Proof. Now we have

$$(53)\qquad A\log[1-\varrho\varphi(s)] = \log\Phi^{+}(s,\varrho)+\log\Phi^{-}(0,\varrho)$$

for $\mathrm{Re}(s)\geqslant 0$ and $|\varrho|<1$. Thus (51) and (52) follow from (43) and (44) respectively.

We can prove (53) for $\mathrm{Re}(s)>0$ if we use the following formula: If $E\{|\zeta|\}<\infty$, then for $\mathrm{Re}(s)>0$ we have

$$(54)\qquad E\{\zeta e^{-s\eta^{+}}\} = \frac{1}{2}E\{\zeta\}+\frac{s}{2\pi i}\lim_{\varepsilon\to 0}\int_{L_\varepsilon}\frac{E\{\zeta e^{-z\eta}\}}{z(s-z)}\,dz,$$

where L_ε, the path of integration, consists of the imaginary axis from $z=-i\infty$ to $z=-i\varepsilon$ and again from $z=i\varepsilon$ to $z=i\infty$. By (54) we can obtain (53) for $\mathrm{Re}(s)>0$. Since (53) is continuous for $\mathrm{Re}(s)\geqslant 0$, we can obtain (53) for $\mathrm{Re}(s)=0$ by continuity.

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Approximation of translation invariant operators

by

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Abstract. The purpose of this paper is to construct approximations to translation invariant operators from $L^p(R^n)$ to $L^q(R^n)$. We give error estimates in the form of rates of convergence on subspaces of L^p.

1. Introduction. The purpose of this paper is to construct a family of approximations A_h, $0<h<\infty$, to a translation invariant operator A from $L^p(R^n)$ to $L^q(R^n)$. We obtain error estimates

$$\|Au-A_h u\|_q \leqslant Ch^s\|\Lambda^s u\|_p$$

for u in the Bessel potential space $L^{p,s}$, $s>0$, where C is independent of h. For the definition of Λ^s see Section 4 below.

First we consider $1<p=q<\infty$. A_h is given by $A_h u(x) = \sum_{\beta\in Z^n} c_{\beta,h}u(x+h\beta)$. An interesting feature is that the coefficients $c_{\beta,h}$ are independent of h if and only if the multiplier $\hat{T}$ corresponding to A is homogeneous of degree zero, that is, $\hat{T}(\lambda\xi)=\hat{T}(\xi)$ for $\lambda>0$ and $0\neq\xi\in R^n$. We also give approximations to singular integral operators with variable kernels.

In Section 7 we construct approximations A_h, where A maps L^p to L^q, $p\leqslant q$. If $p<q$, then A_h cannot be a difference operator as above. However, $A_h u$ is given by convolving a function with u. Certain approximation results for translation invariant operators on locally compact abelian groups are given by Figà-Talamanca and Gaudry [6].

Part of the results presented here appeared in the author's Ph. D. dissertation at Rice University directed by Professor Jim Douglas, Jr.

2. Preliminaries. R^n denotes n-dimensional Euclidean space, Z^n the points in R^n with integer coordinates, and T^n the dual group of Z^n. For $r>0$ we set $Q_r=\{\xi\in R^n\colon -r<\xi_j\leqslant r,\ j=1,\ldots,n\}$ and we identify T^n with Q_π. L^p, l^p, and $L^p(Q_\pi)$ denote the usual L^p spaces of functions on R^n, Z^n, and Q_π respectively. If E is a subset of R^n, CE is the complement of E and χ_E is the characteristic function of E.

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$S(R^n)$ denotes the space of C^∞ functions f on R^n such that $\sup_{R^n} |x^\beta D^\alpha f(x)| < \infty$. $S(Z^n)$ is the space of functions f on Z^n such that $\sup_{\beta \in Z^n} |\beta^\alpha f(\beta)| < \infty$. The Fourier transform $\hat{u}$ of a function $u \in S(R^n)$ is defined by $\hat{u}(\xi) = (2\pi)^{-n/2} \int u(x) e^{-i\langle x, \xi\rangle} dx$, $\langle x, \xi\rangle = \sum_{j=1}^{n} x_j \xi_j$. The Fourier transform $\hat{u}$ of $u \in S(Z^n)$ is defined by $\hat{u}(\xi) = (2\pi)^{-n/2} \sum_{\beta \in Z^n} u(\beta) e^{-i\langle \beta, \xi\rangle}$, $\xi \in Q_\pi$. $\check{u}$ denotes the inverse Fourier transform of u.

A bounded linear operator is said to be translation invariant if it commutes with translations. We refer to [2], [3], and [7] for the fundamental properties of translation invariant operators and multipliers. L_p^q denotes the space of distributions T in $S'(R^n)$ such that

$$\|T * u\|_q \leqslant C\|u\|_p, \quad u \in S(R^n). \tag{2.1}$$

The smallest constant C for which (2.1) holds is $\|T\|_{L_p^q}$. The space of distributions T in $S'(Z^n)$ such that

$$\|T * u\|_{l^q} \leqslant C\|u\|_{l^p}, \quad u \in S(Z^n), \tag{2.2}$$

is l_p^q and $\|T\|_{l_p^q}$ is the smallest constant C for which (2.2) holds.

The space of Fourier transforms $\hat{T}$ of distributions T in L_p^q or l_p^q is denoted by M_p^q or m_p^q respectively. Elements of M_p^q are called multipliers of type (p, q). We write $\|\hat{T}\|_{M_q^p} = \|T\|_{L_p^q}$ and $\|\hat{T}\|_{m_q^p} = \|T\|_{l_p^q}$. Elements of M_p^p or m_p^p are bounded functions on R^n or Q_π.

We refer to [7] for the following facts:

$$L_p^q = L_{q'}^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1; \tag{2.3}$$

$$L_p^\infty = L_1^{p'} = L^{p'} \quad \text{if} \quad p < \infty. \tag{2.4}$$

If $f \in M_p^q$ and $g \in S(R^n)$, then

$$gf \in M_r^q \cap M_p^s \quad \text{for} \quad r \leqslant p \quad \text{and} \quad q \leqslant s. \tag{2.5}$$

3. Periodic multipliers. We shall be concerned with periodic functions which are multipliers. Thus it is of interest to consider the distribution whose Fourier transform is a periodic function. The first lemma is an immediate consequence of the Poisson summation formula.

LEMMA 1. *Suppose $0 < h < \infty$ and $f \in L^\infty$ is periodic with period $2\pi/h$, that is, $f(\xi + 2\pi\beta/h) = f(\xi)$. Then $f = \hat{T}$, where $T \in S'(R^n)$ is given by*

$$T = h^{n/2} \sum_{\beta \in Z^n} a_\beta(f) \delta_{-h\beta} \tag{3.1}$$

with δ_x the Dirac measure supported at x and

$$a_\beta(f) = (h/2\pi)^{n/2} \int_{Q_{\pi/h}} f(\xi) e^{-ih\langle \beta, \xi\rangle} d\xi. \tag{3.2}$$

In certain cases we shall obtain a family of periodic functions from a single periodic function by dilation of R^n. The next lemma characterizes the transforms of such functions.

LEMMA 2. *Suppose $\hat{T}_h$, $0 < h < \infty$, is a family of L^∞ periodic functions and $\hat{T}_h$ has period $2\pi/h$. Then*

$$T_h = \sum_{\beta \in Z^n} a_\beta(\hat{T}_1) \delta_{-h\beta}, \tag{3.3}$$

where

$$a_\beta(\hat{T}_1) = (2\pi)^{-n/2} \int_{Q_\pi} \hat{T}_1(\xi) e^{-i\langle \beta, \xi\rangle} d\xi$$

if and only if

$$\hat{T}_h(\xi) = \hat{T}_1(h\xi) \text{ almost everywhere.} \tag{3.4}$$

Proof. If (3.4) holds, then (3.3) follows from (3.1) and (3.2). For $\varphi \in S(R^n)$ define $\varphi_h(x) = \varphi(hx)$. From (3.3) it follows that $T_h(\hat{\varphi}) = T_1((\hat{\varphi})_h)$. Since $\hat{T}_h$ and $\hat{T}_1$ are L^∞ functions, $\hat{T}_h(\xi) = \hat{T}_1(h\xi)$ almost everywhere.

The next result was proved by Jodeit [8]. We shall use only the case stated here.

THEOREM 3. *Let $1 < p < \infty$ and suppose $f \in M_p^p$ vanishes outside Q_π. Define $f_0 \in L^\infty(Q_\pi)$ by $f_0(\xi) = f(\xi)$, $\xi \in Q_\pi$. Then $f_0 \in m_p^p$ and there is a constant C depending only on p and n such that $\|f_0\|_{m_p^p} \leqslant C\|f\|_{M_p^p}$.*

The following theorem was proved by de Leeuw [2] for $n = 1$. The proof given there is also valid for $n > 1$ if the n-dimensional Fejér kernel (the product of the one-dimensional kernels) is employed. A shorter proof is given in [8].

THEOREM 4. *Let $g \in L^\infty$ be periodic with period 2π. Suppose $f \in L^\infty(Q_\pi)$ is given by $f(\xi) = g(\xi)$, $\xi \in Q_\pi$. Then $g \in M_p^p$ if and only if $f \in m_p^p$, $1 < p < \infty$. If $g \in M_p^p$, then $\|g\|_{M_p^p} = \|f\|_{m_p^p}$.*

4. Continuity and approximation in L^p. Let p be fixed, $1 < p < \infty$. Suppose A is a translation invariant operator from L^p to L^p. By a theorem of [7] there is a unique $T \in S'(R^n)$ such that $Au = T * u$ for all $u \in S(R^n)$. Thus $\hat{T} \in M_p^p$. For $0 < h < \infty$ define $\hat{T}_h$ to be the periodic function with period $2\pi/h$ such that

$$\hat{T}_h(\xi) = \hat{T}(\xi), \quad \xi \in Q_{\pi/h}.$$

THEOREM 5. *$\hat{T}_h \in M_p^p$ and there is a constant C depending only on p and n such that*

$$\|\hat{T}_h\|_{M_p^p} \leqslant C\|\hat{T}\|_{M_p^p}. \tag{4.1}$$

Proof. Define f by $f(\xi) = \hat{T}(\xi/h)$. Then $f \in M_p^p$ and $\|f\|_{M_p^p} = \|\hat{T}\|_{M_p^p}$. Since $\chi_{Q_\pi} \in M_p^p$ and the product of two multipliers in M_p^p is in M_p^p, we see that $f\chi_{Q_\pi} \in M_p^p$ and $\|f\chi_{Q_\pi}\|_{M_p^p} \leqslant C\|\hat{T}\|_{M_p^p}$. By Theorems 3 and 4 the periodic function g with period 2π which agrees with f on Q_π is in M_p^p and $\|g\|_{M_p^p} \leqslant C\|\hat{T}\|_{M_p^p}$. Since $T_h(\xi) = g(h\xi)$ it follows that $\hat{T}_h \in M_p^p$ and (4.1) holds.

Remark. It is easily seen that if $\hat{T}$ is homogeneous of degree zero, then $\hat{T}_h(\xi) = \hat{T}_1(h\xi)$ almost everywhere. Thus $\|\hat{T}_h\|_{M_p^p} = \|\hat{T}_1\|_{M_p^p}$ and all the distributions T_h have the same coefficients by Lemma 2. If $\hat{T}_h(\xi) = \hat{T}_1(h\xi)$ for $0 < h < \infty$, then it follows that $\hat{T}$ is homogeneous of degree zero.

We have seen that T_h belongs to L_p^q and has norm bounded independent of h. Let A_h denote the closure of the mapping

$$L^p \supset S(R^n) \ni u \to (2\pi)^{n/2}(\hat{T}_h\hat{u})^{\check{}} \in L^p. \tag{4.2}$$

A_h is a translation invariant operator from L^p to L^p and we shall see that A_h is an approximation to A.

Let $L^{p,s}$ denote the space of Bessel potentials of L^p functions for $s > 0$ (see [1]). The Bessel potential $J_s f$ of $f \in L^p$ is defined by

$$(J_s f)^{\hat{}} = (1+|\xi|^2)^{-s/2}\hat{f}.$$

Define the operator Λ^s by $(\Lambda^s u)^{\hat{}} = |\xi|^s \hat{u}$. If $u \in L^{p,s}$, then $u \in L^p$, and $\Lambda^s u \in L^p$. C_0^∞ is contained in $L^{p,s}$. We prepare for the estimates of $\|Au - A_h u\|_p$ with the next lemma.

LEMMA 6. *Let $s > 0$ and define $g(\xi) = \chi_{CQ_\pi}(\xi)|\xi|^{-s}$. Then $g \in M_p^p$, $1 < p < \infty$.*

Proof. Let $\varphi \in C^\infty$ be one on CQ_π and zero in a neighborhood of the origin. Set $f(\xi) = \varphi(\xi)|\xi|^{-s}$. It follows from Hörmander's version of Mihlin's multiplier theorem (Theorem 2.5 of [7]) that $f \in M_p^p$, $1 < p < \infty$. Since $\chi_{CQ_\pi} \in M_p^p$ we see that $g = \chi_{CQ_\pi} f \in M_p^p$, $1 < p < \infty$.

THEOREM 7. *Let $s > 0$. There is a constant C depending only on p, n, and s such that*

$$\|Au - A_h u\|_p \leqslant C\|T\|_{L_p^p} h^s \|\Lambda^s u\|_p, \qquad u \in L^{p,s}. \tag{4.3}$$

Proof. Using the definitions of $Q_{\pi/h}$ and $\hat{T}_h$ we obtain

$$[\hat{T}(\xi) - \hat{T}_h(\xi)]\hat{u} = [\hat{T}(\xi) - \hat{T}_h(\xi)]\chi_{CQ_\pi}(h\xi)|h\xi|^{-s}h^s(\Lambda^s u)^{\hat{}}.$$

Since a dilation of R^n preserves multipliers and their norms in M_p^p, (4.3) follows from (4.1), Theorem 5, and Lemma 6.

COROLLARY 8. *If $u \in L^p$, then $\|Au - A_h u\|_p \to 0$ as $h \to 0$.*

Proof. Since the operators A_h have uniformly bounded norms, the result follows from Theorem 7 and the fact that C_0^∞ is contained in $L^{p,s}$.

5. Singular integral operators with variable kernels. We consider in this section an operator A defined by

$$Au(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} k(x, x-y)u(y)\,dy,$$

where $u \in S(R^n)$ and k has the following properties. For each $x \in R^n$, $k(x, z)$ is C^∞ in z for $|z| > 0$, $k(x, z)$ is homogeneous of degree $-n$ in z, and $k(x, z)$ has mean value zero on the sphere $\{z\colon |z| = 1\}$. For a discussion of these operators see [1], where the following results are established. Let $\hat{T}(x, \xi)$ denote the Fourier transform of $k(x, z)$ with respect to z. Then for each $x \in R^n$, $\hat{T}(x, \xi)$ is C^∞ in ξ for $|\xi| > 0$, $\hat{T}(x, \xi)$ is homogeneous of degree zero in ξ, and $\hat{T}(x, \xi)$ has mean value zero on $\{\xi\colon |\xi| = 1\}$. We assume that for each ξ with $|\xi| = 1$, the functions $D_\xi^\alpha \hat{T}(x, \xi)$ are L^∞ in x for $0 \leqslant |\alpha| \leqslant 2n$. Write $\|A\| = \sup\{\|D_\xi^\alpha \hat{T}(x, \xi)\|_\infty\colon |\xi| = 1, 0 \leqslant |\alpha| \leqslant 2n\}$. Then A may be extended to a bounded operator from L^p to L^p for $1 < p < \infty$ and there is a constant C depending only on p and n such that

$$\|Au\|_p \leqslant C\|A\|\,\|u\|_p, \qquad u \in L^p. \tag{5.1}$$

Let $\{Y_{lm}\}$ be a complete orthonormal system of spherical harmonics in $L^2(\Sigma)$, where $\Sigma = \{\xi\colon |\xi| = 1\}$. The positive integer m is the degree of Y_{lm} and the number of Y_{lm} is no more than Cm^{n-2} with C independent of m. Also $|Y_{lm}| \leqslant Cm^{(n-2)/2}$. (5.1) is established by expanding the kernel $k(x, z)$ and the symbol $\hat{T}(x, \xi)$ in series of spherical harmonics,

$$k(x, z) = \sum_{l,m} a_{lm}(x) Y_{lm}(z)|z|^{-n}, \tag{5.2}$$

$$\hat{T}(x, \xi) = \sum_{l,m} b_{lm}(x) Y_{lm}(\xi). \tag{5.3}$$

It follows that the series for $\hat{T}(x, \xi)$ converges uniformly, $b_{lm}(x) = \gamma_m a_{lm}(x)$, $|\gamma_m^{-1}| \leqslant Cm^{n/2}$, and $\|b_{lm}\|_{L^\infty} \leqslant Cm^{-2n}\|A\|$. Using (5.2) the operator A is given by

$$Au(x) = \sum_{l,m} a_{lm}(x) R_{lm}u(x),$$

where R_{lm} is a translation invariant operator from L^p to L^p, $1 < p < \infty$, with norm bounded independent of l and m.

For $0 < h < \infty$, define $\hat{T}_h(x, \xi)$ to be the periodic function with period $2\pi/h$ in ξ such that $\hat{T}_h(x, \xi) = \hat{T}(x, \xi)$ in $Q_{\pi/h}$. Then for each x, $T_h(x)$ is a distribution with support on hZ^n,

$$T_h(x) = h^{n/2}(h/2\pi)^{n/2} \sum_{\beta \in Z^n} \int_{Q_{\pi/h}} \hat{T}(x, \xi) e^{-ih\langle\beta,\xi\rangle}\,d\xi\,\delta_{-h\beta}.$$

For $u \in S(R^n)$ define

$$A_h u(x) = [T_h(x) * u](x), \tag{5.4}$$

where the convolution is over hZ^n.

THEOREM 9. *A_h may be extended to a bounded operator from L^p to L^p, $1 < p < \infty$, and there is a constant C depending only on p and n such that*

$$\|A_h u\|_p \leqslant C\|A\|\,\|u\|_p, \qquad u \in L^p.$$

Proof. For $u \in S(R^n)$,

$$A_h u(x) = \sum_{l,m} a_{lm}(x) R_{lmh} u(x),$$

where R_{lmh} is the approximation to R_{lm} constructed in Section 4. The result follows from Theorem 5 and the estimate for $\|a_{lm}\|_{L^\infty}$.

THEOREM 10. *Let $s > 0$. There is a constant C depending only on p, n, and s such that*

$$\|Au - A_h u\|_p \leqslant C\|A\| h^s \|\Lambda^s u\|_p, \qquad u \in L^{p,s}.$$

Proof. This estimate is a consequence of Theorems 9 and 7.

6. Multipliers with mixed periods. It was seen in the remark following Theorem 5 that the approximating operators A_h have coefficients independent of h if and only if $\hat{T}$ is homogeneous of degree zero. In this section we shall construct the operators A_h so that they have constant coefficients if and only if $\hat{T}$ is mixed homogeneous of degree zero in the sense of Fabes and Rivière [4], [5].

Let $a_j \geqslant 1, j = 1, \ldots, n$. A function f on R^n is said to be mixed homogeneous of degree k if $f(\lambda^a \xi) = \lambda^k f(\xi)$ for $\lambda > 0$ and $0 \neq \xi \in R^n$, where $\lambda^a \xi = (\lambda^{a_1}\xi_1, \ldots, \lambda^{a_n}\xi_n)$. For $0 < h < \infty$, set

$$Q_{\pi/h} = \{\xi \in R^n:\ -\pi < h^{a_j}\xi_j \leqslant \pi,\ j = 1, \ldots, n\}.$$

We say that a function f on R^n is periodic with mixed periods $2L = (2L_1, \ldots, 2L_n)$, $L_j > 0$, if f has period $2L_j$ in the j-th coordinate, $j = 1, \ldots, n$. For $0 < h < \infty$, define $L = (L_1, \ldots, L_n)$ by $h^{a_j} L_j = \pi$, $j = 1, \ldots, n$. Set $L\beta = (L_1\beta_1, \ldots, L_n\beta_n)$ for $\beta \in Z^n$. Then for $\varphi \in S(R^n)$, $\sum_{\beta \in Z^n} \varphi(x + 2L\beta) = (2\pi)^{-n/2} h^{|a|} \sum_{\beta \in Z^n} \hat{\varphi}(h^a\beta) e^{i\langle h^a\beta, x\rangle}$. This replaces the usual Poisson summation formula.

If $\hat{T}_h \in L^\infty$ has mixed periods $2L$, then

$$T_h = h^{|a|/2} \sum_{\beta \in Z^n} a_\beta(\hat{T}_h)\, \delta_{-h^a\beta},$$

where

$$a_\beta(\hat{T}_h) = (2\pi)^{-n/2} h^{|a|/2} \int_{Q_{\pi/h}} \hat{T}_h(\xi) e^{-i\langle h^a\beta, \xi\rangle}\, d\xi.$$

It is easy to see that

$$T_h = \sum_{\beta \in Z^n} a_\beta(\hat{T}_1)\, \delta_{-h^a\beta}$$

if and only if $\hat{T}_h(\xi) = \hat{T}_1(h^a\xi)$ almost everywhere.

THEOREM 11. *Suppose $1 < p < \infty$ and $\hat{T} \in M_p^p$. For $0 < h < \infty$ define $\hat{T}_h$ as the periodic function with mixed periods $2L$, $h^{a_j} L_j = \pi$, such that $\hat{T}_h(\xi) = \hat{T}(\xi)$, $\xi \in Q_{\pi/h}$. Then $\hat{T}_h \in M_p^p$ and there is a constant C depending only on p and n such that*

$$\|\hat{T}_h\|_{M_p^p} \leqslant C\|\hat{T}\|_{M_p^p}.$$

Proof. The proof is identical to that of Theorem 5 except that the dilations $\xi \to h^{-1}\xi$ and $\xi \to h\xi$ are replaced by the affine transformations $\xi \to h^{-a}\xi$ and $\xi \to h^a\xi$ respectively.

It is easy to check that $\hat{T}(\lambda^a\xi) = \hat{T}(\xi)$, $\lambda > 0$, if and only if $\hat{T}_h(\xi) = \hat{T}_1(h^a\xi)$, $h > 0$. Thus T_h has constant coefficients if and only if $\hat{T}$ is mixed homogeneous of degree zero.

Let ϱ be the metric associated with a defined in [4], that is, $\varrho(x)$ is the unique ϱ such that $|\varrho^{-a}x| = 1$. Then for $\lambda > 0$, $\varrho(\lambda^a\xi) = \lambda\varrho(\xi)$ and ϱ is C^∞ except at the origin. Using the proof of Lemma 6 and the mixed homogeneous form of Hörmander's multiplier theorem (see [4]) we obtain the next lemma.

LEMMA 12. *Let $s > 0$ and define $g(\xi) = \chi_{CQ_\pi}(\xi)[\varrho(\xi)]^{-s}$. Then $g \in M_p^p$, $1 < p < \infty$.*

For the definition of the operator P_a and spaces $L_a^{p,s}$ corresponding to Λ and $L^{p,s}$ see [5]. If $u \in L_a^{p,s}$, then $u \in L^p$ and $P_a^s u \in L^p$, where $(P_a^s u) = [\varrho(\xi)]^s \hat{u}$. C_0^∞ is contained in $L_a^{p,s}$.

Let A_h be the closure of the mapping

$$L^p \supset S(R^n) \ni u \to (2\pi)^{n/2}(\hat{T}_h\hat{u})^{\sim} \in L^p.$$

THEOREM 13. *Let $s > 0$. There is a constant C depending only on p, n, and s such that $\|Au - A_h u\|_p \leqslant C\|T\|_{L_p^p} h^s \|P_a^s u\|_p$, $u \in L_a^{p,s}$.*

Proof. Since ϱ is mixed homogeneous of degree one and affine transformations of R^n preserve multipliers and their norms in M_p^p, the result follows from Lemma 12, Theorem 11, and an equation similar to (4.3).

COROLLARY 14. *If $u \in L^p$, then $\|Au - A_h u\|_p \to 0$ as $h \to 0$.*

7. Operators from L^p to L^q. In this section we consider a translation invariant operator A from L^p to L^q, $p \leqslant q$, and either $1 < p < \infty$ or $1 < q < \infty$. We shall construct an approximating operator A_h and give error estimates similar to those of Section 4. First we show that it is not possible to use periodic multipliers to define A_h if $p < q$.

LEMMA 15. *Suppose $f \in M_p^q$ has period 2π and $p < q$. Then $f = 0$.*

Proof. Since $e^{i\langle\beta,\xi\rangle} \in L^1(Q_\pi)$ it follows that $f * e^{i\langle\beta,\cdot\rangle} \in M_p^q$, where the convolution is over Q_π. But

$$(f * e^{i\langle\beta,\cdot\rangle})(\xi) = (2\pi)^{n/2} e^{i\langle\beta,\xi\rangle} a_\beta(f).$$

Since $e^{i\langle\beta,\xi\rangle}$ is not in M_p^q if $p<q$, all the Fourier coefficients of f must be zero.

The approximating multipliers $\hat{T}_h$ will have compact support and by the next lemma T_h will be a function.

LEMMA 16. *Suppose $p\leqslant q$ and either $1<p<\infty$ or $1<q<\infty$. If $\hat{S}\in M_p^q$ and $\hat{S}$ has compact support, then $S\in L^q\cap L^{p'}$.*

Proof. Let $\varphi\in S(R^n)$ be one on the support of $\hat{S}$. It follows from (2.5) that $\hat{S}=\varphi\hat{S}$ is in M_1^q and M_p^∞. Using (2.3) and (2.4) we see that $S\in L^q\cap L^{p'}$.

For $0<r<\infty$ define $Q_r=\{\xi\in R^n\colon -r<\xi_j\leqslant r, j=1,\ldots,n\}$. For $0<h<\infty$ define $\hat{T}_h=\chi_{Q_{\pi/h}}\hat{T}$, where $Au=T*u$ for all $u\in S(R^n)$. Since $\chi_{Q_r}\in M_t^t$ for $1<t<\infty$ with norm independent of r, it follows that $\hat{T}_h\in M_p^q$ and $\|\hat{T}_h\|_{M_p^q}\leqslant C\|\hat{T}\|_{M_p^q}$ with C independent of h. Let A_h denote the closure of the mapping

$$L^p\supset S(R^n)\ni u\to(2\pi)^{n/2}(\hat{T}_h\hat{u})^{\tilde{}}\in L^q.$$

THEOREM 17. *Let $s>0$. There is a constant C independent of h such that*

$$\|Au-A_hu\|_q\leqslant C\|T\|_{L_p^q}h^s\|\Lambda^s u\|_p,\qquad u\in L^{p,s}.$$

Proof. The estimate follows from the fact that

$$(\hat{T}-\hat{T}_h)\hat{u}=h^s\hat{T}\chi_{CQ_\pi}(h\xi)|h\xi|^{-s}(\Lambda^s u)^{\hat{}}.$$

COROLLARY 18. *If $u\in L^p$, then $\|Au-A_hu\|_q\to 0$ as $h\to 0$.*

Note that $\hat{T}_h$ is the best approximation to $\hat{T}$ on $Q_{\pi/h}$. If $\hat{S}$ is any approximation to $\hat{T}$ with compact support, then the error $(\hat{T}-\hat{S})\hat{u}$ must contain a term similar to that appearing in the proof of Theorem 17. Thus A_h is the best approximation to A among operators defined by multipliers with compact support, in the sense that there is no approximating operator with a higher rate of convergence. Similar considerations lead to the conclusion that the difference operator A_h of Section 4 is the best approximation to A among difference operators, that is, A_h yields the highest possible rate of convergence.

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