \leqslant 1$ and if $|f(z, \omega) - b(\omega)|$ is bounded away from zero uniformly with respect to z and ω for $\delta \leqslant |z| \leqslant 1 - \delta$ and $|\arg z| < \alpha$ and if $f(z, \omega) - b(\omega)$ has no zeros in the sector $0 < |z| < 1$, $|\arg z| \leqslant \alpha$, then E is necessarily of measure zero. To prove this result we shall suppose that E is of positive measure and show that this leads to a contradiction.

Whereas in Theorem 1 we used Jensen's theorem, we shall now make use of Green's Theorem. Let D_r be the domain common to the disks $|z| < r$, $|z\cos\alpha - r| < r\sin\alpha$; D_r lies in $0 < |z| < 1$; $|\arg z| \leqslant \alpha$ and so in $|\arg z| < 2\alpha$. The Green's function for this domain with respect to the point $z = r\cos\alpha$ is

$$G_r(z; r\cos\alpha) = -\log\left|\frac{r(z - r\cos\alpha)}{z(z\cos\alpha - r)}\right|.$$

If $f(z, \omega) - b(\omega)$ has no zeros in D_r, then $\log|f(z, \omega) - b(\omega)|$ is harmonic and by Green's theorem

$$\log|f(r\cos\alpha, \omega) - b(\omega)| = \frac{1}{2\pi}\int \log|f(z, \omega) - b(\omega)| \frac{\partial G}{\partial n} ds,$$

where the integral is taken round the boundary of D_r. This integral is in two parts

$$(9.1) \qquad I_1 = \frac{\cos\alpha}{\pi}\int_{-\alpha}^{\alpha} \log|f(re^{i\theta}, \omega) - b(\omega)| \frac{\cos\theta - \cos\alpha}{1 - 2\cos\theta\cos\alpha + \cos^2\alpha} d\theta$$

and

$$(9.2) \qquad I_2 = \frac{r\sin\alpha}{\pi}\int_{\frac{\pi}{2}+\alpha}^{\frac{3\pi}{2}-\alpha} \log|f(z, \omega) - b(\omega)| \frac{\partial G}{\partial n} d\psi,$$

where in the second integral $z = r\sec\alpha + r\tan\alpha\, e^{i\psi}$, and

$$0 \leqslant \frac{\partial G}{\partial n} \leqslant \frac{2\cos\alpha}{r(1 - \sin\alpha)}.$$

If E is the set of positive measure for which $f(z) - b(\omega)$ has no zeros in the given sector, then

$$(9.3) \qquad \int_E I_1(r, \omega) d\mu = \int_E \log|f(r\cos\alpha, \omega) - b(\omega)| d\mu - \int_E I_2(r, \omega) d\mu.$$

We observe that the integrand in I_1 is a logarithm multiplied by a non-negative expression. We can apply the same argument to this integral as we did to the integrals in Lemmas 5.4, 6.3 and 7.3 because the non-negative multiplier plays no role. There is one difference. The function $f(z) - b$ has no zeros in the sector and so there is no need to introduce the Blaschke product and the function $g(z)$. Since the function $g(z)$ does not enter we do not have to pass from (5.15) to (5.16). Hence we shall have for the sequence r_ν of Lemmas 5.4, 6.3, 7.3

$$(9.4) \qquad \int_E I_1(r_\nu, \omega) d\mu \geqslant \left(1 - \frac{2\alpha}{\pi}\right)\mu(E)\log\mathfrak{M}(r_\nu) + C\mu(E)\log\mu(E) - C\big(\mathfrak{M}(r_\nu)\big)^{-1/8}$$

provided ν is large enough.

We choose δ so that $r\cos\alpha < \cos\alpha < 1 - \delta$, then, by our hypothesis that $|f(z) - b|$ is bounded away from zero,

$$(9.5) \qquad \int_E \log|f(r\cos\alpha) - b(\omega)| d\mu \geqslant \mu(E)\log B$$

for some positive number B.

Also since by hypothesis $|f(z)-b|$ is bounded away from zero for $\delta \leqslant |z| \leqslant 1-\delta$ it follows that

$$\int_{\frac{\pi}{2}+2a}^{\frac{3\pi}{2}-2a} \log|f(z)-b(\omega)| \frac{\partial G}{\partial n} d\psi$$

is bounded below, so that in the case of I_2 it is sufficient to consider

$$I_3 = \int_E d\mu \int_{\frac{\pi}{2}+a}^{\frac{\pi}{2}+2a} \log|f(z,\omega)-b(\omega)| \frac{\partial G}{\partial n} d\psi .$$

We shall prove that

(9.6) $$I_3 = o\big(\log\mathfrak{M}(r_\nu)\big)$$

as ν tends to infinity and this is incompatible with (9.3), (9.4) and (9.5) so long as E has positive measure. In this way we shall arrive at a contradiction and so establish Theorem 2.

The argument for I_3 differs in certain respects from that used for Theorem 1. This is because that theorem used inequalities (5.3), (6.3) and (7.2) applicable to Theorem 1 and the integral of (9.1), where $|z|$ was constant in the range of integration. But in I_2 of (9.2) and I_3 of (9.6) $|z|$ varies and inequalities (5.3), (6.3) and (7.2) cannot be used. We consider first the case of § 5. We divide the range $(\frac{1}{2}\pi+a,\ \frac{1}{2}\pi+2a)$ at a point $\beta = \beta_\nu$ chosen as follows. If

(9.7) $$z_{1,\nu} = r_\nu \sec a + r_\nu \tan a\, e^{i(a+\beta)}$$

write $r_1 = r_{1,\nu} = |z_{1,\nu}|$ and choose r_1 so that the following expressions hold as r_ν and $r_{1,\nu}$ tend to unity

(9.8) $$\mathfrak{M}\Big(r_1+4\big(\mathfrak{M}(r_1)\big)^{-1/2}\Big) \leqslant 4\mathfrak{M}(r_1),$$

(9.9) $$\log\mathfrak{M}(r_{1,\nu})/\log\mathfrak{M}(r_\nu) \to 0 .$$

Let

(9.10) $$I_3 = \int_E d\mu \int_{\frac{\pi}{2}+a}^{\frac{\pi}{2}+a+\beta} + \int_E d\mu \int_{\frac{\pi}{2}+a+\beta}^{\frac{\pi}{2}+2a} = I_4 + I_5 .$$

We divide the range $(\frac{1}{2}\pi+a,\ \frac{1}{2}\pi+a+\beta)$ into non-overlapping intervals $(\theta_j-\delta,\ \theta_j+\delta)$, where $\theta_1-\delta = \frac{1}{2}\pi+a$ and $\delta = \big(\mathfrak{M}(r_\nu)\big)^{-2}$. Write $z_j = r\sec a + r\tan a\, e^{i\theta_j}$ and $r_j = |z_j|$. Let

$$E_{1,j} = \{\omega \mid \omega \in E,\ |f(z_j,\omega)| \geqslant 3\},$$
$$E_{2,j} = \Big\{\omega \mid \omega \in E \setminus E_{1,j} \sup_{|z-z_j|\leqslant\delta} |f(z,\omega)-b(\omega)| \geqslant \delta\big(\mathfrak{M}(r_\nu)\big)^{-3}\Big\},$$
$$E_{3,j} = E \setminus E_{1,j} \cup E_{2,j}$$

and

$$I_4 = I_6+I_7+I_8,$$

where for $p = 6, 7, 8$

$$I_p = \sum_j \int_{E_{p,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} .$$

In $E_{1,j}$ we have

$$|f(z)-b(\omega)| \geqslant |f(z_j)| - |f(z_j)-f(z)| - |b(\omega)|$$
$$\geqslant 2 - |z_j - z| \sum_1^\infty n|a_n|(r_\nu+\tau_\nu)^n \geqslant 1 .$$

Hence $I_6 \geqslant 0$.

If $\omega \in E_{2,j}$, then we have

(9.11) $$\mu(E_{2,j}) \leqslant \mathrm{Prob}\big(|f(z_j)| \leqslant 3\big) \leqslant 2\big(\mathfrak{M}(r_{1,\nu})\big)^{-1/3} .$$

Denote by ζ_j the point on $|z-z_j| = \delta$, where $|f(z)-b|$ attains its maximum. We have

$$|f(z)-b| \leqslant C\tau^{-1/2}\,\mathfrak{M}(r_\nu+\tau) \leqslant C\big(\mathfrak{M}(r_\nu)\big)^{5/4} .$$

We apply Lemma 5.3 with $z_0 = \zeta_i$, $\varphi(z)$ replaced by $f(z)-b$, $A = 4\delta$ and $B = 2\delta$. It follows from (5.9) that

$$\log|f(z)-b| \geqslant -C\log\mathfrak{M}(r_\nu) .$$

Therefore, in view of (9.11),

$$I_7 \geqslant -C\log\mathfrak{M}(r_\nu)\cdot\big(\mathfrak{M}(r_{1,\nu})\big)^{-1/3} .$$

If $\omega \in E_{3,j}$ as in the case of (5.14)

$$|f'(z_j,\omega)| \leqslant \big(\mathfrak{M}(r_\nu)\big)^{-3}$$

and so

(9.12) $$\mu(E_{3,j}) \leqslant C\big(\mathfrak{M}(r_\nu)\big)^{-1} .$$

Now $|f(\frac{1}{2}z_j,\omega)-b|$ is bounded away from zero, since $\frac{1}{2}z_j$ is within the sector. Hence, by Lemma 5.3 applied to the disks $|z-\frac{1}{2}z_j| \leqslant \frac{1}{2}|z_j|+\delta$ and $|z-\frac{1}{2}z_j| \leqslant \frac{1}{2}|z_j|+\tau$ we get

(9.13) $$\log|f(z)-b| \geqslant -C\big(\mathfrak{M}(r_\nu)\big)^{1/2}\log\mathfrak{M}(r_\nu)$$

and

$$I_8 \geqslant -C\big(\mathfrak{M}(r_\nu)\big)^{-1/2}\log\mathfrak{M}(r_\nu) .$$

Combining these integrals we get

$$I_4 \geqslant -C\big(\mathfrak{M}(r_{1,\nu})\big)^{-1/3}\log\mathfrak{M}(r_\nu) .$$

Therefore, if $r_{1,\nu}$ is chosen so that

$$\big(\mathfrak{M}(r_{1,\nu})\big)^{-1} \leqslant \big(\mu(E)\big)^{12}$$

we have

$$(9.14) \qquad I_4 \geqslant -C\mu(E)\big(\mathfrak{M}(r_{1,\nu})\big)^{-1/4}\log\mathfrak{M}(r_\nu).$$

For the integral I_5 we divide the range $\left(\dfrac{\pi}{2}+\alpha+\beta,\ \dfrac{\pi}{2}+2\alpha\right)$ into non-overlapping intervals $\theta_j-\delta,\ \theta_j+\delta$, where

$$\delta = \big(\mathfrak{M}(r_{1,\nu})\big)^{-1}.$$

Write

$$z_j = r\sec\alpha + r\tan\alpha\, e^{i\theta_j}$$

and define

$$E_{1,j} = \Big\{\omega\,|\,\omega\in E \sup_{|z-z_j|\leqslant\delta}|f(z)-b| \geqslant \delta\big(\mathfrak{M}(r_{1,\nu})\big)^{-3}\Big\},$$

$$E_{2,j} = E\setminus E_{1,j}.$$

Then write

$$I_5 = \sum_j \int_{E_{1,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} + \sum_j \int_{E_{2,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} = I_9 + I_{10}.$$

In virtue of (9.8) we have

$$\sum_0^\infty |a_n|\big(r_{1,\nu}+\tau(r_{1,\nu})\big) \leqslant \big(\mathfrak{M}(r_{1,\nu})\big)^{5/4}.$$

So if ζ_j is the point on $|z-z_j| = \delta$, where $|f(z)-b|$ takes its maximum we have, just as in the case of I_7, in $E_{1,j}$

$$\log|f(z)-b| \geqslant -C\log\mathfrak{M}(r_{1,\nu})$$

whence

$$I_9 \geqslant -C\mu(E)\log\mathfrak{M}(r_{1,\nu}).$$

If $\omega\in E_{2,j}$ we have, just as in (9.12),

$$\mu(E_{2,j}) \leqslant C\big(\mathfrak{M}(r_{1,\nu})\big)^{-1}$$

further, just as in the case of (9.13),

$$\log|f(z)-b| \geqslant -C\big(\mathfrak{M}(r_{1,\nu})\big)^{1/2}\log\mathfrak{M}(r_{1,\nu})$$

whence

$$I_{10} \geqslant -C\mathfrak{M}(r_{1,\nu})^{-1/2}\log\mathfrak{M}(r_{1,\nu})$$

so that for $r_{1,\nu}$ large enough we have

$$I_5 \geqslant -C\mu(E)\log\mathfrak{M}(r_{1,\nu}).$$

Therefore using (9.14) in the case corresponding to that in § 5 we have

$$I_3 \geqslant -C\mu(E)\Big\{\big(\mathfrak{M}(r_{1,\nu})\big)^{-1/4}\log\mathfrak{M}(r_\nu) + \log\mathfrak{M}(r_{1,\nu})\Big\}$$

so that by (9.9) if $\mu(E) > 0$

$$I_3/\log\mathfrak{M}(r_\nu) \to 0$$

as $r_\nu \to 1$.

We now turn to the case corresponding to that considered in §§ 6 and 7. We choose $r_1 = r_{1,\nu}$ so that, as r_ν and $r_{1,\nu}$ tend to unity, we have

$$(9.15) \qquad \mathfrak{M}\big(\tfrac{1}{2}(1+r_{1,\nu})\big) \leqslant 8\,\mathfrak{M}(r_{1,\nu}),$$

$$(9.16) \qquad \log\mathfrak{M}(r_{1,\nu})/\log\mathfrak{M}(r_\nu) \to 0,$$

$$(9.17) \qquad \log(1-r_{1,\nu})/\log\mathfrak{M}(r_\nu) \to 0.$$

It is clear that these conditions can always be satisfied for an infinity of $r_{1,\nu}$. To each $r_{1,\nu}$ we determine $\beta = \beta_\nu$ to satisfy (9.7) and partition the range of integration as in (9.10). We first consider the integral

$$I = \frac{r\sin\alpha}{2\pi}\int_E d\mu \int_{\frac{\pi}{2}+\alpha+\beta}^{\frac{\pi}{2}+2\alpha} \log|f(z)-b|\,\frac{\partial G}{\partial r}\,d\psi.$$

As before we divide the range of integration into non-overlapping intervals $(\theta_j-\delta,\ \theta_j+\delta)$, where $\delta = (1-r_1)^3$ and write z_j for the point determined by θ_j. Define

$$E_{1,j} = \{\omega\,|\,\omega\in E \sup_{|z-z_j|\leqslant\delta}|f(z)-b| \geqslant (1-r_{1,\nu})^7\},$$

$$E_{2,j} = E\setminus E_{1,j}.$$

Denote by ζ_j the point on $|z-z_j| = \delta$, where $|f(z)-b|$ attains its maximum and apply Lemma 5.3 to this function with $z_0 = \zeta_1$, $A = 4\delta$, $B = 2\delta$. By (5.9) we have

$$\log|f(z)-b| \geqslant -C\log\mathfrak{M}(r_1) + C\log(1-r_1)$$

whence

$$(9.18) \qquad I_1 = \sum_j \int_{E_{1,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} d\psi \geqslant -C\mu(E)\{\log\mathfrak{M}(r_1) - \log(1-r_1)\}.$$

If $\omega\in E_{2,j}$ as in (5.14)

$$|f'(z,\omega)| \leqslant (1-r_1)^4$$

and so

$$\mu(E_{2,j}) \leqslant C(1-r_1)^{4/3}.$$

Applying Lemma 5.3 to $f(z)-b$ with $z_0 = \frac{1}{2}z_j$, $A = \frac{1}{2}|z_j|+\frac{1}{2}(1-r_1)$ and $B = \frac{1}{2}|z_j|+\delta$ we get

$$\log|f(z)-b| \geqslant -C(1-r_1)^{-1}\{\log\mathfrak{M}(r_1)-\log(1-r_1)\}.$$

Therefore

$$I_2 = \sum_j \int_{E_{2,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} d\psi \geqslant -C(1-r_1)^{1/3}\{\log\mathfrak{M}(r_1)-\log(1-r_1)\} \geqslant -C$$

and from this result and (9.18) we deduce, in virtue of (9.16) and (9.17), that

$$I/\log\mathfrak{M}(r_\nu) \to 0$$

as r_ν tends to unity.

Returning to the integral

$$I = \frac{r\sin\alpha}{2\pi}\int_E d\mu \int_{\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}+\alpha+\beta} d\psi$$

we divide the range into intervals $(\theta_j-\delta,\ \theta_j+\delta)$, where $\delta = (1-r_\nu)^{1/2}$ and writing z_j for the point corresponding to θ_j, define

$$F_j = F(z_j, \omega) = \sup_{|z-z_j|\leqslant\delta} |f(z,\omega)-b(\omega)|$$

and

$$E_{1,j} = \{\omega \mid \omega\in E,\ |f(z_j)|\geqslant 3\},$$

$$E_{2,j} = \{\omega \mid \omega\in E\setminus E_{1,j},\ F_j \geqslant (\mathfrak{M}(r_\nu))^{-4}\},$$

$$E_{3,j} = \{\omega \mid \omega\in E\setminus E_{2,j},\ F_j \leqslant (1-r_\nu)^{7}\},$$

$$E_{4,j} = E\setminus E_{1,j}\cup E_{2,j}\cup E_{3,j},$$

Then

$$I = I_1+I_2+I_3+I_4,$$

where

$$I_p = \sum_j \int_{E_{p,j}} d\mu \int_{\theta_j-\delta}^{\theta_j+\delta} d\psi.$$

As before

$$I_1 \geqslant 0.$$

By Applying Lemma 5.3 with z_0 at the point where $|f(z)-b|$ assumes its maximum on $|z-z_j| = \delta$ and with $A = \frac{1}{2}(1-r_\nu)$, $B = 2\delta$ we get, on the arc under consideration,

$$\log|f(z)-b| \geqslant C\log\mathfrak{M}(r_\nu).$$

Now

$$\mu(E_{2,j}) \leqslant (\mathfrak{M}(r_\nu))^{-1/3}$$

and therefore

$$I_2 \geqslant -C(\mathfrak{M}(r_{1,\nu}))^{-1/3}\log\mathfrak{M}(r_\nu).$$

If $\omega\in E_{3,j}$, as in (5.14)

$$|f'(z_j)| \leqslant (1-r_\nu)^{7/2}$$

and so

$$\mu(E_{3,j}) \leqslant C(1-r_\nu)^{7/6}.$$

By applying Lemma 5.3 with z_0 at $\frac{1}{2}z_j$ and with A and B differing by $\frac{1}{2}(1-r_\nu)$ we get on the arc under consideration

$$\log|f(z)-b)| \geqslant -C(1-r_\nu)^{-1}\{\log\mathfrak{M}(r_\nu)-\log(1-r_\nu)\}.$$

Whence

$$I_3 \geqslant -C.$$

This leaves I_4 which we deal with by the method of § 7. Indeed there is little significant change in the argument. The expression corresponding to (7.5) is

$$\mu(E_j\cap E_{j'}) \leqslant CA_{p-1}^{-1/3}\Big[\sum_1^\infty |a_n|^2\Big\{(r_j^n-r_{j'}^n)^2+4r_j^n r_{j'}^n\sin^2\frac{n}{2}(\theta_j-\theta_{j'})\Big\}\Big]^{-1/6}$$

$$\leqslant CA_{p-1}^{-1/3}\Big[\sum_1^\infty |a_n|^2 r_j^n r_{j'}^n \sin^2\frac{n}{2}(\theta_j-\theta_{j'})\Big]^{-1/6}$$

and we consider the two cases (i) when

$$\sum_1^\infty |a_n|^2 r_j^n r_{j'}^n \sin^2\frac{n}{2}(\theta_j-\theta_{j'}) \geqslant 1/A_{p-1}$$

and (ii) when the opposite inequality is satisfied. The remainder of the argument is as before. Combining these results we get

$$I \geqslant -C(\mathfrak{M}(r_{1,\nu}))^{-1/3}\log\mathfrak{M}(r_\nu)-C$$

whence

$$I/\log\mathfrak{M}(r_\nu) \to 0$$

as r_ν tends to unity.

Combining these inequalities and with the notation of (9.6) we have shown that

$$I_3/\log\mathfrak{M}(r_\nu) \to 0$$

as r_ν tends to unity. We have thus established (9.7) and proved Theorem 2.

References

[1] J.-P. Kahane, *Series de Taylor aléatoires Gaussiennes dans le disque unité*, Contemporary problems in the theory of analytic functions, Izdat. Nauk Moscow (1966), pp. 156–176.

[2] — *Some random series of functions*, Heath mathematical monographs, Lexington, Mass. 1968.

[3] J. E. Littlewood and A. C. Offord, *On the distribution of zeros and a-values of a random integral function II*, Annals of Math. 49 (1948), pp. 885–952 and 50 (1949), pp. 990–991.

[4] A. C. Offord, *The distribution of the values of entire functions whose coefficients are independent random variables* (I), Proc. London Math. Soc. (3), 14A (1965), pp. 199–238.

[5] — *The distribution of zeros of power series whose coefficients are independen random variables*, Indian Journal of Math. 9 (1967), pp. 175–196.

[6] G. Polya and G. Szägo, *Aufgaben und Lehrsätze aus der Analysis*, Band I Berlin (1925).

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Uniform algebras satisfying certain extension properties

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Abstract. In this paper we study a uniform algebra A on a compact metric space X. Here follows the main result. If for each closed subset F of X there is a closed neighborhood W of F and a constant k_F, such that for each $f\in A$ there is some $g\in A$ (resp. a sequence (g_n) in A) satisfying $g=f$ on F (resp. $\lim|g_n-f|_F=0$) while $|g|_W < k_F|f|_F$ (resp. $|g_n|_W < k_F|f|_F$), then $A=C(X)$ (resp. A is locally dense in $C(X)$).

Introduction. Let A be a uniform algebra on a compact space X, i. e. A is a closed separating subalgebra of $C(X)$ containing the constants. If F is a closed subset of X and if $f\in C(X)$ we put $|f|_F=\sup\{|f(x)|:\ x\in F\}$. The following two concepts will lead to the problems studied in this paper.

DEFINITION A. Let A be a uniform algebra on a compact space X. We say that A satisfies the local extension property on a closed set F in X if there is a closed neighborhood W of F and a constant C such that: $\forall f\in A$ there is a sequence (g_n) in A with $\lim|g_n-f|_F=0$ while $|g_n|_W\leqslant C|f|_F$ for all n.

DEFINITION B. Let A be a uniform algebra on a compact space X. We say that A satisfies the strong extension property on a closed set F in X if there is a closed neighborhood W of F and a constant C such that: $\forall f\in A$ there is some $g\in A$ satisfying $g=f$ on F while $|g|_W\leqslant C|f|_F$.

Now we can state the main results of this paper.

THEOREM 1. *Let A be a uniform algebra on a compact metric space X. If A satisfies the local extension property on each closed set in X, then A is locally dense in $C(X)$.*

THEOREM 2. *Let A be a uniform algebra on a compact metric space X. If A satisfies the strong extension property on each closed set in X, then $A=C(X)$.*

Finally we study a phenomena closely related to the local extension property. Let A be a uniform algebra with its maximal ideal space M_A and its Silov boundary S_A and let $f\in C(M_A)$. We say that f is boundedly approximable by A on a closed set F in M_A if there is a closed neighborhood