STUDIA MATHEMATICA T. XLI. (1972)

Interpolation of r-Banach spaces

by

YORAM SAGHER* (Buenos Aires)

Abstract. The paper extends the interpolation theory of Lions and Peetre to r-Banach spaces. The extension permits application of the theory to $L(l, \infty)$ as well as to other spaces, not covered by the original theory.

The interpolation between $L^{p,q}$ spaces is carried out also for measure spaces which contain atoms, and this is applied to trigonometric series.

I. Introduction.

DEFINITION 1. Let B be a vector space over C. An r *norm on* B is a function $|\ |_B\colon B \to R^+$ satisfying:

(a) $|b|_B = 0$ iff $b = 0$,

(b) for all $\lambda \in C$, all $b \in B$, $|\lambda b|_B = |\lambda|^r |b|_B$,

(c) $|b_1 + b_2|_B \leqslant |b_1|_B + |b_2|_B$.

An r normed space is a topological vector space, whose topology is given by an r norm. A complete r-normed space is called an *r-Banach space.*

Every r normed space is an r_1 normed space for every $r_1 < r$, for $|\ |_B^{r_1/r}$ is an r_1 norm on B, defining the same topology on it.

From $2^r|b|_B = |2b|_B = |b+b|_B \leqslant 2|b|_B$ we see that non-trivial r normed spaces exist for $r \leqslant 1$ only.

Let $T\colon (B_0, |\ |_{B_0}) \to (B_1, |\ |_{B_1})$ be a linear operator between an r_0 and an r_1 normed space. It is easily seen that T is continuous iff $c > 0$ exists so that for all $b \in B$, $|Tb|_{B_1}^{1/r_1} \leqslant c|b|_{B_0}^{1/r_0}$.

r-Banach spaces occur naturally in analysis. H^r, $0 < r \leqslant 1$, are but one example.

DEFINITION 2. Let B be a vector space over C. A quasi-norm on B is a function $\|\ \|_B\colon B \to R^+$ satisfying:

(a) $\|b\|_B = 0$ iff $b = 0$,

(b) for all $\lambda \in C$, all $b \in B$, $\|\lambda b\|_B = |\lambda|\, \|b\|_B$,

(c) a number $k = k(B)$ exists so that $\|b_1 + b_2\|_B \leqslant k(\|b_1\|_B + \|b_2\|_B)$.

* Research supported by the Multinational Program in Mathematics of the O. A. S.

A quasi-normed space is a topological vector space, whose topology is given by a quasi-norm.

It is easily seen that if $(B, |\ |_B)$ is r-normed, then $(B, \|\ \|_B)$ is quasi-normed, where $\|\ \|_B = |\ |_B^{1/r}$, with $k(B) = 2^{(1/r)-1}$. We also have:

THEOREM 3. (S. Rolewicz [7].) *If B is a quasi-normed space, $2^{1/r} > 2k(B)$, B is r-normed.*

Our work is motivated by the following considerations: L^{pq} spaces (see [2]) which are in $L^1 + L^\infty$ appear as intermediate spaces in the interpolation theory of Lions and Peetre, [4], [5] etc. This, however, leaves out $L^{1\infty}$ (weak L^1), and so Marcinkiewicz's interpolation theorem is not proved in this context, for a case most important in applications. By considering interpolation between r-Banach spaces we shall obtain all L^{pq} spaces $(0 < p < \infty, 0 < q \leqslant \infty; p = q = \infty)$ as intermediate spaces. The theory is also applicable to other problems, e.g. interpolation between H^p spaces $(0 < p \leqslant \infty)$, etc.

Krée [3] has extended the K method of Peetre to quasi-normed spaces, and has obtained the identification of all L^{pq} spaces as intermediate spaces [1]. However, Krée does not make use of r-norms.

Using r norms, we are able to obtain theorems on the topological properties of the intermediate spaces not available in Krée's method, as well as interpolation theorems missing from his theory. As examples of the latter: The reiteration theorem of Lions and Peetre is generalized in full. Another example is the following result:

$$(L^{pq_0}, L^{pq_1})_{\theta q} = L^{pq},$$

$$\text{where} \quad 1/q = (1-\theta)/q_0 + \theta/q_1, \; 0 < p < \infty, \qquad 0 < q_i \leqslant \infty.$$

This generalizes a result of Peetre [5], who proved this under the assumptions $1 < p < \infty$, $1 \leqslant q_i \leqslant \infty$. This result is particularly interesting, for it was previously available only by complex methods.

For an extension of the complex method of Calderón to r-Banach spaces, see Rivière [6].

In Sections II, III we generalize the work of Lions and Peetre [4], [5] in two directions: We consider r-Banach spaces rather than Banach spaces, and our parameters are in $(0, \infty]$ rather than in $[1, \infty]$. For the sake of completeness we have included proofs of all theorems, including

---

(1) We take this opportunity to note that Krée's proof of this fact is valid only when the measure space has no atoms. The theorem is true, however, in the general case (See section IV) and is in fact of interest, since it enables one to obtain various theorems on Fourier coefficients.

those where the generalization of the proofs of Lions and Peetre is straightforward.

In Section IV we present the basic properties of L^{pq} spaces. For an exposition on these spaces using elementary methods, the reader is referred to [2]. We then show that these spaces are intermediate between L^p spaces. From this we proceed to deduce various topological properties, and a generalization of the weak interpolation theorem of Hunt.

In Section V we give a unified account of theorems of Hausdorff–Young, Paley, E. Stein and others, on Fourier coefficients, using L^{pq} spaces and the weak interpolation theorem. The connection between L^{pq} spaces and these theorems was suggested practically since the time that L^{pq} spaces were defined, and this section is of expository nature. Still it may be of some value — if only as a demonstration of the strength of the so-called weak interpolation theory.

II. The spaces $(B_0, B_1)_{\theta, p_0, p_1}$.

THEOREM 1. *Let B be an r-normed space, V a topological vector space, $T: B \to V$ a continuous linear operator. Let R = range T. Define on R $|v|_T = \inf\{|b|_B / Tb = v\}$. Then $(R, |\ |_T)$ is an r-normed space. If $(B, |\ |_B)$ is complete, so is $(R, |\ |_T)$.*

Proof. The last statement is the only one requiring verification $v_n \in R$, $|v_n - v_m|_T \to 0$. Suffices to show the existence of a convergent subsequence. We therefore take a subsequence of the original sequence and can assume $|v_n - v_{n+1}|_T < 2^{-n-1}$. Let $b_n \in B$, $Tb_n = v_n - v_{n+1}$

$$|b_n|_B < |v_n - v_{n+1}|_T + 2^{-n-1} < 2^{-n}.$$

Let $a_1 \in B$ satisfy $Ta_1 = v_1$. Define inductively $a_n = a_{n-1} - b_{n-1}$, $Ta_n = Ta_{n-1} - v_{n-1} + v_n$, and by induction $Ta_n = v_n$. $|a_n - a_{n+1}|_B = |b_n|_B < 2^{-n}$, and so $\{a_n\}$ is a Cauchy sequence $a_n \to a$, $|Ta - v_n|_T \leqslant |a - a_n|_B \to 0$ and so v_n converges.

DEFINITION 2. Let $(B_0, |\ |_{B_0})$, $(B_1, |\ |_{B_1})$ be r_0 and r_1 normed. If both are continuously embedded in a topological vector space B, we shall say that $(B_0, |\ |_{B_0}; B_1, |\ |_{B_1})$ is an *interpolation pair*.

In the sequel, we shall omit the norms and write (B_0, B_1). Also, when $(B, |\ |_B)$ is r-normed, $\|\ \|_B$ will stand for $|\ |_B^{1/r}$.

THEOREM 3. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces. Let $r = \min\{r_0, r_1\}$. Then $|b|_{B_0 \cap B_1} = \max\{\|b\|_{B_0}^r, \|b\|_{B_1}^r\}$ is an r norm on $B_0 \cap B_1$, while $|b|_{B_0+B_1} = \inf\{\max_{i=0,1} \|b_i\|_{B_i}^r / b_0 + b_1 = b\}$ is an r norm on $B_0 + B_1$. Finally, if $(B_0, |\ |_{B_0}), (B_1, |\ |_{B_1})$ are complete, so are $(B_0 \cap B_1, |\ |_{B_0 \cap B_1})$ and $(B_0 + B_1, |\ |_{B_0+B_1})$.*

Proof. The verification for $(B_0 \cap B_1, |\ |_{B_0 \cap B_1})$ is immediate. As for $(B_0 + B_1, |\ |_{B_0+B_1})$ we note that $T: (B_0, |\ |_{B_0}) \times (B_1, |\ |_{B_1}) \to B$ defined by $T(b_0, b_1) = b_0 + b_1$ defines $|\ |_{B_0+B_1}$ as a T induced norm on its range B_0+B_1 as in Theorem 1.

DEFINITION 4. Let $(B, |\ |_B)$ be an r-normed space, $\{\mu_n\}$ a sequence of positive numbers. Define

(a) $\|\{b_n\}\|_{l^p_{\mu_n}(B,|\ |_B)} = \left(\sum_{-\infty}^{\infty} \|\mu_n b_n\|_B^p\right)^{1/p}$, $0 < p < \infty$,

(b) $\|\{b_n\}\|_{l^\infty_{\mu_n}(B,|\ |_B)} = \sup_n \{\|\mu_n b_n\|_B\}$.

Define also $l^p_{\mu_n}(B, |\ |_B)$ as the space of all sequences of elements of B so that $\|\ \|_{l^p_{\mu_n}(B,|\ |_B)} < \infty$.

THEOREM 5. *$l^p_{\mu_n}\big((B, |\ |_B), |\ |_{l^p_{\mu_n}(B,|\ |_B)}\big)$ is an s normed space, where $s = \min\{r, p\}$, and*

$$|\ |_{l^p_{\mu_n}(B,|\ |_B)} = \|\ \|^s_{l^p_{\mu_n}(B,|\ |_B)}.$$

The spaces are complete iff $(B, |\ |_B)$ is.

Proof. The proof proceeds along well-known arguments. We shall write $l^p_{\mu_n}(B)$ for $l^p_{\mu_n}(B, |\ |_B)$; $l^p(B)$ for $l^p_1(B)$.

DEFINITION 6. Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces, $0 < \theta < 1$, $0 < p_i \leqslant \infty$. Denote

$$w(p_0, B_0; p_1, B_1; \theta) = l^{p_0}_{e^{-\theta n}}(B_0) \cap l^{p_1}_{e^{(1-\theta)n}}(B_1),$$

with the s norm:

$$|\{u_n\}|_w = \max_{i=0,1}\{\|\{e^{(i-\theta)n}u_n\}\|^s_{l^{p_i}(B)}\},$$

where $s = \min\{r_0, r_1, p_0, p_1\}$.

THEOREM 7. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 Banach spaces. Then T: $w(p_0, B_0; p_1, B_1; \theta) \to B_0 + B_1$ defined by $T(\{u_n\}) = \sum_{-\infty}^{\infty} u_n$ is a well defined continuous linear transformation.*

Proof. $\sum_{0\leqslant k}^{m} |u_n|_{B_1} = \sum_{k}^{m} |u_n e^{(1-\theta)n}|_{B_1} e^{(\theta-1)r_1 n}$, if now $p_1 \leqslant r_1$

$$\sum_{k}^{m} |u_n e^{(1-\theta)n}|_{B_1} e^{(\theta-1)nr_1} \leqslant \sum_{k}^{m} |u_n e^{(1-\theta)n}|_{B_1} \leqslant \Big(\sum_{k}^{m} \|u_n e^{(1-\theta)n}\|^{p_1}_{B_1}\Big)^{r_1/p_1},$$

while if $r_1 < p_1$

$$\sum_{k}^{m} |u_n e^{(1-\theta)n}|_{B_1} e^{(\theta-1)r_1 n} \leqslant C(k, \theta, p, r)\Big(\sum_{k}^{m} \|u_n e^{(1-\theta)n}\|^{p_1}_{B_1}\Big)^{r_1/p_1}$$

with $\lim_{k\to\infty} C(k, \theta, p, r) = 0$.

In any case, $\sum_{0}^{m} u_n$ is a Cauchy sequence in B_1, and so $\sum_{0}^{\infty} u_n \in B_1$. The proof for $\sum_{-\infty}^{-1} u_n$ is similar. Therefore $\sum_{-\infty}^{\infty} u_n \in B_0 + B_1$. The linearity and continuity of T are now clear.

We can therefore define an s norm on the range of T, where $s = \min\{r_0, r_1, p_0, p_1\}$, with respect to which the range is an s Banach space, T is continuous and from open mapping theorem, $(R, |\ |_T)$ is continuously embedded in $B_0 + B_1$.

DEFINITION 8. Let (B_0, B_1) be an interpolation pair of r_0, r_1 Banach spaces. Define:

$$(B_0, B_1)_{\theta,p_0,p_1} = \Big\{\sum_{-\infty}^{\infty} u_n / \{u_n\} \in w(p_0, B_0; p_1, B_1; \theta)\Big\}$$

with the s norm defined by T above:

$$|b|_{(B_0,B_1)_{\theta,p_0,p_1}} = \inf\Big\{|\{u_n\}|_w \Big/ \sum_{-\infty}^{\infty} u_n = b\Big\}.$$

When B_0, B_1 are not complete we can still make the following definition:

DEFINITION 9. Let (B_0, B_1) be an interpolation pair of r_0 and r_1 normed spaces. Define:

$$(B_0, B_1)^{\theta,p_0,p_1} = \{b \in B_0 + B_1 / \exists \{v_{in}\} \in l^{p_i}_{e^{(i-\theta)n}}(B_i),\ v_{0n} + v_{1n} = b\},$$

$$|b|_{(B_0,B_1)^{\theta,p_0,p_1}} = \inf\{\max_{i=0,1} \|\{e^{(i-\theta)n} v_{in}\}\|^s_{l^{p_i}(B)} / v_{0n} + v_{1n} = b\},$$

where again $s = \min\{r_0, r_1, p_0, p_1\}$. $\big((B_0, B_1)^{\theta,p_0,p_1}, |\ |_{(B_0,B_1)^{\theta,p_0,p_1}}\big)$ is easily seen to be an s normed space.

THEOREM 10. *If (B_0, B_1) is an interpolation pair of r_0, r_1 Banach spaces, then $(B_0, B_1)^{\theta,p_0,p_1} = (B_0, B_1)_{\theta,p_0,p_1}$.*

Proof. Let $b \in (B_0, B_1)^{\theta,p_0,p_1}$. Let $v_{0n} + v_{1n} = b$, $\{e^{(i-\theta)n} v_{in}\} \in l^{p_i}(B_i)$. Take $u_n = v_{0n} - v_{0n-1}$. Clearly

$$|\{u_n\}|_{l^{p_0}_{e^{-\theta n}}(B_0)} \leqslant 2|\{v_{0n}\}|_{l^{p_0}_{e^{-\theta n}}(B_0)}.$$

Since $v_{0n} + v_{1n} = b = v_{0n-1} + v_{1n-1}$, $u_n = v_{0n} - v_{0n-1} = -(v_{1n} - v_{1n-1})$ and so also $|\{u_n\}|_{l^{p_1}_{e^{(1-\theta)n}}(B_1)} \leqslant 2|\{v_{1n}\}|_{l^{p_1}_{e^{(1-\theta)n}}(B_1)}$, $\sum_{-k}^{0} u_n = v_{00} - v_{0-k-1}$. $|\{e^{-\theta n} v_{0n}\}|_{l^{p_0}(B_0)} < \infty$ and so $|v_{0-k-1}|_{B_0} \to 0$. Therefore $\sum_{-k}^{0} u_n \to v_{00}$ in B_0.

Similarly, $\sum_{1}^{k} u_n = v_{10} - v_{1k} \to v_{10}$ in B_1, and so $\sum_{-\infty}^{\infty} u_n = v_{00} + v_{10} = b$ (in $B_0 + B_1$), and

$$|b|_{(B_0,B_1)_{\theta,p_0,p_1}} \leqslant C|b|_{(B_0,B_1)^{\theta,p_0,p_1}}.$$

Conversely, if $b\in(B_0,B_1)_{\theta,p_0,p_1}$, $b=\sum_{-\infty}^{\infty}u_n$, $\{u_n\}\in w(p_0,B_0;p_1,B_1;\theta)$, take $\sum_{-\infty}^{k-1}u_n=v_{0k}$, $\sum_{k}^{\infty}u_n=v_{1k}$, $e^{-\theta k}v_{0k}=\sum_{-\infty}^{k-1}e^{-\theta k}u_n=\sum_{-\infty}^{k-1}e^{-\theta n}u_n e^{\theta(n-k)}$. Therefore:

$$\sum_{-\infty}^{\infty}\|e^{-\theta k}v_{0k}\|_{B_0}^{p_0}\leqslant\sum_{k=-\infty}^{\infty}\Big(\sum_{n=-\infty}^{k-1}|e^{-\theta n}u_n|_{B_0}e^{r_0\theta(n-k)}\Big)^{p_0/r_0}=I.$$

If now $p_0/r_0\geqslant 1$, use Young's inequality for convolutions:

$$I^{r_0/p_0}\leqslant\Big(\sum_{n=-\infty}^{\infty}|e^{-\theta n}u_n|_{B_0}^{p_0/r_0}\Big)^{r_0/p_0}\Big(\sum_{n=-\infty}^{-1}e^{r_0\theta n}\Big)$$

while if $p_0/r_0<1$

$$I\leqslant\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{k-1}|e^{-\theta n}u_n|_{B_0}^{p_0/r_0}e^{p_0\theta(n-k)}=\Big(\sum_{n=-\infty}^{\infty}\|e^{-\theta n}u_n\|_{B_0}^{p_0}\Big)\Big(\sum_{n=1}^{\infty}e^{-p_0\theta n}\Big).$$

In any case $|\{e^{-\theta n}v_{0n}\}|_{l^{p_0}(B_0)}\leqslant C|\{e^{-\theta n}u_n\}|_{l^{p_0}(B_0)}$. Similarly:

$$|\{e^{(1-\theta)n}v_{1n}\}|_{l^{p_1}(B_1)}\leqslant C|\{e^{(1-\theta)n}u_n\}|_{l^{p_1}(B_1)}$$

and the proof is complete.

THEOREM 11. *Let (B_0,B_1) be an interpolation pair of r_0,r_1 normed spaces. If for some $\lambda\neq 0$, there exist sequences $\{v_{in}\}\in l^{p_i}_{e(i-\theta)\lambda n}(B_i)$, $i=0,1$, $v_{0n}+v_{1n}=b$, then for every $\mu\neq 0$ there exist sequences $\{v_{in}^{\mu}\}\in l^{p_i}_{e(i-\theta)\mu n}(B_i)$ so that $v_{0n}^{\mu}+v_{1n}^{\mu}=b$. Further:*

$$|\{e^{(i-\theta)\mu n}v_{in}^{\mu}\}|_{l^{p_i}(B_i)}\leqslant C(\lambda,\mu)|\{e^{(i-\theta)\lambda n}v_{in}\}|_{l^{p_i}(B_i)}\qquad(i=0,1)$$

(and in particular $b\in(B_0,B_1)^{\theta,p_0,p_1}$).

Proof. Since $v_{in}^{-\lambda}=v_{i-n}$ takes care of the case $\mu=-\lambda$, we can assume $0<\lambda,\mu$.

Let $a=\mu/\lambda$, and take $v_{in}^{\mu}=v_{i[an]}$ (where $[an]$ is the largest integer not larger than an). Clearly $v_{0n}^{\mu}+v_{1n}^{\mu}=b$. We have

$$\|e^{(i-\theta)[an]\lambda}v_{i[an]}\|_{B_i}e^{(i-1)\theta\lambda}\leqslant\|e^{(i-\theta)\mu n}v_{in}^{\mu}\|_{B_i}\leqslant\|e^{(i-\theta)[an]\lambda}v_{i[an]}\|_{B_i}e^{i(1-\theta)\lambda}.$$

From these inequalities it is clear that if $0<\mu_1<\mu_2$

$$|\{e^{(i-\theta)\mu_2 n}v_{in}^{\mu_2}\}|_{l^{p_i}(B_i)}\leqslant C|\{e^{(i-\theta)\mu_1 n}v_{in}^{\mu_1}\}|_{l^{p_i}(B_i)}$$

and so suffices to consider the case $a=1/k$.

For such μ, however,

$$\sum_{-\infty}^{\infty}\|e^{(i-\theta)\mu n}v_{in}^{\mu}\|_{B_i}^{p_i}=ke^{(i-\theta)\lambda}\sum_{-\infty}^{\infty}\|e^{(i-\theta)\lambda n}v_{in}\|_{B_i}^{p_i},$$

and the proof is complete.

THEOREM 12. *Let (B_0,B_1) be an interpolation pair of r_0,r_1 normed spaces. Then:*

$$\|b\|_{(B_0,B_1)^{\theta,p_0,p_1}}\sim\inf\{\|\{e^{-\theta n}v_n\}\|_{l^{p_0}(B_0)}^{1-\theta}\|\{e^{(1-\theta)n}v_{1n}\}\|_{l^{p_1}(B_1)}^{\theta}/v_{0n}+v_{1n}=b\}.$$

If B_i are complete,

$$\|b\|_{(B_0,B_1)_{\theta,p_0,p_1}}\sim\inf\Big\{\|\{e^{-\theta n}u_n\}\|_{l^{p_0}(B_0)}^{1-\theta}\|\{e^{(1-\theta)n}u_n\}\|_{l^{p_1}(B_1)}^{\theta}\Big/\sum u_n=b\Big\}.$$

Proof. Let $\{v_{in}\}\in l^{p_i}_{e(i-\theta)n}(B_i)$, $v_{0n}+v_{1n}=b$. Denote $v_{in}^k=v_{in+k}$. We have:

$$\|\{e^{(i-\theta)n}v_{in}^k\}\|_{l^{p_i}(B_i)}=e^{(\theta-i)k}\|\{e^{(i-\theta)n}v_{in}\}\|_{l^{p_i}(B_i)}.$$

Let γ be determined by:

$$e^{\theta\gamma}\|\{e^{-\theta n}v_{0n}\}\|_{l^{p_0}(B_1)}=e^{(\theta-1)\gamma}\|\{e^{(1-\theta)n}v_{1n}\}\|_{l^{p_1}(B_1)}$$

and take $k=[\gamma]+1$.

$$e^{(\theta-i)k}\|\{e^{(i-\theta)n}v_{in}\}\|_{l^{p_i}(B_i)}=e^{(\theta-i)(k-\gamma)}\|\{e^{(1-\theta)n}v_{1n}\}\|_{l^{p_1}(B_1)}^{\theta}\|\{e^{-\theta n}v_{0n}\}\|_{l^{p_0}(B_0)}^{1-\theta}$$

and so:

$$\|b\|_{(B_0,B_1)^{\theta,p_0,p_1}}\leqslant C\inf\{\|\{e^{-\theta n}v_{0n}\}\|_{l^{p_0}(B_0)}^{1-\theta}\|\{e^{(1-\theta)n}v_{1n}\}\|_{l^{p_1}(B_1)}^{\theta}/v_{0n}+v_{1n}=b\}.$$

Since:

$$\|\{e^{-\theta n}v_{0n}\}\|_{l^{p_0}(B_0)}^{1-\theta}\|\{e^{(1-\theta)n}v_{1n}\}\|_{l^{p_1}(B_1)}^{\theta}\leqslant\max_{i=0,1}\{\|\{e^{(i-\theta)n}v_{in}\}\|_{l^{p_i}(B_i)}\},$$

the first claim is proved. The second follows from it via the construction in the proof of Theorem 10, or else can be done directly as above.

THEOREM 13. *If $p_i\leqslant\bar{p}_i$, (B_0,B_1) is an interpolation pair of r_0,r_1 normed spaces, then:*

$$(B_0,B_1)^{\theta,p_0,p_1}\subset(B_0,B_1)^{\theta,\bar{p}_0,\bar{p}_1}.$$

Proof. $l^{p_i}_{e(i-\theta)n}(B_i)\subset l^{\bar{p}_i}_{e(i-\theta)n}(B_i)$.

DEFINITION 14. (See [3]) Let (A_0,A_1), (B_0,B_1) be two interpolation pairs of r_i, ϱ_i normed spaces. T: $A_0+A_1\to B_0+B_1$ will be called a *quasi-linear operator* from (A_0,A_1) to (B_0,B_1) iff for every $a_0+a_1\in A_0+A_1$ we can find $b_i\in B_i$ so that

$$(15)\qquad T(a_0+a_1)=b_0+b_1\quad\text{and}\quad\|b_i\|_{B_i}\leqslant K_i\|a_i\|_{A_i}.$$

Of course if T is a linear operators from A_0+A_1 to B_0+B_1, whose restrictions map A_i continuously into B_i it will satisfy the requirements of the definition.

THEOREM 16. (Interpolation Theorem.) *Let (A_0,A_1), (B_0,B_1) be two interpolation pairs of r_i, ϱ_i normed spaces. T a quasi-linear operator from (A_0,A_1) to (B_0,B_1), then if $0<\theta<1$, $0<p_0,p_1\leqslant\infty$,*

$$T\colon (A_0, A_1)^{\theta,p_0,p_1} \to (B_0, B_1)^{\theta,p_0,p_1}$$

and $\|T\|_\theta \leqslant CK_0^{1-\theta}K_1^\theta$, *where* K_i *are the constants appearing in* (15).

Proof. Let $a \in (A_0, A_1)^{\theta,p_0,p_1}$. $a = v_{0n}+v_{1n}$, $\{e^{(i-\theta)n}v_{in}\} \in l^{p_i}(A_i)$, $i = 0, 1$. $v_{in} \in A_i$ and so for every n we have, w_{0n}, w_{1n} so that

$$Ta = w_{0n}+w_{1n}, \quad \text{and} \quad \|w_{in}\|_{B_i} \leqslant K_i\|v_{in}\|_{A_i}.$$

Therefore $\|\{e^{(i-\theta)n}w_{in}\}\|_{l^{p_i}(B_i)} \leqslant K_i\|\{e^{(i-\theta)n}v_{in}\}\|_{l^{p_i}(A_i)}$ and so

$$\|Ta\|_{(B_0,B_1)^{\theta,p_0,p_1}} \leqslant c\,\|\{e^{-\theta n}w_{0n}\}\|_{l^{p_0}(B_0)}^{1-\theta}\|\{e^{(1-\theta)n}w_{1n}\}|_{l^{p_1}(B_1)}$$
$$\leqslant cK_0^{1-\theta}K_1^\theta\|\{e^{-\theta n}v_{0n}\}\|_{l^{p_0}(A_0)}^{1-\theta}\|\{e^{(1-\theta)n}v_{1n}\}\|_{l^{p_0}(A_1)}^{\theta}.$$

Taking the infimum of last expression over all sequences $\{v_{0n}\}$, $\{v_{1n}\}$ so that $v_{0n}+v_{1n} = a$, we get

$$\|Ta\|_{(B_0,B_1)^{\theta,p_0,p_1}} \leqslant cK_0^{1-\theta}K_1^\theta\|a\|_{(A_0,A_1)^{\theta,p_0,p_1}}.$$

DEFINITION 17. Let (X, Σ, μ) be a σ-finite measure space, $0 \leqslant \mu$. Let $(B, |\ |_B)$ be an r normed space. Denote by $L^p(B)$ the space of all strongly measurable B valued functions on (X, Σ, μ), so that $\|f\|_{L^p(B)} < \infty$, where

(a) $\|f\|_{L^p(B)} = [\int_X \|f\|_B^p\,d\mu]^{1/p}$ if $0 < p < \infty$,

(b) $\|f\|_{L^\infty(B)} = \operatorname{Ess\,Sup}\|f\|_B$.

THEOREM 18. *$L^p(B)$ is s normed with $s = \min\{r, p\}$, $\|\ \|_{L^p(B)}^s = |\ |_{L^p(B)}$. If B is complete, so is $L^p(B)$.*

Proof. Triangle inequality is all we have to verify for the first claim. If $s = p$,

$$|f+g|_{L^p(B)} = \int \|f+g\|_B^p\,d\mu = \int |f+g|_B^{p/r}\,d\mu \leqslant \int |f|_B^{p/r}\,d\mu + \int |g|^{p/r}\,d\mu$$
$$= |f|_{L^p(B)}+|g|_{L^p(B)},$$

while if $s = r$

$$|f+g|_{L^p(B)} = \Big(\int \|f+g\|_B^p\Big)^{r/p} = \Big(\int |f+g|_B^{p/r}\,d\mu\Big)^{r/p}$$
$$\leqslant \Big(\int |f|_B^{p/r}d\mu\Big)^{r/p}+\Big(\int |g|_B^{p/r}\,d\mu\Big)^{r/p}.$$

The last inequality is Minkowski's for $p/r \geqslant 1$.

The proof of completeness of $L^p(B)$, given that of B, follows along the same lines as that of the corresponding proof for Banach space valued functions.

THEOREM 19. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces. Then:*

$$\big(L^{p_0}(B_0), L^{p_1}(B_1)\big)^{\theta,p_0,p_1} = L^p\big((B_0, B_1)^{\theta,p_0,p_1}\big);$$

where $0 < p_0, p_1 \leqslant \infty$ *and* $\dfrac{1}{p} = \dfrac{1-\theta}{p_0}+\dfrac{\theta}{p_1}$.

Proof. Let $f(x) \in \big(L^{p_0}(B_0), L^{p_1}(B_1)\big)^{\theta,p_0,p_1}$, $f(x) = v_{0n}(x)+v_{1n}(x)$, a.e.$x$, where $v_{in}(\cdot) \in L^{p_i}(B_i)$ for every n, and $\{\|e^{(i-\theta)n}v_{in}(x)\|_{L^{p_i}(B_i)}\} \in l^{p_i}$. For almost every x we have

$$\|f(x)\|_{(B_0,B_1)^{\theta,p_0,p_1}} \leqslant c\Big(\sum_{-\infty}^{\infty}\|e^{-\theta n}v_{0n}\|_{B_0}^{p_0}\Big)^{\frac{1-\theta}{p_0}}\Big(\sum_{-\infty}^{\infty}\|e^{(1-\theta)n}v_{1n}\|_{B_1}^{p_1}\Big)^{\frac{\theta}{p_1}}.$$

Using Hölder's inequality:

$$\int_X \|f\|_{(B_0,B_1)^{\theta,p_0,p_1}}^p\,d\mu$$
$$\leqslant c\Big(\int_X\sum_{-\infty}^{\infty}\|e^{-\theta n}v_{0n}\|_{B_0}^{p_0}\,d\mu\Big)^{\frac{p(1-\theta)}{p_0}}\Big(\int_X\sum_{-\infty}^{\infty}\|e^{(1-\theta)n}v_{1n}\|_{B_1}^{p_1}\,d\mu\Big)^{\frac{p\theta}{p_1}}$$

and so:

$$\|f\|_{L^p((B_0,B_1)^{\theta,p_0,p_1}} \leqslant c\|\{e^{-\theta n}v_{0n}\}\|_{l^{p_0}(L^{p_0}(B_0))}^{1-\theta}\|\{e^{(1-\theta)n}v_{1n}\}\|_{l^{p_1}(L^{p_1}(B_1))}^{\theta}.$$

Conversely: Let $f = \sum_k b_k\chi_k$ be in $L^p\big((B_0, B_1)^{\theta,p_0,p_1}\big)$, where χ_k are characteristic functions of disjoint measurable sets $b_k \in (B_0, B_1)^{\theta,p_0,p_1}$ and so we can write $f(x) = v_{0n}(x)+v_{1n}(x)$, where:

$$\|f(x)\|_{(B_0,B_1)^{\theta,p_0,p_1}} \geqslant c\max_{i=0,1}\{\|\{e^{(i-\theta)n}v_{in}\|_{l^{p_i}(B_i)}\}.$$

Let λ be given by: $p_0(1+\lambda\theta) = p$. We then have:

$$p_1\big(1-\lambda(1-\theta)\big) = p.$$

Taking $w_{in} = v_{in+k}$, where $k = [\lambda\log\|f\|_{(B_0,B_1)^{\theta,p_0,p_1}}]$, we have:

$$\sum_{-\infty}^{\infty}\|e^{(i-\theta)n}w_{in}\|_{B_i}^{p_i} \leqslant C\Big(\sum_{-\infty}^{\infty}\|e^{(i-\theta)n}v_{in}\|_{B_i}^{p_i}\Big)\exp\big(\lambda(\theta-i)p_i\log\|f\|_{(B_0,B_1)^{\theta,p_0,p_1}}\big)$$
$$\leqslant c\|f\|_{(B_0,B_1)^{\theta,p_0,p_1}}^{p_i+p_i\lambda(\theta-i)} = c\|f\|_{(B_0,B_1)^{\theta,p_0,p_1}}^{p},$$

and so:

$$\|\{e^{(i-\theta)n}w_{in}\}\|_{l^{p_i}(L^{p_i}(B_i))}^{p_i} = \sum_{-\infty}^{\infty}\int_X\|e^{(i-\theta)n}w_{in}(x)\|_{B_i}^{p_i}\,d\mu$$
$$= \int_X\|\{e^{(i-\theta)n}w_{in}(x)\|_{l^{p_i}(B_i)}^{p_i}\,d\mu$$
$$\leqslant c\int_X\|f\|_{(B_0,B_1)^{\theta,p_0,p_1}}^p\,d\mu$$
$$= c\|f\|_{L^p((B_0,B_1)^{\theta,p_0,p_1})}^p.$$

However:

$$\|f\|_{(L^{p_0}(B_0),L^{p_1}(B_1))^{\theta,p_0,p_1}} \leqslant c\|\{e^{-\theta n}w_{0n}\}\|^{1-\theta}_{l^{p_0}(L^{p_0}(B_0))}\|\{e^{(1-\theta)n}w_{1n}\}\|^{\theta}_{l^{p_1}(L^{p_1}(B_1))}$$

$$\leqslant c\|f\|^{\frac{1-\theta}{p_0}p+\frac{\theta}{p_1}p}_{L^p((B_0,B_1)^{\theta,p_0,p_1})} = c\|f\|_{L^p((B_0,B_1)^{\theta,p_0,p_1})}.$$

Since functions of the form given above are dense in L^p the proof is complete.

THEOREM 20. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces. Then for $b \in B_0 \cap B_1$, we have*

$$\|b\|_{(B_0,B_1)^{\theta,p_0,p_1}} \leqslant c\|b\|^{1-\theta}_{B_0}\|b\|^{\theta}_{B_1}.$$

Proof. Take

$$v_{0n} = \begin{cases} b, & n \geqslant 0, \\ 0, & n < 0, \end{cases} \qquad v_{1n} = \begin{cases} 0, & n \geqslant 0, \\ b, & n < 0, \end{cases}$$

$\|\{e^{(i-\theta)}v_{in}\}\|_{l^{p_i}(B_i)} \leqslant c\|b\|_{B_i}$ and hence the result follows from Theorem 12.

DEFINITION 21. Let (B_0, B_1) be an interpolation pair of r_0, r Banach spaces, $0 < \theta < 1$, $B_0 \cap B_1 \subset B \subset B_0 + B_1$, B an r Banach space. We define: $B \in \underline{K}_\theta(B_0, B_1)$ iff c exists so that for every $b \in B_0 \cap B_1$, $\|b\|_B \leqslant c\|b\|^{1-\theta}_{B_0}\|b\|^{\theta}_{B_1}$.

THEOREM 22. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 Banach spaces, $B_0 \cap B_1 \subset B \subset B_0 + B_1$, B an r Banach space. Then: If for some $p > 0$ $(B_0, B_1)_{\theta,p,p} \subset B$, then $B \in \underline{K}_\theta(B_0, B_1)$.*

If $B \in \underline{K}_\theta(B_0, B_1)$, then $(B_0, B_1)_{\theta,r,r} \subset B$.

Proof. We have shown that for every $0 < p$, $(B_0, B_1)_{\theta,p,p} \in \underline{K}_\theta(B_0, B_1)$. If now $(B_0, B_1)_{\theta,p,p} \subset B$ we have $\|b\|_B \leqslant c\|b\|_{(B_0,B_1)_{\theta,p,p}} \leqslant c\|b\|^{1-\theta}_{B_0}\|b\|^{\theta}_{B_1}$, for every $b \in B_0 \cap B_1$, and so $B \in \underline{K}_\theta(B_0, B_1)$.

If now $B \in \underline{K}_\theta(B_0, B_1)$ is an r Banach space, take

$$b \in (B_0, B_1)_{\theta,r,r}, \qquad b = \sum_{-\infty}^{\infty} u_n,\ \{u_n\} \in w(r, B_0; r, B_1, \theta),$$

and so

$$\Big|\sum_{-\infty}^{\infty} u_n\Big|_B \leqslant \sum_{-\infty}^{\infty} |u_n|_B \leqslant c\sum_{-\infty}^{\infty} \|u_n\|^{r(1-\theta)}_{B_0}\|u_n\|^{r\theta}_{B_1}$$

$$= c\sum_{-\infty}^{\infty} \|e^{-\theta n}u_n\|^{r(1-\theta)}_{B_0}\|e^{(1-\theta)n}u_n\|^{r\theta}_{B_1}$$

$$\leqslant c\|\{e^{-\theta n}u_n\}\|^{r(1-\theta)}_{l^r(B_0)}\|\{e^{(1-\theta)n}u_n\}\|^{r\theta}_{l^r(B_1)}.$$

And so: $\|b\|_B \leqslant c\|b\|_{(B_0,B_1)_{\theta,r,r}}$, and $(B_0, B_1)_{\theta,r,r} \subset B$.

THEOREM 23. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 Banach spaces. Let $(X_i, |\ |_{X_i}) \in \underline{K}_{\theta_i}(B_0, B_1)$, $i = 0, 1$, $\theta_0 < \theta < \theta_1$. Then if $\frac{1}{q_i} = \frac{1-\theta_i}{p_0} + \frac{\theta_i}{p_1}$, we have*

$$(B_0, B_1)_{\theta,p_0,p_1} \subset (X_0, X_1)_{\eta,q_0,q_1}, \quad \textit{where } \eta = (\theta-\theta_0)/(\theta_1-\theta_0).$$

Proof. Let $b \in (B_0, B_1)_{\theta,p_0,p_1}$, $b = \sum_{-\infty}^{\infty} u_n$, $\{u_n\} \in w(p_0, B_0; p_1, B_1; \theta)$.

$$\|u_n\|_{X_i} \leqslant C\|u_n\|^{1-\theta_i}_{B_0}\|u_n\|^{\theta_i}_{B_1},$$

$$\|e^{(\theta_i-\theta)n}u_n\|^{q_i}_{X_i} \leqslant C\|e^{-\theta n}u_n\|^{(1-\theta_i)q_i}_{B_0}\|e^{(1-\theta)n}u_n\|^{\theta_i q_i}_{B_i}.$$

Using Hölder's inequality:

$$\|\{e^{(\theta_i-\theta)n}u_n\}\|_{l^{q_i}(X_i)} \leqslant C\Big(\sum_{-\infty}^{\infty}\|e^{-\theta n}u_n\|^{p_0}_{B_0}\Big)^{\frac{1-\theta_i}{p_0}}\Big(\sum_{-\infty}^{\infty}\|e^{(1-\theta)n}u_n\|^{p_1}_{B_1}\Big)^{\frac{\theta_i}{p_1}}$$

and using Theorem 11:

$$\|b\|_{(X_0,X_1)_{\eta,q_0,q_1}} \leqslant C\|\{e^{(\theta_0-\theta)n}u_n\}\|^{1-\eta}_{l^{q_0}(X_0)}\|\{e^{(\theta_1-\theta)n}u_n\}\|^{\eta}_{l^{q_1}(X_1)}$$

$$\leqslant C\|\{e^{-\theta n}u_n\}\|^{1-\theta}_{l^{p_0}(B_0)}\|\{e^{(1-\theta)n}u_n\}\|^{\theta}_{l^{p_1}(B_1)}.$$

Taking the infimum of last expression over all sequences so that $b = \sum_{-\infty}^{\infty} u_n$ we get

$$\|b\|_{(X_0,X_1)_{\eta,q_0,q_1}} \leqslant C\|b\|_{(B_0,B_1)_{\theta,p_0,p_1}}.$$

DEFINITION 24. Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces, $0 < \theta < 1$. Let $B \subset B_0 + B_1$ be an r normed space. Then we define: $B \in \overline{K}_\theta(B_0, B_1)$ iff a constant c exists so that for every $b \in B$ sequences $\{v_{in}\} \in l^{\infty}_{e(i-\theta)n}(B_i)$, $i = 0, 1$, exist, with $\|\{e^{(i-\theta)n}v_{in}\}\|_{l^\infty(B_i)} \leqslant c\|b\|_B$ and $b = v_{0n} + v_{1n}$.

THEOREM 25. *$B \in \overline{K}_\theta(B_0, B_1)$ iff $B \subset (B_0, B_1)^{\theta,\infty,\infty}$.*

Proof. Follows directly from the definition.

THEOREM 26. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces. $0 < \theta_0 < \theta < \theta_1 < 1$, $\eta = (\theta-\theta_0)/(\theta_1-\theta_0)$, $(X_i, |\ |_{X_i}) \in \overline{K}_{\theta_i}(B_0, B_1)$. Then:*

$$(X_0, X_1)^{\eta,q_0,q_1} \subset (B_0, B_1)^{\theta,p_0,p_1}, \quad \textit{where } \frac{1}{q_i} = \frac{1-\theta_i}{p_0} + \frac{\theta_i}{p_1}.$$

Proof. Let $x \in (X_0, X_1)^{\eta,q_0,p_1}$, $x = v_{0n} + v_{1n}$, where, using Theorem 11, $\|\{e^{(\theta_i-\theta)n}v_{in}\}\|_{l^{q_i}(X_i)} \leqslant c\|x\|_{(X_0,X_1)^{\eta,q_0,q_1}}$. Since $X_i \in \overline{K}_{\theta_i}(B_0, B_1)$, $v_{in} = v_{in0m_i} +$

$+v_{in1m_i}$ with $\|v_{injm_i}\|_{B_j} \leqslant ce^{(\theta_i-j)m_i}\|v_{in}\|_{X_i}$, all m_i. $x = (v_{0m0m_0}+v_{1n0m_1})+$ $+(v_{0n1m_0}+v_{1n1m_1})$, $\|e^{(j-\theta)n}v_{injm_i}\|_{B_j}^{p_j} \leqslant c\|e^{(\theta_i-j)m_i-(\theta-j)n}v_{in}\|_{X_i}^{p_j}$.

Let now $m_i = n+t_i$, we can choose t_i

$$\|e^{(j-\theta)n}v_{injm_i}\|_{B_j}^{p_j} \leqslant ce^{(\theta_i-j)t_ip_j}\|e^{(\theta_i-\theta)n}v_{in}\|_{X_i}^{p_j}.$$

Let ξ_i be chosen so that

$$e^{\theta_i\xi_ip_0}\|e^{(\theta_i-\theta)n}v_{in}\|_{X_i}^{p_0} = cK_i^{p_0\theta_i}\|e^{(\theta_i-\theta)n}v_{in}\|_{X_i}^{q_i} \quad (i = 0, 1)$$

(K_i will be chosen later).

We then have:

$$e^{(\theta_i-1)\xi_ip_1}\|e^{(\theta_i-\theta)n}v_{in}\|_{X_i}^{p_1} = K_i^{p_1(\theta_i-1)}\|e^{(\theta_i-\theta)n}v_{in}\|_{X_i}^{q_i}.$$

Take $t_i = [\xi_i]$:

$$\|e^{(j-\theta)n}v_{injm_i}\|_{B_j}^{p_j} \leqslant cK_i^{p_j(\theta_i-j)}\|e^{(\theta_i-\theta)n}v_{in}\|_{X_i}^{q_i}.$$

Therefore

$$\|\{e^{(j-\theta)n}(v_{0njm_0}+v_{1njm_1})\}\|_{l^{p_j}(B_j)}$$
$$\leqslant c\,[K_0^{(\theta_0-j)}\|\{e^{(\theta_0-\theta)n}v_{0n}\}\|_{l^{q_0}(X_0)}^{q_0/p_j} + K_1^{(\theta_1-j)}\|\{e^{(\theta_1-\theta)}v_{1n}\}\|_{l^{q_1}(X_1)}^{q_1/p_j}]$$
$$\leqslant c\,[K_0^{(\theta_0-j)}\|x\|_{(X_0,X_1)^{\eta,q_0,q_1}}^{q_0/p_j} + K_1^{(\theta_1-j)}\|x\|_{(X_0,X_1)^{\eta,q_0,q_1}}^{q_1/p_j}].$$

Taking $K_i = \|x\|_{(X_0,X_1)^{\eta,q_0,q_1}}^{\left(1-\frac{q_i}{p_0}\right)\frac{1}{\theta_i}}$, we get

$$\|\{e^{(j-\theta)n}v_{0njm_0}+v_{1njm_1}\}\|_{l^{p_j}(B_j)} \leqslant \|x\|_{(X_0,X_1)^{\eta,q_0,q_1}},$$

and: $\|x\|_{(B_0,B_1)^{\theta,p_0,p_1}} \leqslant \|x\|_{(X_0,X_1)^{\eta,q_0,q_1}}$ and the proof is complete.

Definition 27. Let (B_0, B_1) be an interpolation pair of r_0, r_1 Banach spaces. $B_0 \cap B_1 \subset B \subset B_0+B_1$, $0 < \theta < 1$. B an r-Banach space. Then define:

$$B \in K_\theta(B_0, B_1) \quad \text{iff} \quad B \in \underline{K}_\theta(B_0, B_1) \cap \overline{K}_\theta(B_0, B_1).$$

Thus $B \in K_\theta(B_0, B_1)$ iff $(B_0, B_1)_{\theta,r,r} \subset B \subset (B_0, B_1)_{\theta,\infty,\infty}$ (where B is an r-Banach space).

Theorem 28. (Reiteration theorem). *If (B_0, B_1) is an interpolation pair of r_0, r_1 Banach spaces, $X_i \in K_{\theta_i}(B_0, B_1)$, $0 < \theta_0 < \theta < \theta_1 < 1$, then $(X_0, X_1)_{(\theta-\theta_0)/(\theta_1-\theta_0),q_0,q_1} = (B_0, B_1)_{\theta,p_0,p_1}$, where $\dfrac{1}{q_i} = \dfrac{1-\theta_i}{p_0} + \dfrac{\theta_i}{p_1}$.*

The theorem is of course a combination of Theorems 23 and 26. The interpolation theorem can be combined with Theorems 23, 26 to yield the following theorem:

Theorem 29. *Let (A_0, A_1) be an interpolation pair of (r_0, r_1) Banach spaces, (B_0, B_1) of ϱ_0, ϱ_1 normed spaces. Let $0 < \theta_0 < \theta < \theta_1 < 1$, $0 < \eta_0 < \eta < \eta_1 < 1$, $\sigma = (\theta-\theta_0)/(\theta_1-\theta_0) = (\eta-\eta_0)/(\eta_1-\eta_0)$, $X_i \in \underline{K}_{\theta_i}(A_0, A_1)$ and $Y_i \in \overline{K}_{\eta_i}(A_0, A_1)$. Let further $0 < p_0, p_1 \leqslant \infty$ and q_i be determined by $\dfrac{1-\theta_i}{p_0} + \dfrac{\theta_i}{p_1} = \dfrac{1-\eta_i}{q_0} + \dfrac{\eta_i}{q_1}$. If then T is a quasi-linear operator from (X_0, X_1) to (Y_0, Y_1), we have:*

$$\|Ta\|_{(B_0,B_1)^{\eta,q_0,q_1}} \leqslant c\cdot K_0^{1-\sigma}K_1^{\sigma}\|a\|_{(A_0,A_1)_{\theta,p_0,p_1}}.$$

Proof. $a \in (A_0, A_1)_{\theta,p_0,p_1}$. Then $a \in (X_0, X_1)_{\sigma,r_0,r_1}$ with $\dfrac{1}{r_i} = \dfrac{1-\theta_i}{p_0} + \dfrac{\theta_i}{p_1}$, and $\|a\|_{(X_0,X_1)_{\sigma,r_0,r_1}} \leqslant c\|a\|_{(A_0,A_1)_{\theta,p_0,p_1}}$. From interpolation theorem

$$\|Ta\|_{(Y_0,Y_1)^{\sigma,r_0,r_1}} \leqslant cK_0^{1-\sigma}K_1^{\sigma}\|a\|_{(X_0,X_1)_{\sigma,r_0,r_1}}$$

while from Theorem 26

$$\|Ta\|_{(B_0,B_1)^{\eta,q_0,q_1}} \leqslant c\|Ta\|_{(Y_0,Y_1)^{\sigma,r_0,r_1}}.$$

For $\sigma = \dfrac{\eta-\eta_0}{\eta_1-\eta_0}$, $\dfrac{1}{r_i} = \dfrac{1-\eta_i}{q_0} + \dfrac{\eta_i}{q_1}$. Since $\dfrac{1}{r_i} = \dfrac{1-\theta_i}{p_0} + \dfrac{\theta_i}{p_1} = \dfrac{1-\eta_i}{q_0} + \dfrac{\eta_i}{q_1}$.

Combining the three inequalities, the theorem follows.

III. The spaces $(B_0, B_1)_{\theta p}$.

Definition 1. Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces. Define on $B_0 \cap B_1$:

$$M(n, b) = \max\{\|b\|_{B_0}, e^n\|b\|_{B_1}\}$$

and on B_0+B_1:

$$W(n, b) = \operatorname{Inf}\{\max\{\|b_0\|_{B_0}, e^n\|b_1\|_{B_1}\} \mid b_0+b_1 = b\}.$$

These are of course the analogues of the J and K functionals of Peetre [5]. If $r = \min\{r_0, r_1\}$, then $M^r(n, b)$, $W^r(n, b)$ are r norms on $B_0 \cap B_1$, B_0+B_1, both equivalent to the usual ones.

Theorem 2. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces. Then $b \in (B_0, B_1)^{\theta,p,p}$ iff $e^{-\theta n}W(n, b) \in l^p$. Further*

$$\|b\|_{(B_0,B_1)^{\theta,p,p}} \sim \|\{e^{-\theta n}W(n, b)\}\|_{l^p}.$$

Proof. Let $b \in (B_0, B_1)^{\theta,p,p}$, $b = v_{0n}+v_{1n}$, with

$$\|\{e^{(i-\theta)n}v_{in}\}\|_{l^p(B_i)} \leqslant c\|b\|_{(B_0,B_1)^{\theta,p,p}}.$$

$e^{-\theta n}W(n, b) \leqslant \max\limits_{i=0,1}\{\|e^{(i-\theta)n}v_{in}\|_{B_i}\}$, and so

$$\|\{e^{-\theta n}W(n, b)\}\|_{l^p} \leqslant c\max_{i=0,1}\{\|\{e^{(i-\theta)n}v_{in}\}\|_{l^p(B_i)}\} \leqslant c\|b\|_{(B_0,B_1)^{\theta,p,p}}$$

conversely, if $e^{-\theta n}W(n,b)\in l^p$ for every n we can find v_{in} so that $\max\limits_{i=0,1}\{\|v_{in}\|_{B_i}e^{in}\} < cW(n,b)$, $v_{0n}+v_{1n} = b$.

Therefore: $\|b\|_{(B_0,B_1)^{\theta,p,p}} \leqslant c\|\{e^{-\theta n}W(n,b)\}\|_{l^p}$.

THEOREM 3. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 Banach spaces. Then $b\in(B_0,B_1)_{\theta,p,p}$ iff there exists a sequence $\{u_n\}$, $u_n\in B_0\cap B_1$ so that $\sum\limits_{-\infty}^{\infty}u_n = b$ and $\{e^{-\theta n}M(n,u_n)\}\in l^p$. Further:*

$$\|b\|_{(B_0,B_1)_{\theta,p,p}} \sim \operatorname{Inf}\Big\{\|\{e^{-\theta n}M(n,u_n)\}\|_{l^p} \,\Big|\, \sum_{-\infty}^{\infty}u_n = b\Big\}.$$

Proof. Similar to that of Theorem 2.

Our next objective is to show that if (B_0, B_1) is an interpolation pair of r_0, r_1 Banach spaces, then

$$(B_0,B_1)_{\theta,p_0,p_1} = (B_0,B_1)_{\theta,p,p}, \quad \text{where} \quad \frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}$$

we shall need the following result:

THEOREM 4. *Let $0<p\leqslant\infty$, $-\infty<a_i<\infty$, $0<\theta<1$, $a = (1-\theta)a_0+\theta a_1$. Then:*

$$(l^{p_0}_{e^{-a_0 n}}, l^{p_1}_{e^{-a_1 n}})_{\theta,p_0,p_1} = l^p_{e^{-an}}.$$

Proof. Let $\beta = p_1p_0(a_1-a_0)/(p_1-p_0)$. Denote by l^p_β the space of all sequences $\{u_n\}$ so that

$$\|\{u_n\}\|_{l^p_\beta} = \Big(\sum_{-\infty}^{\infty}|u_n|^p e^{\beta n}\Big)^{1/p} < \infty.$$

Define then $I(\{u_n\}) = \Big\{u_n\exp\Big(\dfrac{a_0p_0-a_1p_1}{p_1-p_0}\,n\Big)\Big\}$, we have:

$$\|I\{u_n\}\|_{l^p_\beta} = \Big[\sum_{-\infty}^{\infty}|u_n|^p\exp\Big(-\frac{a_1p_1-a_0p_0}{p_1-p_0}\,pn+p_1p_0\frac{a_1-a_0}{p_1-p_0}\,n\Big)\Big]^{1/p} = \|\{u_n\}\|_{l^p_{e^{-an}}}.$$

Therefore I is an isometry between $l^p_{e^{-an}}$ and l^p_β interpolating the isometry between $l^{p_i}_{e^{-a_i n}}$ and $l^{p_i}_\beta$ we get for $0<\theta<1$:

$$\begin{array}{ccc} (l^{p_0}_{e^{-a_0 n}}, l^{p_1}_{e^{-a_1 n}})_{\theta,p_0,p_1} & \xrightarrow{I} & (l^{p_0}_\beta, l^{p_1}_\beta)_{\theta,p_0,p_1} \\ & & \downarrow \\ l^p_{e^{-an}} & \xleftarrow{I^{-1}} & l^p_\beta \end{array}$$

and the proof is complete.

THEOREM 5. *Let (B_0, B_1) be an interpolation pair of r_0, r_1 Banach spaces. Let $0<p_0, p_1\leqslant\infty$, $0<\theta<1$. Then $(B_0,B_1)_{\theta,p_0,p_1} = (B_0,B_1)_{\theta,p,p}$, where $\dfrac{1}{p} = \dfrac{1-\theta}{p_0}+\dfrac{\theta}{p_1}$.*

Proof. Let $0<\theta_0<\theta<\theta_1<1$, $\lambda = (\theta-\theta_0)/(\theta_1-\theta_0)$, $\dfrac{1}{q_j} = \dfrac{1-\theta_j}{p_0}+\dfrac{\theta_j}{p_1}$, $X_j = (B_0,B_1)_{\theta_j,q_j,q_j}$. By the reiteration theorem we have

$$(6) \qquad (B_0,B_1)_{\theta,p_0,p_1} = (X_0,X_1)_{\lambda,q_0,q_1}.$$

Let now $b\in(B_0,B_1)_{\theta,p_0,p_1}$. Using (6) we have

$$b = v_{0n}+v_{1n}, \quad \|\{e^{(i-\lambda)n}v_{in}\}\|_{l^{q_i}(X_i)} \leqslant c\|b\|_{(B_0,B_1)_{\theta,p_0,p_1}}.$$

Using Theorem 2, $\|\{e^{-\theta_i m}W(m,v_{in})\}\|_{l^{q_i}} \leqslant c\|v_{in}\|_{X_i}$ and so:

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|e^{(i-\lambda)n}e^{-\theta_i m}W(m,v_{in})|^{q_i} \leqslant c\sum_{-\infty}^{\infty}\|e^{(i-\lambda)n}v_{in}\|^{q_i}_{X_i} \leqslant c\|b\|^{q_i}_{(B_0,B_1)_{\theta,p_0,p_1}}.$$

I.e. $\|\{e^{(i-\lambda)n}e^{-\theta_i m}W(m,v_{in})\}\|_{l^{q_i}(l^{q_i})} \leqslant c\|b\|_{(B_0,B_1)_{\theta,p_0,p_1}}$. Since W^r is an r norm on B_0+B_1 we have:

$$W^r(m,b) = W^r(m,v_{0n}+v_{1n}) \leqslant W^r(m,v_{0n})+W^r(m,v_{1n})$$

and so $0\leqslant W(m,b)\leqslant c[W(m,v_{0n})+W(m,v_{1n})]$.

Therefore $W(m,b)\in(l^{q_0}_{e^{-\theta_0 n}}, l^{q_1}_{e^{-\theta_1 n}})_{\lambda,q_0,q_1}$.

By Theorem 4, $W(m,b)\in l^q_{e^{-\theta n}}$, with

$$\frac{1}{q} = \frac{1-\lambda}{q_0}+\frac{\lambda}{q_1} = \frac{(1-\lambda)(1-\theta_0)+\lambda(1-\theta_1)}{p_0}+\frac{(1-\lambda)\theta_0+\lambda\theta_1}{p_1} = \frac{1}{p},$$

$$\theta = (1-\lambda)\theta_0+\lambda\theta_1,$$

and $\|\{e^{-\theta m}W(m,b)\}\|_{l^p} \leqslant c\|b\|_{(B_0,B_1)_{\theta,p_0,p_1}}$. Using Theorem 2 again, $b\in(B_0,B_1)_{\theta,p,p}$ and the injection is continuous.

Conversely, let $b\in(B_0,B_1)_{\theta,p,p}$. We can write $b = \sum\limits_{-\infty}^{\infty}u_m$, with $\|\{e^{-\theta m}M(m,u_m)\}\|_{l^p} \leqslant c\|b\|_{(B_0,B_1)_{\theta,p,p}}$.

$M(m,u_m)\in l^p_{e^{-\theta m}} = (l^{q_0}_{e^{-\theta_0 m}}, l^{q_1}_{e^{-\theta_1 m}})_{\lambda,q_0,q_1}$ and so we can find two sequences $\{v_{min}\}$ so that $v_{mon}+v_{min} = M(m,u_m)$,

$$\|\{e^{(i-\lambda)n}e^{-\theta_i m}v_{min}\}\|_{l^{q_i}(l^{q_i})} \leqslant c\|b\|_{(B_0,B_1)_{\theta,p,p}}.$$

Since $M(m,u_m)\geqslant 0$, we can assume $v_{min}\geqslant 0$. Take now $b_{in} = \sum\limits_{-\infty}^{\infty}u_m\dfrac{v_{min}}{M(m,u_m)}$ $(i = 0,1)$. Clearly $b_{0n}+b_{1n} = b$.

$$e^{(i-\lambda)n}e^{-\theta_i m}M\left(m, u_m\frac{v_{imn}}{M(m,u_m)}\right) = e^{(i-\lambda)n}e^{-\theta_i m}v_{imn} \text{ and so } \|\{e^{(i-\lambda)n}b_{in}\}\|_{l^{q_i}(X_i)}$$

$\leqslant c\|b\|_{(B_0,B_1)_{\theta,p,p}}$ and $\|b\|_{(X_0,X_1)_{\lambda,q_0,q_1}} \leqslant c\|b\|_{(B_0,B_1)_{\theta,p,p}}$.

From (6) the theorem now follows.

We shall from now on denote $(B_0,B_1)_{\theta,p,p}$ by $(B_0,B_1)_{\theta,p}$.

THEOREM 7. *Let (B_0,B_1) be an interpolation pair of r_0, r_1 Banach spaces, $0<\theta,\lambda<1$, $0<p_i, q\leqslant\infty$. $X_i=(B_0,B_1)_{\theta,p_i}$.*

The following conditions are equivalent:

(a) $b\in(X_0,X_1)_{\lambda,q}$,

(b) $b=\sum_{-\infty}^{\infty}u_n$ *(in B_0+B_1), and* $e^{-\theta n}M(n,u_n)\in(l^{p_0},l^{p_1})_{\lambda,q}$,

(c) $e^{-\theta n}W(n,b)\in(l^{p_0},l^{p_1})_{\lambda,q}$.

The corresponding norms are equivalent.

Proof. (a) $\Rightarrow$ (c): Let $b\in(X_0,X_1)_{\lambda,q}$. Then $b=v_{0n}+v_{1n}$ with $\|\{e^{(i-\lambda)n}v_{in}\}\|_{l^q(X_i)}\leqslant c\|b\|_{(X_0,X_1)_{\lambda,q}}$, $\|\{e^{-\theta m}W(m,v_{in})\}\|_{l^{p_i}}\leqslant c\|v_{in}\|_{X_i}$.

Therefore $b=v_{0n}+v_{1n}$ with

(8) $$e^{-\theta m}e^{(i-\lambda)n}W(m,v_{in})\in l^q(l^{p_i}).$$

$W^r(m,\cdot)$ is an r norm, and so $W^r(m,b)\leqslant W^r(m,v_{0n})+W^r(m,v_{1n})$. Therefore $W(m,b)\leqslant c[W(m,v_{0n})+W(m,v_{1n})]$. From (8), $e^{-\theta m}W(m,b)\in(l^{p_0},l^{p_1})_{\lambda,q}$.

(c) $\Rightarrow$ (b). Let $e^{-\theta n}W(n,b)\in(l^{p_0},l^{p_1})_{\lambda,q}$. For every n let $v_{0n}+v_{1n}=b$, with $e^{in}\|v_{in}\|_{B_i}\leqslant cW(n,b)$, $i=0,1$. $(l^{p_0},l^{p_1})_{\lambda,q}\subset l^\infty$, and so $W(n,b)\leqslant ce^{\theta n}$ therefore

$$\|v_{0n}\|_{B_0}\leqslant ce^{\theta n}\to 0 \quad \text{as} \quad n\to-\infty,$$
$$\|v_{1n}\|_{B_1}\leqslant ce^{(\theta-1)n}\to 0 \quad \text{as} \quad n\to\infty.$$

Take $u_n=v_{0n}-v_{0n-1}=v_{1n-1}-v_{1n}$ (see Theorem II 10). $\sum_{-N}^{N}u_n=v_{0N}-v_{0-N-1}=b-v_{1N}-v_{0-N-1}$ and since $\|v_{1N}\|_{B_1}\to 0$, $\|v_{0-N-1}\|_{B_0}\to 0$, $\sum_{-\infty}^{\infty}u_n=b$ (in B_0+B_1),

$$\begin{aligned} M^r(n,u_n) &= \max\{\|u_n\|_{B_0}^r, e^{nr}\|u_n\|_{B_1}^r\} \\ &\leqslant \max\{\|v_{0n}\|_{B_0}^r+\|v_{0n-1}\|_{B_0}^r;\ e^{nr}(\|v_{1n}\|_{B_1}^r+\|v_{1n-1}\|_{B_1}^r)\} \\ &\leqslant \max\{\|v_{0n}\|_{B_0}^r, e^{nr}\|v_{1n}\|_{B_1}^r\}+e^r\max\{\|v_{0n-1}\|_{B_0}^r, e^{(n-1)r}\|v_{1n-1}\|_{B_1}^r\} \\ &\leqslant c[W^r(n,b)+e^r\,W^r(n-1,b)]\leqslant c(1+e^r)\,W^r(n,b). \end{aligned}$$

Therefore $e^{-\theta n}M(n,u_n)\leqslant ce^{-\theta n}W(n,b)\in(l^{p_0},l^{p_1})_{\lambda,q}$.

(b) $\Rightarrow$ (a): Let $b=\sum_{-\infty}^{\infty}u_n$, $e^{-\theta n}M(n,u_n)\in(l^{p_0},l^{p_1})_{\lambda,q}$.

Therefore we have two sequences $\{\bar v_{nim}\}$ $(i=0,1)$, $\{e^{(i-\lambda)m}\bar v_{nim}\}\in l^q(l^{p_i})$, $\bar v_{n0m}+\bar v_{n1m}=e^{-\theta n}M(n,u_n)$. Take $v_{nim}=e^{\theta n}\bar v_{nim}$. We can assume $v_{nim}\geqslant 0$. Take $b_{im}=\sum_{n=-\infty}^{\infty}u_n\frac{v_{nim}}{M(n,u_n)}$. Then $b_{0m}+b_{1m}=b$,

$$e^{(i-\lambda)m}e^{-\theta n}M\left(n, u_n\frac{v_{nim}}{M(n,u_n)}\right)=e^{(i-\lambda)m}e^{-\theta n}v_{nim}\in l^q(l^{p_i}).$$

Therefore $e^{(i-\lambda)m}\|b_{im}\|_{X_i}\in l^q$, and so $b\in(X_0,X_1)_{\lambda,q}$. The proof is complete.

COROLLARY 9. *Since $(l^{p_0},l^{p_1})_{\lambda,p}=l^p$, where $\frac{1-\lambda}{p_0}+\frac{\lambda}{p_1}=\frac{1}{p}$, from Theorem 8 then:*

$$\big((B_0,B_1)_{\theta,p_0},(B_0,B_1)_{\theta,p_1}\big)_{\lambda,p}=(B_0,B_1)_{\theta,p}.$$

IV. L^{pq} spaces. To make this account self contained, we will present the definitions and statements of theorems on L^{pq} spaces. For the missing proofs we refer the reader to Hunt's paper [2].

DEFINITION 1. Let f a complex measurable function defined on a σ-finite measure space (M,Σ,μ), $\mu\geqslant 0$. We assume that f is finite valued a.e. We define

$$E_y=\{x/|f(x)|>y\},\qquad \lambda_f(y)=\mu(E_y).$$

In the following we assume always that $\lambda_f(y)<\infty$ for some $0<y$.

DEFINITION 2. $f^*(t)=\operatorname{Inf}\{y>0/\lambda_f(y)\leqslant t\}$. f^* is called the *non-decreasing rearrangement of f.*

DEFINITION 3.

$$\|f\|_{pq}^*=\begin{cases}\left(\int_0^\infty t^{q/p}(f^*(t))^q\frac{dt}{t}\right)^{1/q}, & 0<p<\infty,\ 0<q<\infty,\\ \operatorname{Sup}_{0<t}t^{1/p}f^*(t), & 0<p\leqslant\infty,\ q=\infty,\end{cases}$$

$$L^{pq}=\{f/\|f\|_{pq}^*<\infty\}.$$

Note that for $p=q$ we get the usual L^p spaces, while for $q=\infty$ we get the weak-L^p spaces, i.e. the spaces of functions satisfy $\lambda_f(y)\leqslant C/y^p$.

DEFINITION 4. Let $f\in L^{pq}$, $r\leqslant q$, $r<p$, $r\leqslant 1$.

Define:

$$f^{**}(t)=f^{**}(t,r)=\begin{cases}\operatorname{Sup}\left\{\left(\frac{1}{\mu(E)}\int_E|f|^r d\mu\right)^{1/r}/\mu(E)>t\right\}, & t<\mu(M),\\ \left(\frac{1}{t}\int_M|f|^r d\mu\right)^{1/r}, & \mu(M)\leqslant t.\end{cases}$$

THEOREM 5. $(f^*)^{**}(t)=\left(\frac{1}{t}\int_0^t(f^*(u))^r du\right)^{1/r}$.

THEOREM 6. $f^*(t) \leqslant f^{**}(t) \leqslant (f^*)^{**}(t)$.

Proof. Let $\varepsilon > 0$. $E_\varepsilon = \{x / |f(x)| \geqslant f^*(t+\varepsilon)\}$, $\mu(E_\varepsilon) \geqslant t+\varepsilon > t$, and so

$$f^{**}(t) \geqslant \left(\frac{1}{\mu(E_\varepsilon)} \int_{E_\varepsilon} |f|^r d\mu\right)^{1/r} \geqslant f^*(t+\varepsilon)$$

and so $f^*(t+\varepsilon) \leqslant f^{**}(t)$. Since f^* is continuous from the right we get $f^*(t) \leqslant f^{**}(t)$,

$$\int_E |f|^r d\mu \leqslant \int_0^{\mu(E)} (|f|^r)^* dt = \int_0^{\mu(E)} (f^*(t))^r dt$$

and so

$$\operatorname*{Sup}_{\mu(E)>t} \left(\frac{1}{\mu(E)} \int_E |f|^r d\mu\right)^{1/r} \leqslant \operatorname*{Sup}_{\mu(E)>t} \left(\frac{1}{\mu(E)} \int_0^{\mu(E)} (f^*)^r du\right)^{1/r}$$
$$\leqslant \left(\frac{1}{t} \int_0^t (f^*(u))^r du\right)^{1/r} = (f^*)^{**}(t).$$

THEOREM 7. $\|f\|_{pq}^* \leqslant \|f^{**}\|_{pq}^* \leqslant \|(f^*)^{**}\|_{pq}^* \leqslant \left(\dfrac{p}{p-r}\right)^{1/r} \|f\|_{pq}^*$.

The first two inequalities follows from Theorem 6. The last inequality follows from Hardy's inequality. Since this inequality will be needed in the sequel again, we state it explicitly:

THEOREM 8. (Hardy's inequality.) *If* $0 \leqslant f$, $1 \leqslant q$, $0 < r$:

(a) $$\left(\int_0^\infty \left(\int_0^t f(s) \frac{ds}{s}\right)^q t^{-r} \frac{dt}{t}\right)^{1/q} \leqslant \frac{q}{r} \left(\int_0^\infty f^q(t) t^{-r} \frac{dt}{t}\right)^{1/q},$$

(b) $$\left(\int_0^\infty \left(\int_t^\infty f(s) \frac{ds}{s}\right)^q t^r \frac{dt}{t}\right)^{1/q} \leqslant \frac{q}{r} \left(\int_0^\infty f^q(t)\, t^r \frac{dt}{t}\right)^{1/q}.$$

Our aim now is to identify the L^{pq} spaces as intermediate spaces between L^a spaces. For $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ this was done by Calderón and by Peetre.

For $0 < p$, $q \leqslant \infty$ this was done by Krée in [3], for measure spaces which contain no atoms.

We present a proof which covers the general case. Our proof is also shorter than Krée's, since we have the reiteration theorem at our disposal.

DEFINITION 9. Let (B_0, B_1) be an interpolation pair of r_0, r_1 normed spaces. Define for $0 < t$, $b \in B_0 + B_1$

$$K(t, b) = \operatorname{Inf}\{\operatorname{Max}\{\|b_0\|_{B_0}, t\|b_1\|_{B_1}\} / b_0 + b_1 = b\}.$$

DEFINITION 10. Denote for $f(t)$ defined for $0 < t$, measurable,

$$\|f\|_{L_*^p} = \left(\int_0^\infty |f(t)|^p \frac{dt}{t}\right)^{1/p}, \qquad \|f\|_{L_*^\infty} = \operatorname*{Sup}_{0<t} |f(t)|.$$

$$L_*^p = \{f / \|f\|_{L_*^p} < \infty\}.$$

THEOREM 11. *$b \in (B_0, B_1)^{\theta, p}$ iff $t^{-\theta} K(t, b) \in L_*^p$ and the corresponding s-norms are equivalent.*

Proof. If $e^n \leqslant t \leqslant e^{n+1}$, we have $W(n, b) \leqslant K(t, b) \leqslant W(n+1, b)$. Therefore:

$$\frac{1-e^{-\theta p}}{\theta p} \cdot (e^{-\theta n} W(n, b))^p \leqslant \int_{e^n}^{e^{n+1}} (t^{-\theta} K(t, b))^p \frac{dt}{t} \leqslant \frac{e^{\theta p}-1}{\theta p} \cdot (e^{-\theta(n+1)} W(n+1, b))^p$$

summing for $-\infty < n < \infty$, and using Theorem 2, the proof is complete.

The case $p = \infty$ is done similarly.

THEOREM 12. *Let now $0 < p < \infty$, $f \in L^p + L^\infty$. Let $K(t, f) = K_{L^p, L^\infty}(t, f)$, $f^{**}(t) = f^{**}(t, r)$, $0 < r < p$, $r \leqslant 1$.*

Then

$$K(t, f) \sim \left(\int_0^{t^p} (f^{**}(u))^p du\right)^{1/p}.$$

Proof. Let $f = f_0 + f_1$, $f_0 \in L^p$, $f_1 \in L^\infty$. For any set E, $\mu(E) > 0$, we have

$$\frac{1}{\mu(E)} \int_E |f|^r d\mu \leqslant \frac{1}{\mu(E)} \int_E |f_0|^r d\mu + \frac{1}{\mu(E)} \int_E |f_1|^r d\mu$$

and so

$$(f^{**}(u))^r \leqslant (f_0^{**}(u))^r + (f_1^{**}(u))^r \leqslant (f_0^{**}(u))^r + \|f_1\|_{L^\infty}^r.$$

Hence

$$c\left(\int_0^{t^p} (f^{**}(u))^p du\right)^{1/p} \leqslant \left(\int_0^{t^p} (f_0^{**}(u))^p du\right)^{1/p} + t\|f_1\|_{L^\infty} \leqslant \|f_0\|_{L^p} + t\|f_1\|_{L^\infty}$$
$$\leqslant 2\operatorname{Max}\{\|f_0\|_{L^p}, t\|f_1\|_{L^\infty}\}.$$

Taking infimum of last expression over all $f_0 \in L^p$, $f_1 \in L^\infty$ so that $f_0 + f_1 = f$, we get

$$\left(\int_0^{t^p} (f^{**}(u))^p du\right)^{1/p} \leqslant cK(t, f).$$

Conversely, let $0 < t$ be given. $E = \{x / |f(x)| > f^*(t^p)\}$, $f_0 = f \cdot \chi_E$, $f_1 = f - f_0$.

Since $|f_0| \leqslant |f|$, $f_0^{**}(u) \leqslant f^{**}(u)$ for all u.

For every $u < t^p$, $\|f_1\|_{L^\infty} \leqslant f^*(t^p) \leqslant f^*(u) \leqslant f^{**}(u)$. Therefore:

(13) $$\mathrm{Max}\Big\{\int_0^{t^p} (f_0^{**}(u))^p\,du,\ t^p \|f_1\|^p_{L^\infty}\Big\} \leqslant \int_0^{t^p} (f^{**}(u))^p\,du.$$

Since f^* vanishes outside E, $\mu(E) = \lambda_f(f^*(t^p)) \leqslant t^p$ we have:

$$\|f_0\|^p_{L^p} = \int_0^\infty (f_0^*(u))^p\,du = \int_0^{t^p} (f_0^*(u))^p\,du \leqslant \int_0^{t^p}(f_0^{**}(u))^p\,du.$$

From (13) we therefore have

$$\mathrm{Max}\{\|f_0\|^p_{L^p},\ t^p\|f_1\|^p_{L^\infty}\} \leqslant \int_0^{t^p}(f^{**}(u))^p\,du$$

and so $K(t,f) \leqslant \big(\int_0^{t^p}(f^{**}(u))^p\,du\big)^{1/p}$, and the proof is complete.

The idea of the proof goes back to Krée [3]. However, the formula

$$\sup_{\mu(E)\leqslant t} \int_E |f|\,d\mu = \int_0^t f^*(t)$$

which is basic there, does not hold if the measure space has atoms.

THEOREM 14. *Let* $0<p<0$, $0<\theta<1$, $p \leqslant q$. *Then* $(L^p, L^\infty)_{\theta q} = L^{\frac{p}{1-\theta}q}$.

Proof. Since $K(t,f) \sim \big(\int_0^{t^p} |f^{**}(u)|^p\,du\big)^{1/p}$, and $f\in(L^p,L^\infty)_{\theta q}$ iff $\int_0^\infty t^{-\theta q}\big(\int_0^{t^p}[f^{**}(u)]^p\,du\big)^{q/p}\frac{dt}{t} < \infty$.

The last integral is equal to

(15) $$\frac1p\int_0^\infty t^{-\frac{\theta q}{p}}\left(\int_0^t [f^{**}(u)]^p\,du\right)^{q/p}\frac{dt}{t}.$$

Using Hardy's inequality (for which we need $q\geqslant p$) the integral in (15) is majorized by

$$\frac1p\left(\frac{q/p}{\theta q/p}\right)^{q/p}\int_0^\infty t^{-\theta q/p}\, t^{q/p}[f^{**}(t)]^q\frac{dt}{t} = \frac1p\left(\frac1\theta\right)^{q/p}\int_0^\infty t^{q/p(1-\theta)}[f^{**}(t)]^q\frac{dt}{t}.$$

The last integral is finite iff $f\in L^{\frac{p}{1-\theta}q}$. The reverse inclusion is easily proved.

THEOREM 16. *Let* $0<p_i<\infty$, $0<q_i$, $q\leqslant\infty$, $p_0\neq p_1$, $0<\theta<1$. *Then*

$$(L^{p_0q_0}, L^{p_1q_1})_{\theta q} = L^{pq}, \quad \textit{where} \quad \frac1p = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

Proof. Take $p_2 < \mathrm{Min}\{1,p_0,p_1,q_0,q_1,q\}$. Let then λ_0,λ_1 be determined by $\frac{1-\lambda_0}{p_2} = \frac1{p_0}$, $\frac{1-\lambda_1}{p_2}=\frac1{p_1}$. We have by Theorem 14:

$$L^{p_0q_0}\in K_{\lambda_0}(L^{p_2},L^\infty),\qquad L^{p_1q_1}\in K_{\lambda_1}(L^{p_2},L^\infty)$$

and since $p_0\neq p_1$, $\lambda_0\neq\lambda_1$, we can use the reiteration theorem:

$$(L^{p_0q_0},L^{p_1q_1})_{\theta q} = (L^{p_2},L^\infty)_{\lambda_0(1-\theta)+\lambda_1\theta,\, q} = L^{pq},$$

where $\frac1p = \frac{1-\lambda_0(1-\theta)-\lambda_1\theta}{p_2} = \frac{(1-\lambda_0)(1-\theta)+(1-\lambda_1)\theta}{p_2} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}$.

THEOREM 17. *Let* $0<p<\infty$ $0<\theta<1$ $(L^{p,q_0}, L^{p,q})\,\theta, q = L^{p,q}$ *where* $\frac1q = \frac{1-\theta}{q_0}+\frac{\theta}{q_1}$.

Proof. Immediate consequence of previous theorems and III. 9.

We can now prove easily a number of results on L^{pq} spaces.

THEOREM 18. *Let* $0<r<p<\infty$, $r\leqslant\min\{q,1\}$, $|f|_{pq} = (\|f^{**}\|^*_{pq})^r$. *Then* $(L^{pq}, |\ |_{pq})$ *is an r-Banach space.*

Proof. Take r as above. $L^{pq} = (L^r,L^\infty)_{\theta q}$ with $\theta = 1-\frac rp\cdot(L^r,L^\infty)_{\theta q}$ is an r-Banach space, with an r-norm equivalent to $\|t^{-\theta}K(t,f)\|^r_{L^q_*}$, which in view of Theorem 14, is equivalent to $|f|_{pq}$, and the proof is complete.

Note that the completeness of $(L^{pq}, |\ |_{pq})$ is an immediate consequence of the completeness of L^r, L^∞.

THEOREM 19. *If* $q_1<q_2$, $0<p<\infty$, *then* $L^{pq_1}\subset L^{pq_2}$.

Proof. Immediate consequence of Theorem II.13.

THEOREM 20. *If* (M,Σ,μ) *contains infinitely many disjoint sets of positive measure, then if* $q_1\leqslant q_2$ $0<p<\infty$, $L^{pq_1}\neq L^{pq_2}$.

Proof. Can be done directly from Definition 3.

We want to compare the interpolation theorems we have for L^{pq} spaces, with the ones in Hunt [2]. For this purpose we specialize the interpolation theorems:

THEOREM 21. *Let* T *be a quasi-linear operator from* $(L^{p_0p_0}, L^{p_1p_1})$ *to* $(L^{\bar p_0\bar p_0}, L^{\bar p_1\bar p_1})$, *then if* $p_0\neq p_1$, $\bar p_0\neq\bar p_1$, $0<\theta<1$, $0<q\leqslant\infty$, *we have*

$$\|Tf\|_{\bar p q}\leqslant B_\theta\|f\|_{pq},\quad \textit{where}\quad \frac1p=\frac{1-\theta}{p_0}+\frac\theta{p_1}$$

$\big(\frac1{\bar p} = \frac{1-\theta}{\bar p_0}+\frac\theta{\bar p_1}$ and $\|g\|$ denotes, for short, $\|g^{**}\|^*\big)$.

This is the weak type theorem of Hunt [2], if quasi-linear operators in the sense used by Hunt ($|T(f+g)|\leqslant K(|Tf|+|Tg|)$), we shall call them *pointwise quasi-linear*) can be connected with quasi-linear operators in the sense of Definition II. 14, when $\|Tf\|_{\bar{p}_j\bar{q}_j} \leqslant M_j\|f\|_{p_jq_j}$.

This follows from the following considerations:

If T is pointwise quasi-linear and $\|Tf\|_{\bar{p}_j\bar{q}_j} \leqslant M_j\|f\|_{p_jq_j}$, we consider $\tilde{T}$ defined by $\tilde{T}f = |Tf|$.

$$\tilde{T}(f_0+f_1) = b_0+b_1, \text{ where } b_j = \frac{\tilde{T}(f_0+f_1)}{\tilde{T}f_0+\tilde{T}f_1}\tilde{T}f_j.$$

Since $\dfrac{\tilde{T}(f_0+f_1)}{\tilde{T}f_0+\tilde{T}f_1} \leqslant K$, $\|b_j\|_{L^{\bar{p}_j\bar{q}_j}} \leqslant K\cdot M_j\|f_j\|_{L^{p_jq_j}}$ and so $\tilde{T}$ is quasi-linear in the sense of Definition II.14.

Hence $\|\tilde{T}f\|_{\bar{p}\bar{q}} \leqslant B_\theta\|f\|_{pq}$, but $\|\tilde{T}f\|_{\bar{p}\bar{q}} = \|Tf\|_{\bar{p}\bar{q}}$ and so Hunt's theorem follows. We also have:

THEOREM 22. *Let T be a quasi-linear operator from $(L^{p_0q_0}, L^{p_1q_1})$ to $(L^{\bar{p}_0\bar{q}_0}, L^{\bar{p}_1\bar{q}_1})$. Then if $0<\theta<1$, $q_i \leqslant \bar{q}_i$, and*

$$\frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \quad \frac{1}{\bar{p}} = \frac{1-\theta}{\bar{p}_0}+\frac{\theta}{\bar{p}_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0}+\frac{\theta}{q_1},$$
$$\frac{1}{\bar{q}} = \frac{1-\theta}{\bar{q}_0}+\frac{\theta}{\bar{q}_1}.$$

Then $\|Tf\|_{\bar{p}\bar{q}} \leqslant B_\theta\|f\|_{pq}$.

Proof. From the interpolation theorem, T is continuous from $(L^{p_0q_0}, L^{p_1q_1})_{\theta q}$ to $(L^{\bar{p}_0\bar{q}_0}, L^{\bar{p}_1\bar{q}_1})_{\theta q}$. By Theorem II.13, $(L^{\bar{p}_0\bar{q}_0}, L^{\bar{p}_0\bar{q}_0})_{\theta q} \subset$ $\subset (L^{\bar{p}_0\bar{q}_0}, L^{\bar{p}_1\bar{q}_1})_{\theta\bar{q}}$, while from Theorems 16, 17

$$(L^{p_0q_0}, L^{p_1q_1})_{\theta q} = L^{pq}, \qquad (L^{\bar{p}_0\bar{q}_0}, L^{\bar{p}_1\bar{q}_1})_{\theta\bar{q}} = L^{\bar{p}\bar{q}}.$$

Without the restriction $q_i \leqslant \bar{q}_i$, and with a B_θ which is bounded when $\theta\to 0, 1$, but for a more restricted class of operators (sublinear), this is proved in [2] (strong type theorem) using complex methods.

Of course Theorem II.16 enables one to interpolate in the manner of Theorems 21, 22 between general interpolation pairs and $(L^{p_0q_0}, L^{p_1q_1})$.

V. Theorems on Fourier coefficients. In this section we make use of the results of previous sections to present some theorems on Fourier coefficients. The exposition is intended as a demonstration of the applicability of the results. The idea is not novel, and is implicit in [2]. Let $\{\varphi_n(x)\}_{1\leqslant n}$ be a bounded orthonormal system on (a, b). We define $\hat{f}(n)$ $= \int_a^b \overline{\varphi_n(x)}f(x)\,dx$, and consider the operator $Tf = \{\hat{f}(n)\}$. Taking the integers as the underlying measure space with the measure 1 carried by each integer, we have $\|Tf\|^*_{\infty\infty} \leqslant M\|f\|^*_{11}$ (boundedness of the system) and $\|Tf\|^*_{22} \leqslant \|f\|^*_{22}$ (Bessel's inequality).

Interpolating, we get $\|Tf\|^*_{p'q} \leqslant c\cdot\|f\|^*_{pq}$ for $\dfrac{1}{p}+\dfrac{1}{p'} = 1$, $1<p<2$. Since we know that $L^{pq} \subset L^{pq_1}$ if $q \leqslant q_1$, we can write

$$(1) \qquad \|Tf\|^*_{p'q_1} \leqslant c\|f\|^*_{pq} \quad \text{for} \quad 0<q\leqslant q_1\leqslant\infty.$$

If now $\{C_n\}_{1\leqslant n}$ is a sequence of complex numbers, we consider it as a function on the integers.

Assume $C_n\to 0$. For such sequences it is easily seen that if $\{C_n^*\}$ is the sequence $\{|C_n|\}$ rearranged in non-increasing order of magnitude,

$$\|\{C_n\}\|^*_{pq} = \Big(\sum_1^\infty C_n^{*q}\int_{n-1}^{n} t^{q/p}\frac{dt}{t}\Big) \quad \text{for} \quad q<\infty,$$

and $\operatorname*{Sup}_{0<t} t^{1/p}C^*_{[t]+1}$ for $q=\infty$.

It can be seen that

$$\|\{C_n\}\|^*_{pq} \sim \Big(\sum_1^\infty C_n^{*q}n^{q/p-1}\Big)^{1/q}, \quad q<\infty,$$
$$\|\{C_n\}\|^*_{p\infty} \sim \operatorname*{Sup}_{1\leqslant n} C_n^* n^{1/p}.$$

Taking now $q = q_1 = p$ in (1) we get

$$\Big(\sum_1^\infty (\hat{f}(n))^{*p}n^{p/p'-1}\Big)^{1/p} \leqslant c\|f\|^*_{pp} = c\|f\|_p,$$

$p/p' = p-1$ and so we got for $1<p\leqslant 2$

$$(2) \qquad \Big(\sum_1^\infty (\hat{f}(n))^{*p}n^{p-2}\Big)^{1/p} \leqslant c\|f\|_p.$$

Taking in (1) $q=p$, $q_1=p'$ (possible, since $p<2<p'$), we get

$$(3) \qquad \Big(\sum_1^\infty |\hat{f}(n)|^{p'}\Big)^{1/p'} \leqslant c\|f\|_p.$$

We know, however, that (2) implies (3), since $L^{p'p} \subset L^{p'p'}$.

We also consider the operators $\tilde{T}_N$: $\{C_n\}\to\sum_{n=1}^N C_n\varphi_n(x)$. We have: $\|\tilde{T}_N\{C_n\}\|^*_{22} \leqslant \|\{C_n\}\|^*_{22}$; $\|\tilde{T}_N\{C_n\}\|^*_{\infty\infty} \leqslant M\|\{C_n\}\|^*_{11}$. Therefore

$$(4) \qquad \|\tilde{T}_N\{C_n\}\|^*_{p'q_1} \leqslant c\|\{C_n\}\|^*_{pq}, \quad 1<p<2,\ 0<q\leqslant q_1\leqslant\infty,$$

and C does not depend on N. From the last remarks, it follows that if $\{C_n\}\in L^{pq}$, $\{\tilde{T}_N(\{C_n\})\}$ is a Cauchy sequence in $L^{p'q_1}$, which we know is complete. Therefore we can define $\tilde{T}\{C_n\} = \lim_{N\to\infty}\tilde{T}_N\{C_n\}$ (in $L^{p'q_1}$), and of course

$$(5) \qquad \|\tilde{T}\{C_n\}\|^*_{p'q_1} \leqslant c\|\{C_n\}\|^*_{pq}, \quad 1<p<2,\ 0<q\leqslant q_1\leqslant\infty.$$

Since $\{C_n\}\in L^{\infty\infty}\cap L^{pq}$ with $p<2$, $\{C_n\}\in L^{22}$ and so $\sum_{n=1}^{N} C_n\varphi_n(x)\to f$ (in L^2) with $\hat{f}(n)=C_n$. We therefore have $\tilde{T}\{C_n\}$ has C_m as its m'th Fourier coefficient.

Summerizing, we have:

Given $\{C_n\}\in L^{pq}$, $1<p<2$, $0<q\leqslant\infty$ (or $p=q=2$) there exists $f\in L^{p'q_1}$, $q\leqslant q_1\leqslant\infty$, so that $\hat{f}(n)=C_n$ and

$$\|f\|^*_{p'q_1}\leqslant c\|\{C_n\}\|^*_{pq}. \tag{6}$$

Taking $q_1=q=p'$ we get:

$$\|f\|_{p'}\leqslant c\Big(\sum_{n=1}^{\infty} C_n^{*p'}n^{p'-2}\Big)^{1/p'}. \tag{7}$$

Taking $q_1=p'$, $q=p$ (possible, for $p\leqslant p'$):

$$\|f\|_{p'}\leqslant c\Big(\sum_{n=1}^{\infty}|C_n|^p\Big)^{1/p}, \tag{8}$$

and we know that (8) is weaker then (7). The results of (2) and (7) are Paley's theorem for Fourier coefficients. Note that we get it immediately in the stronger, rearranged form.

The results of (3) and (8) are Riesz–Hausdorff–Young's theorem for Fourier coefficients. Note that we also have the implication relation between this result and Paley's theorem.

Different choices of parameters in (1) and (6) will yield the "dual" theorems ([10], XII, 5.15). Explicitly:

Taking in (1) $q=q_1=p'$, we get for $1<p\leqslant 2$

$$\Big(\sum_{n=1}^{\infty}|\hat{f}(n)|^{p'}\Big)^{1/p'}\leqslant c\|f\|^*_{pp'}=c\Big(\int_0^{\infty}|f^*(t)|^{p'}t^{p'-2}\,dt\Big)^{1/p'}. \tag{9}$$

(If (a,b) is finite the last integral is from 0 to $b-a$ only, since $f^*(t)$ vanishes for $t\geqslant b-a$.)

Taking in (6) $q=q_1=p$ we get: for $1<p\leqslant 2$

$$\Big(\int_0^{\infty}[f^*(t)]^p t^{p-2}\,dt\Big)^{1/p}\leqslant c\Big(\sum_{1}^{\infty}|C_n|^p\Big)^{1/p}. \tag{10}$$

Again, the theorem presents itself in the stronger, rearranged, form.

E. Stein in [8] has proved essentially (1) and (5), but with some unnecessary restrictions on the parameters, using interpolation with change of measure, and Paley's theorem.

Together with the operator $\tilde{T}$, one can consider also a maximal operator T^*, defined by:

$$T^*\{C_k\}(X)=\sup_{1\leqslant n}\Big|\sum_{k=1}^{n}C_k\varphi_k(x)\Big|. \tag{11}$$

(See [9], Section 5). The importance of this operator is for the proof of a.e. convergence ot the series defined by $\tilde{T}$.

The operator T^* is no longer linear. It is, however, pointwise sublinear. I.e. $|T^*(\{C_k^1\}+\{C_k^2\})|(x)\leqslant|(T^*\{C_k^1\})(x)|+|(T^*\{C_k^2\})(x)|$ a.e. x, and $|(T^*\{\lambda C_k\})(x)|=|\lambda|\,|(T^*\{C_k\})(x)|$ a.e. x. Since there operators are pointwise quasi-linear, we can apply the interpolation theorem.

It is clear that

$$\|T^*\{C_k\}\|^*_{\infty\infty}\leqslant M\,\|\{C_n\}\|^*_{11}. \tag{12}$$

Stein and Weiss in [9, Lemma 6] prove the following: If $\{\varepsilon_v\}$ is a sequence of numbers, N of which are equal to 1, and the rest 0, then if $1\leqslant p<2$, $\|T^*\{\varepsilon_v\}\|^*_{p'p'}\leqslant A_pN^{1/p}$. Sequences as above are of course characteristic functions of sets of finite measure of integers, and $N^{1/p}=\|\{\varepsilon_v\}\|^*_{p1}$. Calderón in [1, Theorem 7] proved that if T is a pointwise sublinear operator defined for simple functions, with values in a Banach lattice B of functions (i.e. a Banach space so that if $f\in B$, $|g|\leqslant|f|$ a.e., then $g\in B$ and $\|g\|_B\leqslant\|f\|_B$), and satisfying $\|T\chi_E\|_B\leqslant C\|\chi_E\|^*_{p1}$, then T can be extended, in a unique way to all of L^{p1} and $\|Tf\|_B\leqslant 2C\|f\|^*_{p1}$. We take for B, $L^{p'p'}$ and get

$$\|T^*\{C_k\}\|_{p'p'}\leqslant A_p\|\{C_k\}\|^*_{p1}. \tag{13}$$

(To be precise, one has to consider as for $\tilde{T}$, operators T_N^*, extend each one of then to L^{p1}, and then consider the limit operator T^*.)

Using the interpolation theorem we get for $1\leqslant p<2$, $0<q\leqslant q_1\leqslant\infty$

$$\|T^*\{C_k\}\|^*_{p'q_1}\leqslant A_p\|\{C_k\}\|^*_{pq}. \tag{14}$$

Taking in particular $q_1=q=p'$ we get

$$\|T^*\{C_k\}\|^*_{p'p'}\leqslant A_p\Big(\sum_{k=1}^{\infty}(C_k^*)^{p'}k^{p'-2}\Big)^{1/p'}. \tag{15}$$

If we take $q_1=q=p$ we get

$$\|T^*\{C_k\}\|^*_{p'p}\leqslant A_p\Big(\sum_{k=1}^{\infty}|C_k|^p\Big)^{1/p}. \tag{16}$$

E. Stein and G. Weiss in [9] have proved 15, using "restricted type" interpolation theorem.

Added in Proof. T. Holmstedt's paper: "*Interpolation of quasi-normed spaces*", Math. Scand. 26 (1970), pp. 177–190, includes better proofs of the reiteration Theorem (Theorem II. 28), and of Theorem III. 5. The range of parameters is the same, but the spaces need not be complete.

References

[1] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), pp. 273–299.

[2] R. A. Hunt, *On $L(p, q)$ spaces*, L'Enseignement Math. 12 (1966), pp. 249–276.

[3] P. Kree, *Interpolation d'espaces vectoriels qui ne sont ni normes, ni complets. Applications*, Ann. Inst. Fourier, Grenoble, 17 (1967), pp. 137–174.

[4] J. L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes Sci. Publ. Math. 19 (1964), pp. 5–68.

[5] J. Peetre, *Sur le nombre de parametres dans la definition de certains espaces d'interpolation*, Ricerche Mat. (1963), pp. 248–261.

[6] N. M. Rivière, *Interpolation theory in s-Banach spaces*, Thesis, University of Chicago, 1965, pp. 1–35.

[7] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci. Cl. III 5 (1957), pp. 471–473.

[8] E. M. Stein, *Interpolation of linear operators*, Trans. A. M. S. 83 (1956), pp. 482–492.

[9] — and G. Weiss, *An extension of a theorem of Marcinkiewicz and some of its applications*, J. Math. Mech. 8 (1959), pp. 263–284.

[10] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge 1968.

UNIVERSITY OF ILLINOIS CHICAGO CIRCLE
UNIVERSIDAD DE BUENOS AIRES

Received October 27, 1970 (261)

STUDIA MATHEMATICA T. XLI. (1972)

The distribution of the values of a random function in the unit disk

by
A. C. OFFORD (London)

Abstract. Let $f(z)$ be defined in the unit disk by a power series whose coefficients are independent random variables and let $n(t, b)$ denote the number of zeros of $f(z)-b$ in $|z| \leq t$. It is shown that, for almost all functions of the family considered, $\inf_{|b|\leq K} \int_{1/2}^{r} \frac{n(t,b)}{t}\,dt$ has a well defined asymptotic behaviour. Furthermore $f(z)$ almost surely takes every finite value in every open sector of the unit disk. The paper contains some inequalities for $\int_E \log|X|\,d\mu$, where X is a random variable defined on a measure space $(\Omega, \mathscr{M}, \mu)$ and E belongs to $\mathscr{M}$ but is otherwise arbitrary.

§ 1. Introduction and principal results. This paper is concerned with the behaviour of functions

$$f(z) = \sum_{0}^{\infty} a_n z^n \tag{1.1}$$

defined in the unit disk for which the coefficients a_n are independent random variables. Our object is to show that the family (1.1) has certain properties almost surely. This implies some statistical basis and it becomes necessary to define this statistical basis precisely. Many years ago Littlewood and Offord [3] studied a similar problem for the family of entire functions

$$\sum_{0}^{\infty} \varepsilon_n a_n z^n \tag{1.2}$$

in which the coefficients a_n were given and the ε_n took the values ± 1 with equal probability. In 1964 one of the authors [4] returned to this problem and established the basic results of Littlewood and Offord under very general conditions on the distribution functions of the coefficients a_n. From this it followed that the behaviour of the family of entire functions was largely independent of the particular distribution functions chosen for the coefficients a_n. For this reason in the present investigation we