Interpolation of $r$-Banach spaces

by

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Abstract. The paper extends the interpolation theory of Lions and Peetre to $r$-Banach spaces. The extension permits application of the theory to $L(1, \infty)$ as well as to other spaces, not covered by the original theory.

The interpolation between $L^{p,q}$ spaces is carried out also for measure spaces which contain atoms, and this is applied to trigonometric series.

I. Introduction.

Definition 1. Let $B$ be a vector space over $C$. An $r$ norm on $B$ is a function $\| \cdot \|_r: B \to \mathbb{R}^+$ satisfying:

(a) $\|b\|_r = 0$ iff $b = 0$,

(b) for all $\lambda \in C$, all $b \in B$, $\|b\|_r = |\lambda| \|b\|_r$, 

(c) $\|b + b\|_r \leq \|b\|_r + \|b\|_r$.

An $r$ normed space is a topological vector space, whose topology is given by an $r$ norm. A complete $r$-normed space is called an $r$-Banach space.

Every $r$ normed space is an $r_1$ normed space for every $r_1 < r$, for $\| \cdot \|_{r_1}$ is an $r_1$ norm on $B$, defining the same topology on it.

From $\|a+b\|_r = \|a\|_r + \|b\|_r + \|2\|_r$ we see that non-trivial $r$ normed spaces exist for $r < 1$ only.

Let $T: (B_1, \| \cdot \|_r) \to (B_2, \| \cdot \|_r')$ be a linear operator between an $r_1$ and an $r_1$ normed space. It is easily seen that $T$ is continuous iff $c > 0$ exists so that for all $b \in B$, $\|Tb\|_r' \leq c \|b\|_r$.

$r$-Banach spaces occur naturally in analysis, $H^r$, $0 < r \leq 1$, are but one example.

Definition 2. Let $B$ be a vector space over $C$. A quasi-norm on $B$ is a function $\| \cdot \|_q: B \to \mathbb{R}^+$ satisfying:

(a) $\|b\|_q = 0$ iff $b = 0$,

(b) for all $\lambda \in C$, all $b \in B$, $\|b\|_q = |\lambda| \|b\|_q$, 

(c) a number $k = k(B)$ exists so that $\|b_1 + b_2\|_q \leq k \|b_1\|_q + \|b_2\|_q$.

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A quasi-normed space is a topological vector space, whose topology is given by a quasi-norm. It is easily seen that if \((B, || \cdot ||)\) is a \(r\)-normed space, then \((B, || \cdot ||)\) is quasi-normed, where \(||b|| = ||b||_r\), with \(k(B) = 2^{in(r-1)}\). We also have:

**Theorem 3.** (S. Rolewicz [7].) If \(B\) is a quasi-normed space, \(2^{in(r)} > 2k(B)\), \(B\) is \(r\)-normed.

Our work is motivated by the following considerations: \(L^p\) spaces (see [2]) which are in \(L^1+L^\infty\) appear as intermediate spaces in the interpolation theory of Lions and Peetre [4], [5] etc. This, however, leaves out \(L^\infty\) (weak \(L^1\)), and so Marcinkiewicz’s interpolation theorem is not proved in this context, for a case most important in applications. By considering interpolation between \(r\)-Banach spaces we shall obtain all \(L^p\) spaces \((0 < p < \infty, 0 < q < \infty; p = q = \infty)\) as intermediate spaces. The theory is also applicable to other problems, e.g. interpolation between \(L^\infty\) spaces \((0 < p < \infty)\), etc.

Kréô [3] has extended the \(K\) method of Peetre to quasi-normed spaces, and has obtained the identification of all \(L^p\) spaces as intermediate spaces [1]. However, Kréô does not make use of \(r\)-norms.

Using \(r\) norms, we are able to obtain theorems on the topological properties of the intermediate spaces not available in Kréô’s method, as well as interpolation theorems missing from his theory. As examples of the latter: The reiteration theorem of Lions and Peetre is generalized in full. Another example is the following result:

\[(L^{p_1}, L^{p_2})_{\alpha, \beta} = L^p,\]

where \(1/\alpha = (1-\beta)/p_1 + \beta/q_1, 0 < \beta < \infty, 0 < q_1 < \infty, 0 < q_1 < \infty\).

This generalizes a result of Peetre [5], who proved this under the assumptions \(1 < p < \infty, 1 < q_1 < \infty\). This result is particularly interesting, for it was previously available only by complex methods.

For an extension of the complex method of Calderón to \(r\)-Banach spaces, see Rivière [6].

In Sections II, III we generalize the work of Lions and Peetre [4], [5] in two directions: We consider \(r\)-Banach spaces rather than Banach spaces, and our parameters are in \([0, \infty)\) rather than in \([1, \infty)\). For the sake of completeness we have included proofs of all theorems, including those where the generalization of the proofs of Lions and Peetre is straightforward.

In Section IV we present the basic properties of \(L^p\) spaces. For an exposition on these spaces using elementary methods, the reader is referred to [2]. We then show that these spaces are intermediate between \(L^p\) spaces. From this we proceed to deduce various topological properties, and a generalization of the weak interpolation theorem of Hunt.

In Section V we give a unified account of theorems of Hausdorff-Young, Paley, Stein and others, on Fourier coefficients, using \(L^p\) spaces and the weak interpolation theorem. The connection between \(L^p\) spaces and these theorems was suggested practically since the time that \(L^p\) spaces were defined, and this section is of expository nature. Still it may be of some value — if only as a demonstration of the strength of the so-called weak interpolation theory.

**II. The spaces \((L_{\alpha, \beta}, B)\).**

**Theorem 1.** Let \(B\) be an \(r\)-normed space, \(V\) a topological vector space, \(T: B \to V\) a continuous linear operator. Let \(R = \text{range } T\). Define on \(R\) \(||\cdot||_T = \inf \{||\cdot||: f T = \cdot\}\). Then \((R, || \cdot ||)\) is an \(r\)-normed space. If \((B, || \cdot ||)\) is complete, so is \((R, || \cdot ||)\).

**Proof.** The last statement is the only one requiring verification \(\forall \alpha, \beta, \gamma, \delta, \epsilon \in B\). \(\alpha, \beta, \gamma, \delta \to 0\). Suffices to show the existence of a convergent subsequence. We therefore take a subsequence of the original sequence and can assume \(\alpha, \beta, \gamma, \delta < 2^{-n-1}\). Let \(b_1, b_\epsilon, b_\delta, b_\gamma, b_\beta, b_\alpha < 2^{-n} < 2^{-n}.\)

Let \(a_1, b_1\) satisfy \( Ta_1 = b_1.\) Define inductively \(a_n = a_{n-1} - b_{n-1}, T a_n = T a_{n-1} - b_{n-1} + b_n,\) and by induction \(T a_n = b_n, a_n - a_{n-1} = |b_n| < 2^{-n},\) and so \(a_n\) is a Cauchy sequence \(a_n \to a, Ta_n - a = a_n b_n \to 0\) and so \(a_n\) converges.

**Definition 2.** Let \((B_{\alpha}, || \cdot ||_\alpha), (B_{\beta}, || \cdot ||_\beta)\) be \(r\) and \(r\) normed. If both are continuously embedded in a topological vector space \(B\), we shall say that \((B_{\alpha}, || \cdot ||_\alpha, B_{\beta}, || \cdot ||_\beta)\) is an interpolation pair.

In the sequel, we shall omit the norms and write \((B_{\alpha}, B_{\beta})\). Also, when \((B, || \cdot ||)\) is \(r\)-normed, \(|| \cdot ||\) will stand for \(|| \cdot ||_r\).

**Theorem 3.** Let \((B_{\alpha}, B_{\beta})\) be an interpolation pair of \(r\)-normed spaces. Let \(r = \min(r_\alpha, r_\beta)\). Then \(||\cdot||_{B_{\alpha} \cap B_{\beta}} = \max(||\cdot||_{B_{\alpha}}, ||\cdot||_{B_{\beta}})\) is an \(r\) norm on \((B_{\alpha} \cap B_{\beta})\), while \(||\cdot||_{B_{\alpha} \cap B_{\beta}} = \max(||\cdot||_{B_{\alpha}}, ||\cdot||_{B_{\beta}})\) is an \(r\) norm on \((B_{\alpha} \cap B_{\beta})\).

**Proof.** The verification for \((B_{\alpha} \cap B_{\beta}), || \cdot ||_{B_{\alpha} \cap B_{\beta}}\) is immediate. As for \((B_{\alpha} \cap B_{\beta}, || \cdot ||_{B_{\alpha} \cap B_{\beta}})\) we note that \(T: (B_{\alpha}, || \cdot ||_{B_{\alpha}}) \times (B_{\beta}, || \cdot ||_{B_{\beta}}) \to B\) defined
by \(T(b_1, b_2) = b_1 + b_2\) defines \(\|B_1 + B_2\|\) as a \(T\) induced norm on its range \(B_1 + B_2\) as in Theorem 1.

**Definition 4.** Let \((B_1, \|\cdot\|_p)\) be an \(r\)-normed space, \(\langle u_n \rangle\) a sequence of positive numbers. Define

\[
\begin{align*}
\|u_n\|_{p, \langle a_i \rangle} & = \left(\sum_{i=1}^{n} |a_i u_n|^{p_i}\right)^{1/p_i}, 0 < p < \infty, \\
\|u_n\|_{p, \langle a_i \rangle} & = \sup_{i} |a_i u_n|.
\end{align*}
\]

Define also \(P_{\langle a_i \rangle} (B_1, \|\cdot\|)\) as the space of all sequences of elements of \(B\) so that \(\|u_n\|_{p, \langle a_i \rangle} < \infty\).

**Theorem 5.** \(P_{\langle a_i \rangle} (B_1, \|\cdot\|)\) is a \(s\)-normed space, where \(s = \min \{r, p\}\), and

\[
\|u_n\|_{s, \langle a_i \rangle} = \|u_n\|_{p, \langle a_i \rangle}.
\]

The spaces are complete if \((B_1, \|\cdot\|)\) is.

**Proof.** The proof proceeds along well-known arguments. We shall write \(P_{\langle a_i \rangle} (B_1, \|\cdot\|) = P(B)\) for \(P_{\langle a_i \rangle} (B_1, \|\cdot\|) = P(B)\).

**Definition 6.** Let \((B_1, B_2)\) be an interpolation pair of \(r_1, r_2\) normed spaces, \(0 < \theta < 1, 0 < p_1 < \infty\). Denote

\[
w(p_1, B_1; p_2, B_2; \theta) = P_{\theta, \infty} (B_1) \cap P_{1-\theta, \infty} (B_2),
\]

with the \(s\) norm:

\[
\|(u_n)\|_s = \max \left\{\left(\sum_{i=1}^{n} |a_i u_n|^{p_i}\right)^{1/p_i}\right\},
\]

where \(s = \min \{r_1, r_2, p_1, p_2\}\).

**Theorem 7.** Let \((B_1, B_2)\) be an interpolation pair of \(r_1, r_2\) Banach spaces. Then \(T: w(p_1, B_1; p_2, B_2; \theta) \to B_1 + B_2\) defined by \(T((u_n)) = \sum_{n} u_n\) is a well defined continuous linear transformation.

**Proof.**

\[
\begin{align*}
\sum_{n} |u_n|_{B_1} & = \sum_{n} |u_n| d^{\theta s_1}(p_1, B_1) d^{(1-\theta)s_2}(p_2, B_2), \quad \text{if now} \quad p_1 \leq r_1, \\
\sum_{n} |u_n| d^{\theta s_1}(p_1, B_1) d^{(1-\theta)s_2}(p_2, B_2) & \leq \sum_{n} |u_n| d^{\theta s_1}(p_1, B_1) d^{(1-\theta)s_2}(p_2, B_2), \\
\text{while if} \quad r_1 < p_1, \\
\sum_{n} |u_n| d^{\theta s_1}(p_1, B_1) d^{(1-\theta)s_2}(p_2, B_2) & \leq C(k, \theta, p_1, \theta) \sum_{n} |u_n| d^{\theta s_1}(p_1, B_1) d^{(1-\theta)s_2}(p_2, B_2),
\end{align*}
\]

with \(\lim_{k \to \infty} C(k, \theta, p_1, \theta) = 0\).
Conversely, if \( b = (B_1, B_2) \), \( p \in (p_r, p_t) \), take \( \sum_{n=0}^{\infty} u_n = v_a \), \( \sum_{n=0}^{\infty} v_n = v_b \), \( e^{-\alpha t} v_n = \sum_{n=0}^{\infty} e^{-\alpha n} u_n = \sum_{n=0}^{\infty} e^{-\alpha n + \alpha n} = \sum_{n=0}^{\infty} e^{-\alpha n} v_n, \)

Therefore:
\[
\sum_{n=0}^{\infty} e^{-\alpha n} u_n = \sum_{n=0}^{\infty} e^{-\alpha n} v_n = \sum_{n=0}^{\infty} e^{-\alpha n} v_n + \sum_{n=0}^{\infty} e^{-\alpha n} v_n.
\]

If now \( p_a < r < 1 \), use Young's inequality for convolutions:
\[
P_{r\beta} \left( \sum_{n=0}^{\infty} e^{-\alpha n} u_n \right) \left( \sum_{n=0}^{\infty} e^{-\alpha n} v_n \right) \leq \left( \sum_{n=0}^{\infty} e^{-\alpha n} u_n \right)^r \left( \sum_{n=0}^{\infty} e^{-\alpha n} v_n \right)^\beta.
\]

In any case \( |e^{\alpha n} v_n|, |e^{\alpha n} u_n| \leq \mathcal{C} |e^{\alpha n} u_n|, |e^{\alpha n} u_n| \). Similarly:
\[
|e^{\alpha n} u_n| |e^{\alpha n} v_n| \leq \mathcal{C} |e^{\alpha n} u_n| |e^{\alpha n} u_n| \]

and the proof is complete.

**Theorem 12.** Let \((B_1, B_2)\) be an interpolation pair of \( r_1, r_2 \) normed spaces. Then:
\[
\|v\|_{L^p(B_1) \cap L^q(B_2)} \leq \min \{ \|v\|_{L^p(B_1)}^{1-p} \|v\|_{L^q(B_2)}^{1-q}, \|v\|_{L^q(B_2)}^{1-q} \|v\|_{L^p(B_1)}^{1-p} \}.
\]

If \( B \) is complete, then:
\[
\|v\|_{L^p(B_1) \cap L^q(B_2)} \leq \min \{ \|v\|_{L^p(B_1)}^{1-p} \|v\|_{L^q(B_2)}^{1-q}, \|v\|_{L^q(B_2)}^{1-q} \|v\|_{L^p(B_1)}^{1-p} \}.
\]

**Proof.** Let \( (v_n) e^{\alpha n} u_n(B_1), v_n + v_m = b \). Denote \( v_n = v_{n+m} \). We have:
\[
\|e^{\alpha n} u_n\|_{L^p(B_1)} = \|e^{\alpha n} u_n\|_{L^q(B_2)}.
\]

Let \( \gamma \) be determined by:
\[
e^{\alpha \gamma} \|e^{\alpha n} u_n\|_{L^p(B_1)} = e^{\alpha \gamma} \|e^{\alpha n} u_n\|_{L^q(B_2)}
\]

and take \( k = [\gamma] + 1 \).
\[
e^{\alpha k} \|e^{\alpha n} u_n\|_{L^p(B_1)} = e^{\alpha k} \|e^{\alpha n} u_n\|_{L^q(B_2)}
\]

and so:
\[
\|v\|_{L^p(B_1) \cap L^q(B_2)} \leq \mathcal{C} \|v\|_{L^p(B_1)}^{1-p} \|v\|_{L^q(B_2)}^{1-q}.
\]

Since:
\[
\|e^{\alpha n} u_n\|_{L^p(B_1)} \leq \max \{ \|e^{\alpha n} u_n\|_{L^p(B_1)}, \|e^{\alpha n} u_n\|_{L^q(B_2)} \},
\]

the first claim is proved. The second follows from it via the construction in the proof of Theorem 10, or else we can be done directly as above.

**Theorem 13.** If \( p_1 < \frac{1}{\lambda}, (B_1, B_2) \) is an interpolation pair of \( r_1, r_2 \) normed spaces, then:

\[
(B_1, B_2) \in \mathcal{I}_{\mu, \nu}^p(B_1, B_2).
\]

**Proof.**

**Definition 15.** (see [3]) Let \((A_1, A_2), (B_1, B_2)\) be two interpolation pairs of \( r_1, r_2 \) normed spaces. \( \Gamma \) is a linear operator from \((A_1, A_2)\) to \((B_1, B_2)\) iff for every \( a \in A_1, A_2 \), we can find \( b \in B_1 \) so that:
\[
\|T(a) + b\|_{B_1} \leq K\|a\|_{A_1}.
\]

Of course if \( T \) is a linear operator from \((A_1, A_2)\) to \((B_1, B_2)\), then if \( 0 < \eta < 1 \), \( 0 < \eta < \eta \), \( 0 < \eta \leq \eta \),
T: \((A_{n}, A_{n})^{p_{n}, q_{n}} \rightarrow (B_{n}, B_{n})^{p_{n}, q_{n}}\)
and \(|T|_{0} \leq C K_{n}^{1/q_{n}} K_{n}^{1/q_{n}}\), where \(K_{n}\) are the constants appearing in (15).

Proof. Let \(a \in (A_{n}, A_{n})^{p_{n}, q_{n}}\), \(a = v_{n} + w_{n}\), \((\ell^{R_{n}}v_{n}, w_{n})_{\ell^{R_{n}}(A_{n})}, i = 0, 1, v_{n} + A_{n}\) and so for every \(n\) we have, \(v_{n} + w_{n}\) so that

\[ T_{n} = w_{n} + v_{n}, \text{ and } \|w_{n}\|_{0} \leq K_{n}\|v_{n}\|_{A_{n}}. \]

Therefore \(\|\langle \ell^{R_{n}}u_{n}, v_{n}\rangle\|_{\ell^{R_{n}}(A_{n})} \leq K_{n}\|\langle \ell^{R_{n}}u_{n}, v_{n}\rangle\|_{\ell^{R_{n}}(A_{n})}\) and so

\[ \|T_{n}\|_{(\ell^{R_{n}}B_{n}, \ell^{R_{n}}B_{n})^{p_{n}, q_{n}}} \leq c\|\langle \ell^{R_{n}}u_{n}, v_{n}\rangle\|_{\ell^{R_{n}}(A_{n})}^{1/q_{n}} \|\langle \ell^{R_{n}}u_{n}, v_{n}\rangle\|_{\ell^{R_{n}}(A_{n})}^{1/p_{n}} \leq cK_{n}^{1/q_{n}} K_{n}^{1/q_{n}} \|\langle \ell^{R_{n}}u_{n}, v_{n}\rangle\|_{\ell^{R_{n}}(A_{n})}^{1/q_{n}} \|\langle \ell^{R_{n}}u_{n}, v_{n}\rangle\|_{\ell^{R_{n}}(A_{n})}^{1/p_{n}}. \]

Taking the infimum of last expression over all sequences \(\{v_{n}\}\), \(\{w_{n}\}\) so that \(v_{n} + w_{n} = a\), we get

\[ \|T\|_{(\ell^{R_{n}}B_{n}, \ell^{R_{n}}B_{n})^{p_{n}, q_{n}}} \leq cK_{n}^{1/q_{n}} K_{n}^{1/q_{n}} \|a\|_{A_{n}} \|a\|_{A_{n}}. \]

**Definition 11.** Let \((X, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, \(0 \leq \mu\).

Let \((B_{1}, \|\cdot\|)\) be a \(\tau\) normed space. Denote by \(L^{p}(B)\) the space of all strongly measurable \(B\) valued functions on \((X, \Sigma, \mu)\) so that \(\|f\|_{L^{p}(B)} < \infty\), where

\[(a) \quad \|f\|_{L^{p}(B)} = \left\{ \int |f|^{p} \text{d} \mu \right\}^{1/p}, \quad 0 < p < \infty, \]

\[(b) \quad \|f\|_{L^{1}(B)} = \text{Ess. Sup.} \|f\|_{B}. \]

**Theorem 18.** \(L^{p}(B)\) is a \(\sigma\)-space with \(\psi = \min\{r, p\}, \|\cdot\|_{L^{p}(B)} = |\|\cdot\|_{B}|_{L^{p}(B)}\).

If \(B\) is complete, so is \(L^{p}(B)\).

Proof. Triangle inequality is all we have to verify for the first claim.

If \(s = p\),

\[ |f + g|_{L^{p}(B)} = \left( \int |f + g|^{p} \text{d} \mu \right)^{1/p} \leq \left( \int |f|^{p} \text{d} \mu + \int |g|^{p} \text{d} \mu \right)^{1/p} = |\|f\|_{L^{p}(B)} + |\|g\|_{L^{p}(B)}|, \]

while if \(s = \tau\)

\[ |f + g|_{L^{p}(B)} = \left( \int |f + g|^{\tau} \text{d} \mu \right)^{1/\tau} \leq \left( \int |f^{\tau} \text{d} \mu + \int |g^{\tau} \text{d} \mu \right)^{1/\tau} \leq \left( |\|f\|_{B}|^{\tau} \right)^{1/\tau} + \left( |\|g\|_{B}|^{\tau} \right)^{1/\tau}. \]

The last inequality is Minkowski's for \(p \geq 1\).

The proof of completeness of \(L^{p}(B)\), given that of \(B\), follows along the same lines as the corresponding proof for Banach space valued functions.

**Theorem 19.** Let \((B_{1}, B_{1})\) be an interpolation pair of \(r_{1}\), \(r_{1}\) normed spaces. Then:

\[ L^{p_{1}}(B_{1})^{p_{1}, q_{1}} = L^{p_{1}}(B_{1})^{p_{1}, q_{1}}, \]

where \(0 < p_{1}, p_{1} \leq \infty\) and \(\frac{1}{p_{1}} = \frac{1}{p} - \frac{q}{1 - p_{1}} = \frac{1}{p}. \]

Proof. Let \(f(x) \in (L^{p_{1}}(B_{1}), L^{p_{1}}(B_{1}))^{p_{1}, q_{1}}\), \(f(x) = v_{n}(x) + v_{n}(x)\) a.e.,

where \(v_{n}(x) \in L^{p_{1}}(B_{1})\) for every \(n\), and \(\|v_{n}(x) + v_{n}(x)\|_{L^{p_{1}}(B_{1})} \leq v_{n}(x)\) for almost all \(\xi\).

Using Hölder's inequality:

\[ \int f(x) \|v_{n}, p_{1}, q_{1}, p_{1}, q_{1} \|^{1/q_{1}} \quad \leq \quad c \left( \int \sum_{n} |v_{n}(x)|^{q_{1}p_{1}} \right)^{1/q_{1}} \left( \int \sum_{n} |v_{n}(x)|^{p_{1}q_{1}} \right)^{1/q_{1}}. \]

and so:

\[ \|f(x)\|_{L^{p_{1}}(B_{1})^{p_{1}, q_{1}}} \leq c \left( \sum_{n} |v_{n}(x)|^{q_{1}p_{1}} \right)^{1/q_{1}} \left( \sum_{n} |v_{n}(x)|^{p_{1}q_{1}} \right)^{1/q_{1}}. \]

Conversely: Let \(f = \sum_{n} b_{n}x_{n}\) be in \(L^{p_{1}}(B_{1})^{p_{1}, q_{1}}\), where \(x_{n}\) are characteristic functions of disjoint measurable sets \(B_{x_{n}}(B_{1})^{p_{1}, q_{1}}\) and so we can write \(f(x) = v_{n}(x) + v_{n}(x)\), where:

\[ \int \|f(x)\|_{L^{p_{1}}(B_{1})^{p_{1}, q_{1}}} \geq \max_{i = 1, \ldots, n} \left( \|v_{n}(x)\|_{L^{p_{1}}(B_{1})} \right). \]

Let \(\lambda\) be given by: \(p_{1}(1 - \lambda(1 - \theta)) = \theta. \) We then have:

\[ p_{1}(1 - \lambda(1 - \theta)) = \theta. \]

Taking \(w_{n} = v_{n},\) where \(k = \lambda \log \|f(x)\|_{L^{p_{1}}(B_{1})^{p_{1}, q_{1}}},\) we have:

\[ \sum_{n} |v_{n}(x)|^{q_{1}p_{1}} \leq c \sum_{n} |v_{n}(x)|^{p_{1}q_{1}} \exp \left( \lambda \theta - \theta \right) p_{1} \log \|f(x)\|_{L^{p_{1}}(B_{1})^{p_{1}, q_{1}}} \]

and so:

\[ \|v_{n}(x)\|_{L^{p_{1}}(B_{1})^{p_{1}, q_{1}}} \leq c \left( \sum_{n} |v_{n}(x)|^{q_{1}p_{1}} \right)^{1/q_{1}} \left( \sum_{n} |v_{n}(x)|^{p_{1}q_{1}} \right)^{1/q_{1}}. \]

The proof of completeness of \(L^{p}(B)\), given that of \(B\), follows along the same lines as the corresponding proof for Banach space valued functions.
However:
\[
\|f\|_{L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1)))^{\omega p_0}} \leq \epsilon \|f\|_{L^p(a_0,a_1)}^\omega \|f\|_{L^p(a_0,a_1)}^{1-\omega} \\|f\|_{L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1)))^{\omega p_0}}.
\]

Since functions of the form given above are dense in \(L^p\) the proof is complete.

**Theorem 20.** Let \((B_2, B_1)\) be an interpolation pair of \(r_0, r_1\) normed spaces. Then for \(b \in B_2 \cap B_1\), we have
\[
\|b\|_{(B_0, B_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1))))} \leq \|b\|_{B_0}^\omega \|b\|_{B_1}^{1-\omega}.
\]

Proof. Take
\[
v_m = \begin{cases} 1 & m \geq 0, \\ 0 & m < 0, \end{cases}
\]

and hence the result follows from Theorem 12.

**Definition 21.** Let \((B_0, B_1)\) be an interpolation pair of \(r_0, r_1\) Banach spaces, \(r_0 < \theta < r_1\), \(B_0 \cap B_1 \subset B_0 \cap B_1\). We define: \(B_0, B_1\) is \(r\) Banach space if there exists a constant \(c\) such that for every \(b \in B_0 \cap B_1\),
\[
\|b\|_\theta \leq c \|b\|_0 \|b\|_1.
\]

**Theorem 22.** Let \((B_0, B_1)\) be an interpolation pair of \(r_0, r_1\) Banach spaces, \(B_0 \cap B_1 \subset B_0 \cap B_1\). Then: If for some \(p > 0\), \(B_0H_p \cap B_1H_p \subset B_0 \cap B_1\), then \(B_0, B_1\) is \(r\) Banach space.

If \(b \in B_0 \cap B_1\), then \(B_0, B_1\) is \(r\) Banach space. If \(p > 0\), \(B_0H_p \cap B_1H_p \subset B_0 \cap B_1\).

**Theorem 23.** Let \((B_0, B_1)\) be an interpolation pair of \(r_0, r_1\) Banach spaces. Let \((X_0, X_1) \in \mathcal{K}(B_0, B_1), \ i = 0, 1, \eta \in (0, \theta_0)\). Then if
\[
\frac{1}{q_i} = \frac{1 - \eta_i}{p_0} + \frac{\eta_i}{p_1},
\]

we have
\[
\|f\|_{L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1)));(X_0, X_1))} \leq \|f\|_{L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1)));(X_0, X_1))}.
\]

Proof. Let \(b \in B_0 \cap B_1\), then
\[
\|b\|_{(L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1))))^{\omega p_0}} \leq \|b\|_{B_0}^\omega \|b\|_{B_1}^{1-\omega}.
\]

Using Hölder's inequality:
\[
\|f\|_{L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1))))} \leq \|f\|_{L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1))))}.
\]

and using Theorem 11:

**Theorem 24.** Let \((B_0, B_1)\) be an interpolation pair of \(r_0, r_1\) normed spaces, \(r_0 < \theta < r_1\). Let \(B \subset B_0 \cap B_1\) be a \(r\) normed space. Then we define:
\[
B_0, B_1\) is an \(r\) Banach space if there exists a constant \(c\) such that for every \(b \in B_0 \cap B_1\),
\[
\|b\|_\theta \leq c \|b\|_0 \|b\|_1.
\]

**Theorem 25.** Let \((B_0, B_1)\) be an interpolation pair of \(r_0, r_1\) normed spaces, \(r_0 < \theta < r_1\). Then:
\[
(X_0, X_1) \in \mathcal{K}(B_0, B_1), \ i = 0, 1, \eta \in (0, \theta_0)\). Then:
\[
\|f\|_{L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1))))} \leq \|f\|_{L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1))))}.
\]

Proof. Let \(x \in (X_0, X_1)\), then
\[
\|b\|_{(X_0, X_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1;L^p(a_0,a_1))))} \leq \|b\|_{X_0}^\omega \|b\|_{X_1}^{1-\omega}.
\]

And so:
\[
\|b\|_\theta \leq \|b\|_{X_0}^\omega \|b\|_{X_1}^{1-\omega}.
\]
Theorem 29. Let \((A_0, A_1)\) be an interpolation pair of \((r_0, r_1)\) Banach spaces, \((B_0, B_1)\) of \(q_0, q_1\) normed spaces. Let \(0 < q_0 < q_1 < 1\), \(0 < q_0 \leq q_1 < 1\), \(\sigma = (q_0 - q_1)/(q_1 - q_0) = (q_1 - q_0)/(q_1 - q_0)\), \(X \in K_{r_0}(A_0, A_1)\), and \(Y \in K_{r_1}(A_0, A_1)\). Let further \(0 < p_0, p_1 \leq \infty\) and \(q_i\) be determined by \[\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{q_0} + \frac{1}{q_1}.\] If then \(T\) is a quasi-linear operator from 
\((X_0, X_1)\) to \((Y_0, Y_1)\), we have:

\[\|T\|_{(B_0, B_1)_{r_0} \to (B_0, B_1)_{r_1}} \leq c\|K_{r_0}^{\sigma - q_0} T K_{r_1}^{q_1 - q_0}\|_{(A_0, A_1)_{r_0} \to (A_0, A_1)_{r_1}}.\]

Proof. a \((\sigma (A_0, A_1)_{r_0 + r_1})\). Then \(a \in X_0, X_1 \in r_0 + r_1 \) with \[\|a\|_{r_0 + r_1} \leq c\|a\|_{(A_0, A_1)_{r_0 + r_1}}.\]

From interpolation theorem:

\[\|T\|_{(r_0, r_1)_{r_0 + r_1} \to (l_0, l_1)_{r_0 + r_1}} \leq c\|K_{r_0}^{\sigma - q_0} T K_{r_1}^{q_1 - q_0}\|_{(A_0, A_1)_{r_0} \to (A_0, A_1)_{r_1}}.\]

while from Theorem 26

\[\|T\|_{(X_0, X_1)_{r_0 + r_1} \to (Y_0, Y_1)_{r_0 + r_1}} \leq c\|T\|_{(X_0, X_1)_{r_0 + r_1} \to (Y_0, Y_1)_{r_0 + r_1}}.\]

For \(\sigma = q_1 - q_0 = \frac{1}{r_1} - \frac{1}{r_0}\), Since \(\sigma = \frac{1}{r_0}\), \(\|a\|_{r_0 + r_1} \leq c\|a\|_{(A_0, A_1)_{r_0 + r_1}}.\)

Combining the three inequalities, the theorem follows.

III. The spaces \((B_0, B_1)_{r_0 + r_1}\).

Definition 1. Let \((B_0, B_1)\) be an interpolation pair of \(r_0, r_1\) normed spaces. Define on \(B_0 + B_1\):

\[M(a, b) = \max \{\|a\|_{B_0}, \|b\|_{B_1}\}\]

and on \(B_0 + B_1\):

\[W(a, b) = \max \{\|a\|_{B_0}, \|b\|_{B_1}\}\].

These are the coarse analogues of the \(J\) and \(K\) functionals of Peetre [5]. If \(r = \min\{r_0, r_1\}\), then \(M(a, b), W(a, b)\) are \(r\) norms on \(B_0 + B_1, B_0 + B_1\), both equivalent to the usual ones.

Theorem 2. Let \((B_0, B_1)\) be an interpolation pair of \(r_0, r_1\) normed spaces. Then \(b \in (B_0, B_1)^{r_0, r_1}\) iff \(e^{r_0} W(a, b) \in p\). Further

\[\|b\|_{(B_0, B_1)^{r_0, r_1}} \leq c\|e^{r_0} W(a, b)\|_p.\]

Proof. Let \(b \in (B_0, B_1)^{r_0, r_1}\), \(b = a + v, v_1\), with

\[B_0 + B_1 = c\|v\|_{B_0 + B_1}.\]

Theorem 28 (Reiteration theorem). If \((B_0, B_1)\) is an interpolation pair of \(r_0, r_1\) Banach spaces, \(X \in K_{r_0}(B_0, B_1), 0 < q_0 < q_1 < 1\), then

\[(X_0, X_1)_{(r_0, r_1)_{r_0 + r_1}} = (B_0, B_1)_{r_0 + r_1}, \quad \text{where} \quad \frac{1}{q_i} = \frac{1}{p_i} + \frac{1}{p_i}.\]

The theorem is of course a combination of Theorems 23 and 26. The interpolation theorem can be combined with Theorems 23, 26 to yield the following theorem:
conversely, if \( e^{-\alpha n} W(n, b) i p \) for every \( n \) we can find \( v_n \) so that
\[
\max_{i \leq n} (\|v_i\|_p) < c \|W(n, b)\|_p, \quad v_n + v_{n-1} = b.
\]
Therefore:
\[
\|b\|_{p, p}^p \leq c \|e^{-\alpha n} W(n, b)\|_p.
\]

**Theorem 3.** Let \((B_1, B_2)\) be an interpolation pair of \( r_1, r_1 \) Banach spaces. Then \( b \in (B_1, B_2)_{h(p, p)} \) if there exists a sequence \( \{u_n\} \), \( u_i \neq B_1 \cap B_2 \), so that
\[
\sum_{i=1}^{\infty} u_i = b \quad \text{and} \quad (e^{-\alpha n} M(n, u_n)) \in p.
\]
Further:
\[
\|b\|_{h(p, p)} \sim \text{Inf} \left( \|e^{-\alpha n} M(n, u_n)\|_p \right) \sum_{i=1}^{\infty} u_i = b.
\]

Proof. Similar to that of Theorem 2.

Our next objective is to show that if \((B_1, B_2)\) is an interpolation pair of \( r_1, r_1 \) Banach spaces, then
\[
(B_1, B_2)_{h(p, p)} = (B_1, B_2)_{h(p, p)}, \quad \text{where} \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}
\]
we shall need the following result:

**Theorem 4.** Let \( 0 < p < \infty, -\infty < a_1 < \infty, 0 < \theta < 1, a = (1 - \theta) a_0 + \theta a_1. \) Then:
\[
(\ell_{-a_0}^a, \ell_{-a_1}^a)_{h(p, p)} = \ell_{-a}^a.
\]

Proof. Let \( b = p_1 (a_1 - a_0) / (p_1 - p_0) \). Denote by \( d \) the space of all sequences \( \{u_n\} \) so that
\[
\|u_n\|_p = \left( \sum_{n=1}^\infty |u_n|^p e^{\alpha n} \right)^{1/p} < \infty.
\]
Define then \( I(\{u_n\}) = \left\{ \frac{e^{\alpha n} u_n}{e^{\alpha n} \alpha_0 / e^{\alpha n} \alpha_0} \right\}, \) we have:
\[
\|I(\{u_n\})\|_p = \left( \sum_{n=1}^\infty \left| \frac{e^{\alpha n} u_n}{e^{\alpha n} \alpha_0 / e^{\alpha n} \alpha_0} \right|^p \right)^{1/p} = \|u_n\|_p.
\]
Therefore \( I \) is an isometry between \( \ell_{-a_0}^a \) and \( \ell_{-a}^a \) if interpolating the isometry between \( \ell_{-a_0}^a \) and \( \ell^p \) we get for \( 0 < \theta < 1: \)
\[
(\ell_{-a_0}^a, \ell_{-a_1}^a)_{h(p, p)} \stackrel{I}{\longrightarrow} (\ell^p, \ell^p)_{h(p, p)}, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}
\]
and the proof is complete.

**Theorem 5.** Let \((B_1, B_2)\) be an interpolation pair of \( r_1, r_1 \) Banach spaces. Let \( 0 \leq p_0, p_1 \leq \infty, 0 < \theta < 1 \). Then \((B_1, B_2)_{h(p_0, p_1)} = (B_1, B_2)_{h(p_1, p_0)} \)
where \( \frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_0} \).

Proof. Let \( 0 < \theta < \theta < 1, a = (1 - \theta), a_0 = \theta \). Then:
\[
\frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}
\]

Using Theorem 2, \( \|e^{-\alpha n} W(m, v_n)\|_q \leq c \|v_n\|_q \) and so:
\[
\sum_{n=1}^\infty \sum_{n=1}^\infty |e^{-\alpha n} W(m, v_n)| q \leq c \sum_{n=1}^\infty \|v_n\|_q \leq c \|\{v_n\}\|_{h(p_0, p_1)}.
\]
I.e. \( \|e^{-\alpha n} W(m, v_n)\|_q \|v_n\|_q \leq c \|\{v_n\}\|_{h(p_0, p_1)} \).

Since \( W \) is an \( r \) norm on \( B_1 \) and \( B_2 \) we have:
\[
W'(m, b) = W'(m, v_n + v_{n-1}) = W'(m, v_n) + W'(m, v_{n-1})
\]
and so \( 0 \leq W'(m, b) \leq c \|W'(m, v_n) + W'(m, v_{n-1})\|_q \).

Therefore \( W'(m, b) \in (\ell_{-a_0}^a, \ell_{-a_1}^a)_{h(p_0, p_1)} \).

By Theorem 4, \( W'(m, b) \in \ell_{-a}^a \), with
\[
\frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{where} \quad \theta = (1 - \lambda) \theta_1 + \lambda \theta_0.
\]

and \( \|e^{-\alpha n} W'(m, b)\|_q \leq c \|\{v_n\}\|_{h(p_0, p_1)} \). Using Theorem 2 again, \( b \in (B_1, B_2)_{h(p_0, p_1)} \) and the injection is continuous.

Conversely, let \( b \in (B_1, B_2)_{h(p_0, p_1)} \). We can write \( b = \sum_{n=1}^\infty u_n \), with
\[
\|e^{-\alpha n} M(m, u_n)\|_p \leq c \|\{v_n\}\|_{h(p_0, p_1)}
\]
and so we can find two sequences \( \{v_n\} \) so that \( v_n + v_{n-1} = M(m, u_n) \)

Since \( M(m, u_n) \geq 0 \), we can assume \( v_n \geq 0 \). Take now \( b_n = \sum_{n=1}^\infty M(m, u_n) \) \( (i = 0, 1) \). Clearly \( b_0 + b_1 = b \).
\[ e^{(λ−β)m}M\left( m, u_m, e_m\right) = e^{(λ−β)m}e_m e_m \text{ and so } \|e^{(λ−β)m}e_m\| = e_m \]

From (8) the theorem now follows.

\[ \left\|e^{(λ−β)m}\right\| = e_m \]

From Theorem 1. Let \( B, B_1, B_2, B_3 \) be an interpolation pair of \( r_1, r_2 \) Banach spaces, \( 0 < α < β < 1, 0 < p, q \leq \infty, X = (B_1, B_2, h) \).

The following conditions are equivalent:

(a) \( b \in (X_1, X_1)_h \).

(b) \( b = \sum_{n=0}^{∞} u_n (b_n + B_1), \) and \( e^{-\lambda m}M(\alpha, u_m) \epsilon (P_\alpha, P_\beta)_h \).

(c) \( e^{-\lambda m}W(m, b) \epsilon (P_\alpha, P_\beta)_h \).

The corresponding norms are equivalent.

Proof. (a) \( \Rightarrow \) (c): Let \( b \in (X_1, X_1)_h \). Then \( b = \sum_{n=0}^{∞} u_n (b_n + B_1) \), and \( e^{-\lambda m}M(\alpha, u_m) \epsilon (P_\alpha, P_\beta)_h \).

Therefore \( b = \sum_{n=0}^{∞} u_n (b_n + B_1) \epsilon (P_\alpha, P_\beta)_h \).

\( W(m, b) \epsilon (P_\alpha, P_\beta)_h \).

The corresponding norms are equivalent.

Proof. (b) \( \Rightarrow \) (a): Let \( b \in (X_1, X_1)_h \). Then \( b = \sum_{n=0}^{∞} u_n (b_n + B_1) \), and \( e^{-\lambda m}M(\alpha, u_m) \epsilon (P_\alpha, P_\beta)_h \).

Therefore \( b = \sum_{n=0}^{∞} u_n (b_n + B_1) \epsilon (P_\alpha, P_\beta)_h \).

\( W(m, b) \epsilon (P_\alpha, P_\beta)_h \).

The corresponding norms are equivalent.

IV. \( L^p \) spaces. To make this account self-contained, we will present the definitions and statements of theorems on \( L^p \) spaces. For the missing proofs we refer the reader to Hunt and Davis [3].

Definition 1. Let \( f \) be a \( <p,p> \) measurable function defined on a \( \sigma \)-finite measure space \( (\Omega, \Sigma, \mu) \), \( \mu > 0 \). We assume that \( f \) is \( \mu \)-finite valued a.e. We define

\[ E_f = \{ \omega | f(\omega) > 0 \}, \lambda_f(\mu) = \mu(E_f) \]

In the following we assume as usual that \( \lambda_f(\mu) < \infty \) for some \( 0 < \mu \).

Definition 2. \( f^*(\mu) = \inf \{ \lambda_f(\mu) : f(\mu) < \lambda_f(\mu) \} \).

\( f^* \) is called the non-decreasing rearrangement of \( f \).

Definition 3.

\[ \|f\|_p = \left\{ \int_0^{\infty} \left( \int_0^{|f|} t^{\delta-p} \right)^{\frac{q}{p}} d\mu \right\} \]

\( 0 < p < \infty, 0 < q < \infty, \)

\( \sup_{t > 0} f^*(t), \quad 0 < p < \infty, q = \infty, \)

\( L^p = \{ f | |f|^p < \infty \} \).

Note that for \( p = q \) we get the usual \( L^p \) spaces, while for \( q = \infty \) we get the weak \( L^p \) spaces, i.e. the spaces of functions satisfy \( \lambda_f(\mu) < C/|f|^p \).

Definition 4. Let \( f \in L^p, r < q, r < p, r < 1 \).

Define:

\[ f^{**}(t, r) = \left\{ \begin{array}{ll}
\sup \left\{ \frac{1}{r} \int_0^t \left( \int_0^{|f|} d\mu \right)^{\frac{q}{r}} d\mu \right\} & \text{if } \mu(M) > 0, \\
1 & \text{if } \mu(M) = 0
\end{array} \right.
\]

\( t < \mu(M) \),

\( f^{**}(t, r) = \left\{ \begin{array}{ll}
\frac{1}{r} \int_0^t \left( \int_0^{|f|} d\mu \right)^{\frac{q}{r}} d\mu & \text{if } \mu(M) > 0, \\
1 & \text{if } \mu(M) = 0
\end{array} \right.
\]

\( \mu (M) < t \).

Theorem 5. \( (f^*)^{**}(t, r) = \left\{ \frac{1}{r} \int_0^t \left( \int_0^{|f^*(\mu)|} d\mu \right)^{\frac{q}{r}} d\mu \right\} \)

\( \tau < t \).
THEOREM 6. $f^*(t) \leq f^{**}(t) \leq (f^*)^{**}(t)$.  
Proof. Let $\varepsilon > 0$, $E_\varepsilon = \{x | |f(x)| \geq \varepsilon f(t + \varepsilon)\}$, $\mu(E_\varepsilon) \geq t + \varepsilon > t$, and so

$$f^{**}(t) \geq \left(\frac{1}{\mu(E_\varepsilon)}\right) \int_{E_\varepsilon} |f^*| \, d\mu \geq f^*(t + \varepsilon)$$

and so $f^*(t + \varepsilon) \leq f^{**}(t)$. Since $f^*$ is continuous from the right we get $f^*(t) \leq f^{**}(t)$.

$$\int \frac{1}{\mu(E_\varepsilon)} \left(\int f^* \, d\mu\right) \leq \int \frac{1}{\mu(E_\varepsilon)} \left(\int f^{**}(t) \, d\mu\right)$$

and so

$$\sup_{\mu(E_\varepsilon) > 0} \left(\frac{1}{\mu(E_\varepsilon)}\right) \int f^* \, d\mu \leq \int f^{**}(t) \, d\mu$$

**Theorem 7.** $||f||_{L^p} \leq ||f^{**}||_{L^p} \leq ||f^*||^* ||f||_{L^p}$.

The first two inequalities follow from Theorem 6. The last inequality follows from Hardy's inequality. Since this inequality will be needed in the sequel again, we state it explicitly:

**Theorem 8.** (Hardy's inequality.) If $0 \leq f$, $1 \leq q$, $0 < q$:

(a) $$\left(\frac{1}{\mu(E_\varepsilon)}\right) \int_{E_\varepsilon} |f(t)| \, d\mu \leq \frac{1}{\mu(E_\varepsilon)} \int_{E_\varepsilon} |f^*(t)| \, d\mu$$

(b) $$\left(\frac{1}{\mu(E_\varepsilon)}\right) \int_{E_\varepsilon} |f(t)| \, d\mu \leq \frac{1}{\mu(E_\varepsilon)} \int_{E_\varepsilon} |f^{**}(t)| \, d\mu$$

Our aim now is to identify the $L^{p,p}$ spaces as intermediate spaces between $L^p$ spaces. For $1 < p < \infty$, $1 \leq q \leq \infty$ this was done by Calderón and by Peetre.

For $0 < p$, $q < \infty$ this was done by Krée in [3], for measure spaces which contain no atoms.

**Definition 9.** Let $(\mathcal{B}_1, \mathcal{B}_2)$ be an interpolation pair of $r_0$, $r_1$ normed spaces. Define for $0 < r < r_0 + r_1$,

$$K(t, b) = \inf \{\max \{|b_0|_{L_0}, t|b_1|_{L_1}\}|b_0 + b_1 = b\}.$$ 

**Definition 10.** Denote for $f(t)$ defined for $0 < t$, measurable,

$$||f||_{L^p} = \left(\int f(t)^p \, dt \right)^{1/p}, \quad ||f||_{L^\infty} = \sup_{t > 0} |f(t)|.$$

$$L^p = \{f ||f||_{L^p} < \infty\}.$$

**Theorem 11.** $b \in (\mathcal{B}_1, \mathcal{B}_2)^{kp}$ iff $t \in \mathcal{K}(t, b) \in L^p$, and the corresponding $s$-norms are equivalent.

Proof. If $c^q \leq t \leq c^{q+1}$, we have $W(n, b) \subset K(t, b) \subset W(n + 1, b)$. Therefore,

$$\frac{1}{\theta_p} \leq (e^{-\infty} M(n, b))^p \leq \int_{k} |t^{-1/2} K(t, b)|^P \, dt \leq \frac{e^{p-1}}{\theta_p} \cdot (e^{-\infty} M(n + 1, b))^p$$

summing for $-\infty < n < \infty$, and using Theorem 2, the proof is complete.

The case $p = \infty$ is done similarly.

**Theorem 13.** Let now $0 < p < \infty, f \in L^p + L^\infty$. Let $K(t, f) = K_{L^p, L^\infty}(t, f)$.

Then

$$K(t, f) \approx \left(\int \frac{1}{p} |f^{**}(t)| \, dt\right)^{1/p}.$$

Proof. Let $f = f_0 + f_1$, $f_0 \in L^p$, $f_1 \in L^\infty$. For any set $E$, $\mu(E) > 0$, we have

$$\frac{1}{\mu(E)} \int f^* \, d\mu \leq \frac{1}{\mu(E)} \int f_0^* \, d\mu + \frac{1}{\mu(E)} \int f_1^* \, d\mu$$

and so

$$|f^{**}(t)| \leq |f_0^{**}(t)| + |f_1^{**}(t)| \leq ||f_0^{**}||_{L^p} + ||f_1^{**}||_{L^p}.$$

Hence

$$\epsilon \left(\int \frac{1}{p} |f^{**}(u)|^p \, du \right)^{1/p} \leq \epsilon \left(\int \frac{1}{p} |f_0^{**}(u)|^p \, du \right)^{1/p} + \epsilon \left(\int \frac{1}{p} |f_1^{**}(u)|^p \, du \right)^{1/p} \leq 2 \max \{||f_0||_{L^p}, ||f_1||_{L^p}\}.$$

Taking infimum of last expression over all $f_0 \in L^p, f_1 \in L^\infty$ so that $f_0 + f_1 = f$, we get

$$\left(\int \frac{1}{p} |f^{**}(u)|^p \, du \right)^{1/p} \leq cK(t, f).$$

Conversely, let $0 < t$ be given. $E = \{x | |f(x)| > f^*(t)\}$, $f_0 = f \cdot 1_E, f_1 = f - f_0$.

Since $|f_0| \leq |f|, |f_1^{**}(t)| \leq |f^{**}(t)|$ for all $u$. 
For every $u \leq \rho$, \( \|f\|_{L^p} \leq \|f^*(u)\|_{L^{p,\rho}} \leq f^*(u) \leq \|f^*(u)\|_{L^{p,\rho}} \). Therefore:

\[
\max \left\{ \rho \left( f^*(u) \right)_{L^{p,\rho}}, \rho \left( f^*(u) \right)_{L^{p,\rho}} \right\} \leq \rho \left( f^*(u) \right)_{L^{p,\rho}} du.
\]

Since \( f^* \) vanishes outside \( E, m(E) = \lambda f^*(\rho) \leq \rho \) we have:

\[
\|f\|_{L^{p,\rho}} = \rho \left( f^*(u) \right)_{L^{p,\rho}} du = \rho \left( f^*(u) \right)_{L^{p,\rho}} du \leq \rho \left( f^*(u) \right)_{L^{p,\rho}} du.
\]

From (13) we therefore have

\[
\max \left\{ \rho \|f\|_{L^{p,\rho}}, \rho \|f\|_{L^{p,\rho}} \right\} \leq \rho \left( f^*(u) \right)_{L^{p,\rho}} du
\]

and so \( K(t, f) \leq \rho \left( f^*(u) \right)_{L^{p,\rho}} \), and the proof is complete.

The idea of the proof goes back to Kröe [3]. However, the formula

\[
\sup_{t \in \mathbb{R}} \left( f^*(u) \right)_{L^{p,\rho}} = \frac{1}{p} \int f(t) dt
\]

which is basic there, does not hold if the measure space has atoms.

**Theorem 14.** Let \( 0 < p < \infty, \ 0 < \theta < 1, \ p < \infty \). Then \( (L^p, L^\theta)_{\theta} = L^{p,\theta} \).

**Proof.** Since \( K(t, f) = \rho \left( f^*(u) \right)_{L^{p,\rho}} du \), and \( f \in (L^p, L^\theta)_{\theta} \iff \rho \left( f^*(u) \right)_{L^{p,\rho}} du \frac{dt}{t} < \infty \).

The last integral equals to

\[
\frac{1}{p} \int \rho \left( f^*(u) \right)_{L^{p,\rho}} du \frac{dt}{t}.
\]

Using Hardy’s inequality (for which we need \( q \geq \rho \)) the integral in (15) is majorized by

\[
\frac{1}{p} \left( \frac{q}{\rho(q)} \right)^{1/\theta} \int \rho \left( f^*(u) \right)_{L^{p,\rho}} du \frac{dt}{t} = \frac{1}{p} \left( \frac{q}{\rho(q)} \right)^{1/\theta} \int \rho \left( f^*(u) \right)_{L^{p,\rho}} du \frac{dt}{t}.
\]

The last integral is finite iff \( f \in L^{p,\theta} \). The reverse inclusion is easily proved.

**Theorem 16.** Let \( 0 < p_1 < \infty, \ 0 < q_1 \leq \infty, \ p_0 \neq p_1, \ 0 < \theta < 1 \).

Then

\[
(L^{p_0,\theta}, L^{p_1,\theta})_{\theta} = L^{p_1,\theta}, \text{ where } \frac{1}{p_0} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.
\]

**Proof.** Take \( p_4 \leq \min \{1, p_0, p_1, q_1, q_2, q_3 \} \). Let then \( \lambda_4, \lambda_1 \) be determined by

\[
\frac{1 - \theta}{p_0} = \frac{1}{p_4} \left( \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \right) = \frac{1}{p_4}.
\]

We have by Theorem 14:

\[
L^{p_0,\theta} \times K_{\theta}(L^p, L^{\rho}), \quad L^{p_1,\theta} \times K_{\theta}(L^p, L^{\rho})
\]

and since \( p_4 \neq p_0, p_1 \neq \lambda_0 \), we can use the reiteration theorem:

\[
(L^{p_0,\theta}, L^{p_1,\theta})_{\theta} = (L^p, L^{\rho})_{\theta} \times (L^p, L^{\rho})_{\theta}, \quad q = L^p, q = L^p
\]

where \( q \left( f^*(u) \right)_{L^{p_0,\theta}} = q \left( f^*(u) \right)_{L^{p_1,\theta}} = 1 - \theta + \theta q \left( f^*(u) \right)_{L^{p_0,\theta}} + q \left( f^*(u) \right)_{L^{p_1,\theta}}
\]

**Theorem 17.** Let \( 0 < p < \infty, \ 0 < \theta < 1, \ (L^{p_0,\theta}, L^{p_1,\theta})_{\theta} \leq L^p, q = L^p
\]

where \( q \left( f^*(u) \right)_{L^{p_0,\theta}} = q \left( f^*(u) \right)_{L^{p_1,\theta}} = 1 - \theta + \theta
\]

**Proof.** Immediate consequence of previous theorems and III. 9. We can now prove easily a number of results on \( L^{p,\theta} \) spaces.

**Theorem 18.** Let \( 0 < r < p < \infty, \ 0 < \theta < 1 \). \( |f|_{\theta} = \left( \left| f^*(u) \right|_{L^{p_0,\theta}} \right)^{1/\theta}
\]

Then \( (L^{p,\theta})_{\theta} \) is an \( r \)-Banach space.

**Proof.** Take \( r \) as above. \( L^\infty = (L, L^\infty)_{\lambda_0} \) is an \( r \)-Banach space, with an \( r \)-norm equivalent to \( |f|_{\theta} \), which in view of Theorem 14, is equivalent to \( |f|_{\theta} \), and the proof is complete.

**Theorem 19.** If \( q_1 \leq q_2, \ 0 < q_1 \leq \infty \), then \( L^{p,\theta} \subseteq L^{q,\theta} \).

**Proof.** Immediate consequence of Theorem 18.

**Theorem 20.** If \( (M, \Sigma, \mu) \) contains infinitely many disjoint sets of positive measure, then \( L^{q,\theta} \subseteq L^{q,\theta} \).

**Proof.** Can be done directly from Definition 3.

We want to compare the interpolation theorems we have for \( L^{p,\theta} \) spaces, with the ones in Hunt [2]. For this purpose we specialize the interpolation theorems:

**Theorem 21.** Let \( T \) be a quasi-linear operator from \( (L^{p_0,\theta}, L^{p_1,\theta})_{\theta} \) to \( (L^{p_2,\theta}, L^{p_3,\theta})_{\theta} \), then if \( p_0 \neq p_1, p_0 \neq p_3, \ 0 < \theta < 1, \ 0 < q < \infty \), we have

\[
|Tf|_{\theta} \leq \|T\|_{\theta} \|f|_{\theta}, \quad \text{where } \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}
\]

This is the weak type theorem of Hunt [2,3], if quasi-linear operators in the sense used by Hunt \( (|Tf|_{\theta}) \approx K(|f|_{\theta}) \left| f \right|_{\theta} \), we shall call them...
pointwise quasi-linear) can be connected with quasi-linear operators in the sense of Definition II.14, when \( \|T_f\|_{L^p_{\mathbb{R}^n}} \leq M \|f\|_{L^p_{\mathbb{R}^n}} \).

This follows from the following considerations:

If \( T \) is pointwise quasi-linear and \( \|T_f\|_{L^p_{\mathbb{R}^n}} \leq M \|f\|_{L^p_{\mathbb{R}^n}} \), we consider \( \widetilde{T} \) defined by \( \widetilde{T}f = |Tf| \).

\[
\widetilde{T}(f_1 + f_2) = b_1 + b_2,
\]

where \( b_1 = \frac{\widetilde{T}(f_1 + f_2)}{\widetilde{T}f_1 + \widetilde{T}f_2} \).

Since \( \frac{\widetilde{T}(f_1 + f_2)}{\widetilde{T}f_1 + \widetilde{T}f_2} \leq K, \|\frac{\widetilde{T}f_1 + \widetilde{T}f_2}{\widetilde{T}f_1 + \widetilde{T}f_2} \|_{L^p_{\mathbb{R}^n}} \leq K \cdot M \|f_1\|_{L^p_{\mathbb{R}^n}} \) and so \( \widetilde{T} \) is quasi-linear in the sense of Definition II.14.

Hence \( \|\widetilde{T}f\|_{L^p_{\mathbb{R}^n}} \leq B_0 \|f\|_{L^p_{\mathbb{R}^n}} \) but \( \|\widetilde{T}f\|_{L^q_{\mathbb{R}^n}} = \|Tf\|_{L^q_{\mathbb{R}^n}} \) and so Hunt’s theorem follows. We also have:

**Theorem 22.** Let \( T \) be a quasi-linear operator from \( (L^{p_0}_{\mathbb{R}^n}, L^{p_1}_{\mathbb{R}^n}) \) to \( (L^{q_0}_{\mathbb{R}^n}, L^{q_1}_{\mathbb{R}^n}) \). Then if \( 0 < \theta < 1, \|T\|_{L^{p_0}_{\mathbb{R}^n}} \leq B_0 \|T\|_{L^{p_1}_{\mathbb{R}^n}} \).

If \( T \) is continuous from \( (L^{p_0}_{\mathbb{R}^n}, L^{p_1}_{\mathbb{R}^n}) \) to \( (L^{q_0}_{\mathbb{R}^n}, L^{q_1}_{\mathbb{R}^n}) \). By Theorem II.13, \( (L^{p_0}_{\mathbb{R}^n}, L^{p_1}_{\mathbb{R}^n}) \subset \subset (L^{q_0}_{\mathbb{R}^n}, L^{q_1}_{\mathbb{R}^n}) \), while from Theorems 16, 17

\[
(L^{q_0}_{\mathbb{R}^n}, L^{q_1}_{\mathbb{R}^n}) \subset \subset (L^{p_0}_{\mathbb{R}^n}, L^{p_1}_{\mathbb{R}^n}).
\]

Without the restriction \( q_1 \leq q, \) and with a \( B_0 \) which is bounded when \( \theta \to 0, 1, \) but for a more restricted class of operators (sublinear), this is proved in [2] (strong type theorem) using complex methods.

Of course Theorem II.14 enables one to interpolate in the manner of Theorems 21, 22 between general interpolation pairs and \( (L^{p_0}_{\mathbb{R}^n}, L^{p_1}_{\mathbb{R}^n}) \).

V. Theorems on Fourier coefficients. In this section we make use of the results of previous sections to present some theorems on Fourier coefficients. The exposition is intended as a demonstration of the applicability of the results. The ideas are new, and implicit in [3]. Let \( \{\varphi_n(\omega)\}_{n=1}^{\infty} \) be a bounded orthonormal system on \( (a, b) \). We define \( \tilde{f}(n) = \frac{1}{a} \varphi_n(\omega) f(\omega) d\omega, \) and consider the operator \( Tf = (\tilde{f}(n)). \)

Taking the integers as the underlying measure space with the measure 1 carried by each integer, we have \( \|Tf\|_{L^\infty_{\mathbb{R}^n}} \leq M \|f\|_{L^1_{\mathbb{R}^n}} \) (boundedness of the system) and \( \|Tf\|_{L^p_{\mathbb{R}^n}} \leq \|f\|_{L^p_{\mathbb{R}^n}}^{\infty} \) (Bessel’s inequality).

Interpolating, we get \( \|Tf\|_{L^p_{\mathbb{R}^n}}^{\infty} \leq c \|f\|_{L^p_{\mathbb{R}^n}}^{\infty} \) for \( \frac{1}{p} + \frac{1}{p'} = 1, 1 < p < 2. \)

Since we know that \( L^p_{\mathbb{R}^n} \subset L^q_{\mathbb{R}^n} \) if \( q \leq q_1, \) we can write

\[
(1) \quad \|Tf\|_{L^p_{\mathbb{R}^n}}^{\infty} \leq c \|f\|_{L^p_{\mathbb{R}^n}}^{\infty} \quad 0 < q \leq q_1 < \infty.
\]

If now \( (O_n)_{n=1}^{\infty} \) is a sequence of complex numbers, we consider it as a function on the integers.

Assume \( O_n \to 0. \) For such sequences it is easily seen that if \( \{O_n\} \) is the sequence \( \{O_n\} \) rearranged in non-increasing order of magnitude,

\[
\|O_n\|_{L^q_{\mathbb{R}^n}}^{\infty} = \left( \sum_{k=1}^\infty O_k^{q} \right)^{1/q} \quad q < \infty,
\]

and \( \text{Sup}^{1/q} \Omega_{n+1}^{q} \text{ for } q = \infty. \)

It can be seen that

\[
\|O_n\|_{L^q_{\mathbb{R}^n}}^{\infty} \sim \left( \sum_{n=1}^\infty O_n^{q} \right)^{1/q}, \quad q < \infty
\]

\[
\|O_n\|_{L^\infty_{\mathbb{R}^n}}^{\infty} \sim \text{Sup}^{1/q} \Omega_{n+1}^{q} \text{ for } q = \infty.
\]

Taking now \( q = q_1 = p \) in (1) we get

\[
\left( \sum_{n=1}^\infty \|f(n)\|_{L^p_{\mathbb{R}^n}}^{q} \right)^{1/p} \leq c \|f\|_{L^p_{\mathbb{R}^n}}^{\infty} \leq c \|f\|_{L^p_{\mathbb{R}^n}}
\]

\( p/p' = p - 1 \) and so we get for \( 1 < p < 2 \)

\[
(2) \quad \left( \sum_{n=1}^\infty \|f(n)\|_{L^p_{\mathbb{R}^n}}^{q} \right)^{1/p} \leq c \|f\|_{L^p_{\mathbb{R}^n}}
\]

Taking in (1) \( q = p, \) \( p = p' \) (possible, since \( p < 2 < p' \)), we get

\[
(3) \quad \left( \sum_{n=1}^\infty \|f(n)\|_{L^p_{\mathbb{R}^n}}^{q} \right)^{1/p} \leq c \|f\|_{L^p_{\mathbb{R}^n}}
\]

We know, however, that (2) implies (3), since \( L^p_{\mathbb{R}^n} \subset L^{p'}_{\mathbb{R}^n}. \)

We also consider the operators \( \tilde{T}_n: (O_n) \to \sum_{n=1}^\infty O_n \varphi_n(\omega). \) We have:

\[
\|\tilde{T}_n (O_n)\|_{L^p_{\mathbb{R}^n}} \leq \|O_n\|_{L^p_{\mathbb{R}^n}}^{\infty} \quad \|\tilde{T}_n (O_n)\|_{L^\infty_{\mathbb{R}^n}} \leq M \|O_n\|_{L^\infty_{\mathbb{R}^n}}^{\infty}.
\]

Thus

\[
\|\tilde{T}_n (O_n)\|_{L^p_{\mathbb{R}^n}} \leq \|O_n\|_{L^p_{\mathbb{R}^n}}^{\infty} \quad 1 < p < 2, 0 < q \leq q_1 < \infty
\]

and \( C \) does not depend on \( N. \) From the last remarks, it follows that if \( \{O_n\}_{n=1}^\infty \) is a Cauchy sequence in \( L^p_{\mathbb{R}^n}, \) which we know is complete. We can therefore define \( \tilde{T}_{\lim} (O_n) = \lim \tilde{T}_n (O_n) \) in \( L^{p\infty}_{\mathbb{R}^n}, \) and of course

\[
(5) \quad \|\tilde{T}_{\lim} (O_n)\|_{L^p_{\mathbb{R}^n}} \leq \|O_n\|_{L^p_{\mathbb{R}^n}}^{\infty} \quad 1 < p < 2, 0 < q \leq q_1 < \infty.
\]
Since \( (C_0) \ast L^\infty \cap L^p \) with \( p < 2 \), \( (C_0) \ast L^2 \) and so \( \sum_{n=1}^{\infty} C_n \varphi_n (x) \rightarrow f \) (in \( L^p \)) if \( f (x) = C_n \). We therefore have \( \hat{f} (x) = C_n \) as its \( m \)th Fourier coefficient.

Summarizing, we have:

Given \( (C_0) \ast L^p \), \( 1 < p < 2 \), \( 0 < q < \infty \) (or \( p = q = 2 \)) there exists \( f \in L^p \), \( q \leq q_1 \leq \infty \), so that \( f (x) = C_n \) and

\[
(6) \quad \| f \|_{L^p} \leq \| C_n \|_{L^q}.
\]

Taking \( q_1 = q = p \) we get:

\[
(7) \quad \| f \|_{L^p} \leq \left( \sum_{n=1}^{\infty} C_n^{p^*} \right)^{1/p^*}.
\]

Taking \( q_1 = p, \quad q = q \) (possible, for \( p < p^* \):

\[
(8) \quad \| f \|_{L^p} \leq \left( \sum_{n=1}^{\infty} C_n^{p^*} \right)^{1/p^*}.
\]

and we know that (8) is weaker than (7). The results of (2) and (7) are Paley's theorem for Fourier coefficients. Note that we get it immediately in the stronger, rearranged form.

The results of (3) and (8) are Riesz–Hausdorff–Young's theorem for Fourier coefficients. Note that we also have the implication relation between this result and Paley's theorem.

Different choices of parameters in (1) and (6) will yield the “dual” theorems ([10], XII, 5.15). Explicitly:

Taking in (1) \( q_1 = p \), \( p < 2 \) we get for \( 1 < p < 2 \)

\[
(9) \quad \left( \sum_{n=1}^{\infty} | \hat{f} (n) |^{p^*} \right)^{1/p^*} \leq \| f \|_{L^p} = \| f \|_{L^p} = \left( \int_{-\infty}^{\infty} | f (t) |^p e^{-\lambda t} \, dt \right)^{1/p}.
\]

(If \( a, b \) is finite the last integral is from 0 to \( b - a \) only, since \( f^* (t) \) vanishes for \( t > b - a \).)

Taking in (6) \( q_1 = q = p \) we get: for \( 1 < p < 2 \)

\[
(10) \quad \left( \int_{-\infty}^{\infty} | f^* (t) |^p e^{-\lambda t} \, dt \right)^{1/p} \leq \left( \sum_{n=1}^{\infty} C_n^{p^*} \right)^{1/p^*}.
\]

Again, the theorem presents itself in the stronger, rearranged form.

E. Stein in [8] has proved essentially (1) and (5), but with some unnecessary restrictions on the parameters, using interpolation with change of measure, and Paley's theorem.

Together with the operator \( \hat{T} \), one can consider also a maximal operator \( T^* \) defined by:

\[
T^* (C_0) (X) = \text{Sup}_{1 \leq n} \left( \sum_{n=1}^{\infty} C_n \varphi_n (x) \right).
\]

(See [9], Section 5). The importance of this operator is for the proof of a.e. convergence of the series defined by \( \hat{T} \).

The operator \( T^* \) is no longer linear. It is, however, pointwise sublinear. i.e. \( | T^* (C_0) + (C_0) | (x) \leq | T^* (C_0) (x) | + | T^* (C_0) (x) | \) a.e. \( x \) and \( | T^* (C_0) (x) | = \text{Ae} \| T^* (C_0) (x) \| \) a.e. \( x \). Since those operators are pointwise quasi-linear, we can apply the interpolation theorem.

It is clear that

\[
(12) \quad \| T^* (C_0) \|_{L^\infty} \leq M \| (C_0) \|_{L^1}.
\]

Stein and Weiss in [9, Lemma 6] prove the following: If \( \{ \{ \} \} \) is a sequence of numbers, \( N \) of which are equal to 1, and the rest 0, then if \( 1 < p < 2 \), \( \| T^* (\{ \} ) \|_{L^p} = \sum_{n=1}^{\infty} N^{1/p} \). Sequences as above are of course characteristic functions of sets of finite measure of integers, and \( \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \). Calderon in [1, Theorem 7] proved that if \( T \) is a pointwise sublinear operator defined for simple functions, with values in a Banach lattice \( B \) of functions (i.e. a Banach space so that \( f \in B, \| f \| \leq \| f \| \), then \( T \) can be extended, in a unique way to all of \( L^p \) and \( \| T \|_{L^p} \leq 2 \| f \|_{L^p} \). We take for \( B \), \( L^p \) and get

\[
(13) \quad \| T^* (C_0) \|_{L^p} \leq A_p \| (C_0) \|_{L^p}.
\]

To be precise, one has to consider as \( \hat{T} \), operators \( T_{\frac{1}{d}} \), extend each one of them to \( L^p \), and then consider the limit operator \( T^* \).

Using the interpolation theorem we get for \( 1 < p < 2 \), \( 0 < q < \infty \)

\[
(14) \quad \| T^* (C_0) \|_{L^p} \leq A_p \| (C_0) \|_{L^q}.
\]

Taking in particular \( q_1 = q = p^* \) we get

\[
(15) \quad \| T^* (C_0) \|_{L^p} \leq A_p \left( \sum_{n=1}^{\infty} C_n^{p^*} \right)^{1/p^*}.
\]

If we take \( q_1 = q = p \) we get

\[
(16) \quad \| T^* (C_0) \|_{L^p} \leq A_p \left( \sum_{n=1}^{\infty} C_n^{p^*} \right)^{1/p^*}.
\]

E. Stein and G. Weiss in [9] have proved 15, using “restricted type” interpolation theorem.

Added in Proof. T. Holmestad's paper: “Interpolation of quasi-normed spaces”, Math. Scand. 26 (1970), pp. 177-190, includes better proofs of the interpolation Theorem (Theorem II. 28), and of Theorem III. 5. The range of parameters is the same, but the spaces need not be complete.
The distribution of the values of a random function in the unit disk

by

A. C. OFFORD (London)

Abstract. Let \( f(z) \) be defined in the unit disk by a power series whose coefficients are independent random variables and let \( \alpha(t, b) \) denote the number of zeros of \( f(z) - b \) in \( |z| < t \). It is shown that, for almost all functions of the family considered, \( \inf_{t} \int \alpha(t, b) \, dt \) has a well defined asymptotic behaviour. Furthermore \( f(z) \) almost surely takes every finite value in every open sector of the unit disk. The paper contains some inequalities for \( \int \log |X| \, d\mu \), where \( X \) is a random variable defined on a measure space \( (\Omega, \mathcal{M}, \mu) \) and \( \mathcal{M} \) belongs to \( \mathcal{M} \) but is otherwise arbitrary.

§ 1. Introduction and principal results. This paper is concerned with the behaviour of functions

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]  

defined in the unit disk for which the coefficients \( a_n \) are independent random variables. Our object is to show that the family (1.1) has certain properties almost surely. This implies some statistical basis and it becomes necessary to define this statistical basis precisely. Many years ago Littlewood and Offord [3] studied a similar problem for the family of entire functions

\[ \sum_{n=0}^{\infty} \alpha_n a_n z^n \]

in which the coefficients \( \alpha_n \) were given and the \( \alpha_n \) took the values 0, 1 with equal probability. In 1964 one of the authors [4] returned to this problem and established the basic results of Littlewood and Offord under very general conditions on the distribution functions of the coefficients \( a_n \). From this it followed that the behaviour of the family of entire functions was largely independent of the particular distribution functions chosen for the coefficients \( a_n \). For this reason in the present investigation we