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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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A theorem on B-splines

by

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Abstract. In this paper we investigate a special family of partitions of unity on $I = \langle 0, 1\rangle$. Each partition of unity is formed by B-splines of order m, $m \geqslant 0$, corresponding to a given dyadic partition of I. The partitions of unity are linearly independent sets of functions and therefore their Gram matrices are invertible. The aim of this work is to give exponential estimates of elements of the inverse matrices. This result plays the central role in the construction of bases in $C^m(I^d)$ and $W_p^m(I^d)$, [3].

1. Introduction. In this paper we investigate a special sequence of partitions of unity on $I = \langle 0, 1\rangle$. The nth partition of unity is the (linearly independent) set of B-splines of order m corresponding to the nth dyadic partition of I for $n \in \mathcal{N} = \{1, 2, \ldots\}$. Thus the Gram matrix $\boldsymbol{G}_n^{(m)}$ of the nth partition of unity is non-singular and therefore is invertible. Let $\boldsymbol{A}_n^{(m)}$ denote the matrix inverse to $\boldsymbol{G}_n^{(m)}$.

The $(n+m+1)$-dimensional space spanned by elements of the nth spline partition of unity is denoted by $\mathscr{S}_n^m(I)$ for $n \in \mathcal{N}$.

Let us further denote by $\mathscr{S}_n^m(I)$, $n = -m, \ldots, 0$, the subspace spanned by the functions $1, t, \ldots, t^{n+m}$. Applying the Schmidt orthonormalization procedure to the sequence of functions $\{h_n^{(m)}\}$, defined as follows: $h_n^{(m)} \in \mathscr{S}_n^m(I) \setminus \mathscr{S}_{n-1}^m(I)$ for $n \geqslant -m+1$ and $h_{-m}^{(m)} = 1$, we obtain an orthonormal complete in $L_2(I)$ set $\{f_n^{(m)}: n \geqslant -m\}$. The exponential estimate of the elements of $\boldsymbol{A}_n^{(m)}$ established in Theorem 1 (cf. Section 3) can be used to obtain exponential bounds for the Dirichlet kernel of the orthonormal set $\{f_n^{(m)}\}$. By means of this argument it was shown in [3] that the d-fold tensor product of $\{f_n^{(m)}\}$ is an orthogonal basis in $C^m(I^d)$ and $W_p^m(I^d)$ for $d \geqslant 1, m \geqslant 0, 1 \leqslant p < \infty$.

Theorem 1 was conjectured by Ciesielski and the result was announced at the Conference on Constructive Function Theory in Varna, May 1970 [1].

2. The B-splines. Let $S = \{s_i: i \in \mathscr{Z}\}$, where $\mathscr{Z} = \{0, \pm 1, \ldots\}$, be a partition of $(-\infty, \infty)$, i.e. let $s_{i+1} > s_i$ for $i \in \mathscr{Z}$ and $\lim\limits_{i\to\infty} s_i = -\lim\limits_{i\to\infty} s_{-i} = \infty$.

DEFINITION 1. The function $f \in C^m(-\infty, \infty)$ is said to be a *spline (-function)* of order m corresponding to S, whenever all the restrictions

$f|_{\langle s_{i-1}, s_i\rangle}$ are polynomials of degree not exceeding $m+1$. The elements of S are called the *knots* of the spline functions.

The splines were introduced by Schoenberg [7]. Their basic properties are given in [4], [8–9]. The set of all splines of order m corresponding to the partition S is a linear space with usual definitions of addition and multiplication by scalars. Let this space be denoted by $\mathscr{S}^m(S)$. The functions $h_i(t) = (s_i - t)_+^{m+1}$, where $x_+ = \max\{0, x\}$, belong to $\mathscr{S}^m(S)$, whenever $s_i \in S$. Thus also the functions

$$N_i^{(m)}(t) = (s_{i+m+1} - s_i)[s_{i-1}, \ldots, s_{i+m+1}; (s-t)_+^{m+1}], \tag{1}$$

where $[s_1, \ldots, s_r; f(s)]$ is the divided difference of f in points $s_1, \ldots, s_m$, are elements of $\mathscr{S}^m(S)$. The functions $N_i^{(m)}$ have the following properties (cf. [4], [8]):

(P.1) $$N_i^{(m)}(t) \geqslant 0 \quad \textit{for } t \in (-\infty, \infty),\ i \in \mathscr{Z},$$

(P.2) $$\operatorname{supp} N_i^{(m)} = \langle s_{i-1}, s_{i+m+1}\rangle \quad \textit{for} \quad i \in \mathscr{Z},$$

(P.3) $$\sum_{i \in \mathscr{Z}} N_i^{(m)}(t) = 1 \quad \textit{for} \quad t \in (-\infty, +\infty),$$

(P.4) $$\int_{-\infty}^{+\infty} N_i^{(m)}(t)\,dt = \frac{s_{i+m+1} - s_{i-1}}{m+2} \quad \textit{for} \quad i \in \mathscr{Z},$$

(P.5) *The sequence of functions $\{N_i^{(m)} : \operatorname{supp} N_i^{(m)} \cap (a, b) \neq \emptyset\}$ is a basis in the finite-dimensional space*

$$\mathscr{S}^m_{\langle a,b\rangle}(S) = \{f|_{\langle a,b\rangle} : f \in \mathscr{S}^m(S)\}$$

of restrictions of splines of order m to $\langle a, b\rangle \subset (-\infty, +\infty)$. In particular, if $a = s_0$ and $b = s_n$, then the sequence $\{N_i^{(m)} : i = -m, \ldots n\}$ is a basis in the space $\mathscr{S}_n^m(S) = \mathscr{S}^m_{\langle s_0, s_n\rangle}(S)$.

LEMMA 1. *If $S = \{s_i : i \in \mathscr{Z}\}$ and $S' = \{s_i' : i \in \mathscr{Z}\}$ are two partitions of $(-\infty, \infty)$ such that*

$$s_i' = c \cdot s_i + d \quad \textit{for } i \in \mathscr{Z},$$

where c and d are independent of i, $c > 0$, then the equality

$$N_i^{(m)}(t) = N_i'^{(m)}(c \cdot t + d) \tag{2}$$

holds for $t \in (-\infty, \infty)$, $i \in \mathscr{Z}$, $m \geqslant 0$.

Proof. According to (1) the left-hand side of (2) equals (cf. [6], p. 17)

$$L_{(2)} = (s_{i+m+1} - s_{i-1}) \sum_{l=-1}^{m+1} \frac{(s_{i+l} - t)_+^{m+1}}{D\omega_i^{(m)}(s_{i+l})}$$
$$= (s_{i+m+1}' - s_{i-1}') \sum_{l=-1}^{m+1} \frac{(s_{i+l}' - ct - d)_+^{m+1}}{D\omega_i'^{(m)}(s_{i+m}')} = N_i'^{(m)}(ct + d),$$

where $\omega_i^{(m)}(s) = (s - s_{i-1}) \ldots (s - s_{i+m+1})$, $\omega_i'^{(m)}(s) = (s - s_{i-1}') \ldots (s - s_{i+m+1}')$ and $\mathrm{D}f(s) = \dfrac{\mathrm{d}f}{\mathrm{d}s}(s)$.

Properties (P.1)–(P.3) and (P.5) allow us to introduce

DEFINITION 2. The sequence $\{N_i^{(m)} : i \in \mathscr{Z}\}$ of splines of order m corresponding to the partition $S = \{s_i\}$ of $(-\infty, \infty)$ according to formula (1) is called a *spline partition of unity*; the elements $N_i^{(m)}$ are called *B-splines* (*basic*) *corresponding to S*.

The following result proved in [4] will be used later.

LEMMA 2. *The unique spline of order m with a support contained in $\langle s_i, s_{i+m+1}\rangle$ is the null function $f(s) \equiv 0$, for any $i \in \mathscr{Z}$.*

It follows that Properties (P.2) and (P.5) define uniquely the spline partition of unity. In the case of equidistant knots, i.e. if

$$s_i - s_{i-1} = h \quad \text{for } i \in \mathscr{Z}, \tag{3}$$

the *B*-splines have special property important for our considerations. According to Lemma 1 we find that the numbers

$$G_{i,j}^{(m)} = \int_{-\infty}^{+\infty} N_i^{(m)}(t) N_j^{(m)}(t)\,dt \tag{4}$$

corresponding to $S = \{s_i\}$ satisfying (3) can be expressed in terms of

$$G_l^{(m)} = G_{0,l}^{(m)} \quad \text{for } l \in \mathscr{Z} \tag{5}$$

as follows:

$$G_{i,j}^{(m)} = G_{j-i}^{(m)} \quad \text{for } i, j \in \mathscr{Z}. \tag{6}$$

The important property of $N_i^{(m)}$-s may now be formulated like this (cf. [8], p. 182):

LEMMA 3. *The roots $\gamma_j^{(m)}$, $j = \pm 1, \ldots, \pm(m+1)$, of the function*

$$g^{(m)}(z) = \sum_{l \in \mathscr{Z}} G_l^{(m)} z^l \equiv \sum_{l=-m-1}^{m+1} G_l^{(m)} z^l \quad \textit{for } z \in \mathscr{C},\ z \neq 0, \tag{7}$$

are all simple and negative. Moreover, they may be enumerated as follows:

$$\gamma_{m+1}^{(m)} < \ldots < \gamma_1^{(m)} < -1 < \gamma_{-1}^{(m)} < \ldots < \gamma_{-m-1}^{(m)} < 0 \tag{8}$$

and

$$\gamma_i^{(m)} = (\gamma_{-i}^{(m)})^{-1} \quad \textit{for } i = \pm 1, \ldots, \pm(m+1). \tag{9}$$

3. The family of partitions of unity. For each $n \in \mathscr{N}$, $\mathscr{N} = \{1, 2, \ldots\}$, we define the sequence $S_n \equiv S_n^{[0]} = \{s_{n;i}^{[0]} : i = 0, \pm 1, \ldots\}$ by

$$s_{1;i}^{[0]} = i \quad \text{for } i \in \mathscr{Z} \tag{10'}$$

and for $n \geqslant 2$

$$(10'') \qquad s_{n;i}^{[0]} = \begin{cases} \dfrac{i}{2^{\mu+1}} & \text{for } i = \ldots, -1, 0, \ldots, 2\nu, \\[2ex] \dfrac{i-\nu}{2^{\mu}} & \text{for } i = 2\nu+1, \ldots, n, \ldots, \end{cases}$$

where μ and ν are non-negative integers defined by the unique decomposition: $n = 2^{\mu}+\nu$, $\mu \geqslant 0$, $1 \leqslant \nu \leqslant 2^{\mu}$. Notice that for each $n \in \mathcal{N}$ we have

$$(11) \qquad s_{n;0}^{[0]} = 0 \quad \text{and} \quad s_{n;n}^{[0]} = 1.$$

The set $\{s_{n;i}^{[0]}: i = 0, 1, \ldots, n\}$ is called the *n-th dyadic partition of* $I = \langle 0, 1\rangle$. The spaces $\mathcal{S}_n^m(S_n^{[0]})$ of restrictions to the interval $\langle s_{n;0}, s_{n;n}\rangle = I$ of splines corresponding to $S_n^{[0]}$ are defined uniquely by the dyadic partitions. These spaces are denoted shortly as $\mathcal{S}_n^m(I)$. Notice that $\mathcal{S}_n^m(I) \subset \mathcal{S}_{n+1}^m(I)$ and that $\dim \mathcal{S}_n^m(I) = n+m+1$ for $n \in \mathcal{N}$. According to (P.5) the system

$$(12) \qquad \boldsymbol{N}_n^{(m)} = \{N_{n;i}^{(m)[0]}: i = -m, \ldots, n\} \subset \mathcal{S}_n^m(I)$$

is a basis in $S_n^m(I)$. Thus for the Gram matrix $\boldsymbol{G}_n^{(m)} = \{G_{n;i,j}^{(m)}: i, j \in J_n^m\}$ with $J_n^m = \{-m, \ldots, n\}$ and $G_{n;i,j}^{(m)} = \int_I N_{n;i}^{(m)} N_{n;j}^{(m)}$ for $i, j \in J_n^m$, there exists the inverse matrix $\boldsymbol{A}_n^{(m)} = \{A_{n;i,j}^{(m)}: i, j \in J_n^m\} = (\boldsymbol{G}_n^{(m)})^{-1}$, $n \in \mathcal{N}$. The main result now can be formulated as follows:

THEOREM 1. *For each $m \geqslant 0$ there exists constants C_m and q_m, $0 < q_m < 1$, independent of n, such that*

$$(13) \qquad |A_{n;i,j}^{(m)}| \leqslant nC_m q_m^{|i-j|} \quad \textit{for } i, j \in J_n^m,\ n \in \mathcal{N}.$$

In order to prove Theorem 1 we have to develop a technic of estimating the inverse matrices.

4. Some properties of matrices. The letters J, K, L equipped possibly in additional indices denote finite or infinite subsets of the set $\mathcal{Z} = \{0, \pm 1, \ldots\}$. Any function defined on J is called a *J-system*. Any function defined on $J \times K$ is called a *$J \times K$-table*. The systems and tables are denoted in the sequel by capital bold-faced Latin or Greek letters. The domain of a system or a table $\boldsymbol{M}$ is denoted by $\mathcal{D}(\boldsymbol{M})$.

The real- (complex-) valued systems and tables are called *real* (*complex*) *vectors* and *matrices*, respectively. The set of all real (complex) $\boldsymbol{J}$-vectors is denoted by $\mathcal{R}^J$ ($\mathcal{C}^J$). Similarly $\mathcal{R}^{J\times K}$ ($\mathcal{C}^{J\times K}$) denotes the set of all real (complex) $J \times K$-matrices.

The value $\boldsymbol{M}(j)$ ($\boldsymbol{M}(j, k)$) of the J-system ($J \times K$-table) $\boldsymbol{M}$ is called *j-th* (*(j, k)-th*) *element* of $\boldsymbol{M}$ whenever $j \in \mathcal{D}(\boldsymbol{M})$ ($(j, k) \in \mathcal{D}(\boldsymbol{M})$) and is denoted by M_j ($M_{j,k}$). If some subindices occur to denote the system (table), then the subindex j (subindices j, k) is (are) proceded by a semicolon, unless it is (they are) preceded by a parenthesis, e.g. $M_{n;j,k}$ and $M_{(a;b)j,k}^{[l]}$ denote the (j, k)-th element of, say, matrices $\boldsymbol{M}_n$ and $\boldsymbol{M}_{(a;b)}^{[l]}$, respectively.

In the sequel we idenfity $\boldsymbol{J}$-systems with $\boldsymbol{J} \times \{0\}$-tables in the following way: $\boldsymbol{X}$ is indentified with $\boldsymbol{Y}$, $\mathcal{D}(\boldsymbol{X}) = J$, $\mathcal{D}(\boldsymbol{Y}) = J \times \{0\}$, if $X_j = Y_{j,0}$ for $j \in J$.

To complete the list of definitions let $\boldsymbol{M}$ denote a $J \times K$-matrix. The *main diagonal of* $\boldsymbol{M}$ is the set $\{M_{j,k}: j = k, (j, k) \in J \times K\}$. It may be the empty set. $\boldsymbol{M}$ is said to be diagonal if all its elements are equal zero unless they belong to the main diagonal. The $K \times J$-matrix $\boldsymbol{M}^T$ is the *transposition of* $\boldsymbol{M}$ if $M_{k,j}^T = M_{j,k}$ for $(j, k) = J \times K$. $\boldsymbol{M}$ is said to be *symmetric* if $\boldsymbol{M}^T = \boldsymbol{M}$.

The restriction $\boldsymbol{M}_{J'\times K'}$ of $\boldsymbol{M}$ to the set $J' \times K'$, where $J' \subset J$, $K' \subset K$, is called the *$J' \times K'$-submatrix of* $\boldsymbol{M}$. The submatrices $\boldsymbol{M}_{J\times\{k\}}$ and $\boldsymbol{M}_{\{j\}\times K}$ are called *k-th column* and *j-th row of* $\boldsymbol{M}$, respectively. If the sets J and K are decomposed into e.g. three parts J^a, J^b, J^c, and K^a, K^b, K^c respectively, then $\boldsymbol{M}_{J^\alpha\times K^\beta}$ is denoted shortly by $\boldsymbol{M}_{(\alpha;\beta)}$ for $\alpha, \beta = a, b, c$. In more complicated cases of submatrices the following examples explain the rule of the abbreviation of notation. We write e.g. $\boldsymbol{M}_{(;b,c)}$, $\boldsymbol{M}_{(a,c;c)}$, $\boldsymbol{M}_{(;b)}$ and $\boldsymbol{M}_{(a,b;)}$ for $\boldsymbol{M}_{J\times(K^b\cup K^c)}$, $\boldsymbol{M}_{(J^a\cup J^c)\times K^c}$, $\boldsymbol{M}_{J\times K^b}$ and $\boldsymbol{M}_{(J^a\cup J^b)\times K}$, respectively. For $J'' \supset J$ and $K'' \supset K$ the $J'' \times K''$-matrix $\boldsymbol{M}^{J''\times K''}$, where

$$M_{j,k}^{J''\times K''} = \begin{cases} M_{j,k} & \text{for } (j, k) \in J \times K, \\ 0 & \text{for } (j, k) \in (J'' \times K'') \setminus (J \times K), \end{cases}$$

is called the *null $J'' \times K''$-extension of* $\boldsymbol{M}$.

$\boldsymbol{M}$ is said to be the *null $J \times K$-matrix* if $M_{j,k} = 0$ for $(j, k) \in J \times K$. $\boldsymbol{M}$ is said to be the *identity $J \times J$-matrix* if $J = K$ and if $M_{j,k} = \delta_{j,k}$ for $(j, k) \in J \times J$, where

$$(14) \qquad \delta_{j,k} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

for $j, k \in \mathcal{Z}$.

The null (identity) matrices are denoted by $\boldsymbol{O}$ ($\boldsymbol{E}$) with a subindex denoting their domain. Thus, e.g. $\boldsymbol{O}_{J\times K}$, $\boldsymbol{E}_{(b;b)}$ and $\boldsymbol{E}_{J\times J}$ denote the null $J \times K$-matrix, the identity $J^b \times J^b$-matrix and the identity $J \times J$-matrix, respectively.

The *sum* $\boldsymbol{M}+\boldsymbol{N}$ of two $J \times K$-matrices, the *product* $\lambda\boldsymbol{M}$ of $\boldsymbol{M}$ by a scalar λ and the *product* $\boldsymbol{M}\circ\boldsymbol{N}$ of the $J \times K$-matrix $\boldsymbol{M}$ and the $K \times L$-matrix $\boldsymbol{N}$ are defined as usual; namely, $(\boldsymbol{M}+\boldsymbol{N})(j, k) = M_{j,k}+N_{j,k}$, $(\lambda\boldsymbol{M})(j, k) = M_{j,k}$ and $(\boldsymbol{M}\circ\boldsymbol{N})(j, l) = \sum_{k\in K} M_{j,k}N_{k,l}$, whenever all the sums converge. The $K \times J$-matrix $\boldsymbol{N}$ is inverse to $\boldsymbol{M}$ if $\boldsymbol{M}\circ\boldsymbol{N} = \boldsymbol{E}_{J\times J}$ and $\boldsymbol{N}\circ\boldsymbol{M} = \boldsymbol{E}_{K\times K}$. $\boldsymbol{M}$ is said to be *non-singular* if it has (exactly one) inverse matrix.

Let $\overline{\overline{J}} = \overline{\overline{K}} < \infty$, where $\overline{\overline{\mathscr{A}}}$ denotes the cardinality of the set $\mathscr{A}$. $\Pi(J, K)$ denotes then the set of all (one-to-one) mappings of J onto K. Moreover, there exists exactly one $\pi_0 \in \Pi(J, K)$ which is order preserving, i.e. $\pi_0(j_1) \leqslant \pi_0(j_2)$ if and only if $j_1 \leqslant j_2$. This enables us to define the determinant of $\boldsymbol{M}$ according to the formula

$$\text{Det}\,\boldsymbol{M} = \sum_{\pi \in \Pi(J,K)} \operatorname{sgn}\pi \prod_{j \in J} M_{j,\pi(j)}, \tag{15}$$

where $\operatorname{sgn}\pi$ equals 1 whenever the permutation $p \in \Pi(K, K)$, $p = \pi \circ (\pi_0)^{-1}$, is even and -1 for odd p.

Now let $\boldsymbol{M} \in \mathscr{R}^{J \times J}$. $\boldsymbol{M}$ is said to be *non-negative* (*positive*) *definite* if for each $\boldsymbol{X} \in \mathscr{R}^J$, $\boldsymbol{X} \neq \boldsymbol{O}_J$, $\boldsymbol{X} \circ \boldsymbol{M} \circ \boldsymbol{X} \geqslant 0$ (> 0), where one-element matrices are identified with numbers. The complex number λ is said to be an *eigenvalue* of $\boldsymbol{M}$ if there exists $\boldsymbol{X} \in \mathscr{C}^J$, $\boldsymbol{X} \neq \boldsymbol{O}_J$, such that $\boldsymbol{M} \circ \boldsymbol{X} = \lambda \boldsymbol{X}$.

The *Euclidean norm* of a real J-vector $\boldsymbol{X}$, $\overline{\overline{J}} < \infty$, is defined as follows

$$|\boldsymbol{X}| = \Big(\sum_{j \in J} X_j^2\Big)^{1/2} = (\boldsymbol{X}^T \circ \boldsymbol{E}_{J \times J} \circ \boldsymbol{X})^{1/2} \quad \text{for } \boldsymbol{X} \in \mathscr{R}^J. \tag{16}$$

Now we may list the known properties of matrices which are used in the sequel. To do this let $\boldsymbol{M} \in \mathscr{R}^{J \times J}$, $\overline{\overline{J}} < \infty$.

If $\boldsymbol{N} \in \mathscr{R}^{J \times J}$ and if $\boldsymbol{N}$ is non-singular, then

$$\boldsymbol{M} \text{ is p.d.} \Leftrightarrow \boldsymbol{N}^T \circ \boldsymbol{M} \circ \boldsymbol{N} \text{ is p.d.}, \tag{17}$$

where p.d. stands for positive definite. If $\boldsymbol{M}$ is symmetric, then

$$\lambda \quad \text{is an eigenvalue of } \boldsymbol{M} \Rightarrow \lambda \in \mathscr{R} \tag{18}$$

$$\lambda_m(\boldsymbol{M}) = \min\{\boldsymbol{X}^T \circ \boldsymbol{M} \circ \boldsymbol{X}\colon \boldsymbol{X} \in \mathscr{R}^J, |\boldsymbol{X}| = 1\}, \tag{19'}$$

$$\lambda_M(\boldsymbol{M}) = \max\{\boldsymbol{X}^T \circ \boldsymbol{M} \circ \boldsymbol{X}\colon \boldsymbol{X} \in \mathscr{R}^J, |\boldsymbol{X}| = 1\}, \tag{19''}$$

where $\lambda_m(\boldsymbol{M})$ and $\lambda_M(\boldsymbol{M})$ denote the minimal and the maximal eigenvalues of $\boldsymbol{M}$ respectively. According to (19) we have for a $J' \times J'$-submatrix $\boldsymbol{M}'$ of $\boldsymbol{M}$, $J' \subset J$,

$$\lambda_m(\boldsymbol{M}) \leqslant \lambda_m(\boldsymbol{M}') \leqslant \lambda_M(\boldsymbol{M}') \leqslant \lambda_M(\boldsymbol{M}) \tag{20}$$

and if $\boldsymbol{N} \in \mathscr{R}^{J \times J}$ is symmetric, then

$$\lambda_m(\boldsymbol{M}) + \lambda_m(\boldsymbol{N}) \leqslant \lambda_m(\boldsymbol{M}+\boldsymbol{N}) \leqslant \lambda_M(\boldsymbol{M}+\boldsymbol{N}) \leqslant \lambda_M(\boldsymbol{M}) + \lambda_M(\boldsymbol{N}). \tag{21}$$

It also follows from (19) that

$$\boldsymbol{M} \text{ is p.d.} \Leftrightarrow \lambda_m(\boldsymbol{M}) > 0, \tag{22'}$$

$$\boldsymbol{M} \text{ is n-n.d.} \Leftrightarrow \lambda_m(\boldsymbol{M}) \geqslant 0, \tag{22''}$$

where n-n.d. stands for non-negative definite. The following estimate holds for any symetric $J \times J$-matrix $\boldsymbol{M}$ (cf. [5], p. 240):

$$\max\{|\lambda_M(\boldsymbol{M})|, |\lambda_m(\boldsymbol{M})|\} \leqslant \max\Big\{\sum_{k \in J} |M_{j,k}|\colon j \in J\Big\}. \tag{23}$$

Let $\boldsymbol{B}$ denote a finite J-system of elements of a Hilbert space $\langle \mathscr{H}, (\cdot, \cdot)\rangle$, where $(\cdot, \cdot)$ denotes the inner product. The complex $J \times J$-matrix $\boldsymbol{G}$ is called the *Gram matrix* of $\boldsymbol{B}$ if $G_{i,j} = (B_i, B_j)$ for $i, j \in J$. If $\mathscr{H}$ is a real Hilbert space, then the following statements are all equivalent

(24′) $\boldsymbol{B}$ *is linearly independent*;

(24″) $\boldsymbol{G}$ *is non-singular*;

(24‴) $\boldsymbol{G}$ *is p.d.*

To complete the list of properties of matrices we give a proof of the following

Lemma 4. *Let $\boldsymbol{\Gamma}, \boldsymbol{M}, \boldsymbol{N}$ be real $J \times J$-matrices such that $\overline{\overline{J}} < \infty$ and $\boldsymbol{N} = \boldsymbol{M} - \boldsymbol{\Gamma} \circ \boldsymbol{M} \circ \boldsymbol{\Gamma}$. If $\boldsymbol{\Gamma}$ is diagonal, $0 < |\Gamma_{j,j}| < 1$ for $j \in J$, and $\boldsymbol{N}$ is p.d., then also $\boldsymbol{M}$ is p.d.*

Proof. Denoting $\boldsymbol{\Gamma}^n = \underbrace{\boldsymbol{\Gamma} \circ \ldots \circ \boldsymbol{\Gamma}}_{n}$ for $n \in \mathscr{N}$ and $\boldsymbol{\Gamma}^0 = \boldsymbol{E}_{J \times J}$ we may assert: $\boldsymbol{\Gamma}^n$ is symmetric and non-singular for $n \geqslant 0$. Hence $\boldsymbol{\Gamma}^n \circ \boldsymbol{N} \circ \boldsymbol{\Gamma}^n$ is p.d. (cf. (17)), i.e.

$$\boldsymbol{X}^T \circ (\boldsymbol{\Gamma}^n \circ \boldsymbol{M} \circ \boldsymbol{\Gamma}^n - \boldsymbol{\Gamma}^{n+1} \circ \boldsymbol{M} \circ \boldsymbol{\Gamma}^{n+1}) \circ \boldsymbol{X} > 0 \quad \text{for } \boldsymbol{X} \in \mathscr{R}^J, \boldsymbol{X} \neq \boldsymbol{O}_J, n \geqslant 0. \tag{25}$$

It follows that for any $n \in \mathscr{N}$,

$$\boldsymbol{X}^T \circ \boldsymbol{\Gamma} \circ \boldsymbol{M} \circ \boldsymbol{\Gamma} \circ \boldsymbol{X} > \boldsymbol{X}^T \circ \boldsymbol{\Gamma}^{n+1} \circ \boldsymbol{M} \circ \boldsymbol{\Gamma}^{n+1} \circ \boldsymbol{X} \quad \text{for } \boldsymbol{X} \in \mathscr{R}^J, \quad \boldsymbol{X} \neq \boldsymbol{O}_J.$$

But

$$\begin{aligned} \boldsymbol{X}^T \circ \boldsymbol{\Gamma}^{n+1} \circ \boldsymbol{M} \circ \boldsymbol{\Gamma}^{n+1} \circ \boldsymbol{X} &\geqslant -|\boldsymbol{X}^T \circ \boldsymbol{\Gamma}^{n+1} \circ \boldsymbol{M} \circ \boldsymbol{\Gamma}^{n+1} \circ \boldsymbol{X}| \\ &\geqslant -\Big|\sum_{i,j \in J} X_i (\Gamma_{i,i})^{n+1} M_{i,j} (\Gamma_{j,j})^{n+1} X_j\Big| \\ &\geqslant -\Gamma^{2n+2} \sum_{i,j} |X_i M_{i,j} X_j| \quad \text{for } n \in \mathscr{N}, \end{aligned}$$

where $\Gamma \equiv \max\{|\Gamma_{j,j}|\colon j \in J\} < 1$. Hence

$$\boldsymbol{X}^T \circ \boldsymbol{\Gamma} \circ \boldsymbol{M} \circ \boldsymbol{\Gamma} \circ \boldsymbol{X} \geqslant 0 \quad \text{for } \boldsymbol{X} \in \mathscr{R}^J. \tag{26}$$

Inequalities (25) (for $n = 0$) and (26) give

$$\boldsymbol{X}^T \circ \boldsymbol{M} \circ \boldsymbol{X} > 0 \quad \text{for } \boldsymbol{X} \in \mathscr{R}^J, \boldsymbol{X} \neq \boldsymbol{O}_J.$$

DEFINITION 3. If $J = \{j_0, \ldots, j_0+d_1\}$ and $K = \{k_0, \ldots, k_0+d_2\}$, then the $J\times K$-matrix $\boldsymbol{M}$ is said to be *rotatively symmetric* (r.s.) whenever

$$M_{j_0+j,k_0+k} = M_{j_0+d_1-j,k_0+d_2-k} \quad \text{for } 0 \leqslant j \leqslant d_1,\ 0 \leqslant k \leqslant d_2.$$

It is easy to verify

LEMMA 5. *If $\boldsymbol{M}$ and $\boldsymbol{N}$ are r.s., then their sum (product) is r.s. whenever the sum (product) exists.*

5. Diagonally exponential matrices.

DEFINITION 4. The family $\{\boldsymbol{M}_a\colon a \in \mathscr{A}\}$, $\mathscr{A}$ — any set of indices, of finite or infinite matrices $\boldsymbol{M}_a$ is said to be *diagonally exponential* (d.e.) if there exist constants C and q, $0<q<1$, independent of a such that

$$|\boldsymbol{M}_{a;j,k}| \leqslant Cq^{|j-k|} \quad \text{for } (j,k) \in \mathscr{D}(\boldsymbol{M}_a),\ a \in \mathscr{A}.$$

In the sequel we shall make use of the following convention: The constants C and q occurring in the above definition will be endowed with a subindex corresponding to the capital letter denoting the family of matrices, e.g. if $\{\boldsymbol{A}_n\colon n \in \mathscr{N}\}$ is d.e., then the appropriate constants are denoted by C_A and q_A respectively. It is obvious that

LEMMA 6. *Finite sum of d.e. families of matrices is d.e.*

LEMMA 7. *If $\{\boldsymbol{M}_a\colon a \in \mathscr{A}\}$ is d.e., then any family $\{\boldsymbol{N}_\beta\colon \beta \in \mathscr{B}\}$ such that each $\boldsymbol{N}_\beta$ is a null extension of a submatrix of some $\boldsymbol{M}_a$, $\beta \in \mathscr{B}$, is d.e.*

LEMMA 8. *If $\{\boldsymbol{M}_a\colon a \in \mathscr{A}\}$ and $\{\boldsymbol{N}_a\colon a \in \mathscr{A}\}$ are d.e. families of matrices with $\mathscr{D}(\boldsymbol{M}_a) = \mathscr{D}(\boldsymbol{N}_a)$ for $a \in \mathscr{A}$, then $\{\boldsymbol{M}_a+\boldsymbol{N}_a\colon a \in \mathscr{A}\}$ is d.e. too.*

LEMMA 9. *If $\{\boldsymbol{M}_a\colon a \in \mathscr{A}\}$ is a family of matrices such that*

$$N \equiv \sup\{|j-k|\colon (j,k) \in \mathscr{D}(\boldsymbol{M}_a),\ a \in \mathscr{A}\} < \infty, \tag{27}$$

$$C \equiv \sup\{|M_{a;j,k}|\colon (j,k) \in \mathscr{D}(\boldsymbol{M}_a),\ a \in \mathscr{A}\} < \infty, \tag{28}$$

then $\{\boldsymbol{M}_a\}$ is d.e.

Note that (27) cannot be replaced by the following condition: $\sup\{\overline{\overline{\mathscr{D}(\boldsymbol{M}_a)}}\colon a \in \mathscr{A}\} < \infty$.

DEFINITION 5. A family of matrices $\{\boldsymbol{M}_a\colon a \in \mathscr{A}\}$ is said to be *almost diagonal* (a.d.) if

$$\sup\{|j-k|\colon (j,k) \in \mathscr{D}(\boldsymbol{M}_a),\ M_{a;j,k} \neq 0,\ a \in \mathscr{A}\} < \infty.$$

LEMMA 10. *An a.d. family of matrices with uniformly bounded elements is d.e.*

DEFINITION 6. We say that the family of matrices $\{\boldsymbol{M}_a\colon a \in \mathscr{A}\}$ with $\mathscr{D}(\boldsymbol{M}_a) = J_a \times K_a$ is of *almost null rows* (a.n.r.) or of *almost null columns* (a.n.c.) if

$$\sup\{\overline{\overline{Z_a(j,\,)}}\colon j \in J_a,\ a \in \mathscr{A}\} < \infty, \tag{29'}$$

resp.

$$\sup\{\overline{\overline{Z_a(\,,k)}}\colon k \in K_a,\ a \in \mathscr{A}\} < \infty \tag{29''}$$

holds; here

$$Z_a(j,\,) = \{k \in K_a\colon M_{a;j,k} \neq 0\} \quad \text{for } j \in J_a,\ a \in \mathscr{A}, \tag{30'}$$

and

$$Z_a(\,,k) = \{j \in J_a\colon M_{a;j,k} \neq 0\} \quad \text{for } k \in K_a,\ a \in \mathscr{A}. \tag{30''}$$

It is obvious that an a.d. family of matrices is simultaneously of a.n.r. and of a.n.c.

Now we are ready to formulate the rule of multiplication of d.e. matrices.

LEMMA 11. *Let $\{\boldsymbol{M}_a\colon a \in \mathscr{A}\}$ and $\{\boldsymbol{N}_a\colon a \in \mathscr{A}\}$ be two families of matrices such that $\mathscr{D}(\boldsymbol{M}_a) = J_a \times K_a$, $\mathscr{D}(\boldsymbol{N}_a) = K_a \times L_a$, where $J_a, K_a, L_a \subset \mathscr{Z}$ for $a \in \mathscr{A}$. Moreover, let $\{\boldsymbol{M}_a\}$ and $\{\boldsymbol{N}_a\}$ be d.e. If in addition $\{\boldsymbol{M}_a\}$ is of a.n.r. or $\{\boldsymbol{N}_a\}$ is of a.n.c., then the family $\{\boldsymbol{M}_a \circ \boldsymbol{N}_a\}$ is d.e.*

Proof. According to Lemma 6 we may assume that $C_M = C_N = C$ and $q_M = q_N = q$, $0<q<1$. Let $Z_a(j,l) = \{k \in K_a\colon M_{a;j,k}N_{a;k,l} \neq 0\}$ for $(j,l) \in J_a \times L_a$, $a \in \mathscr{A}$. By the assumptions

$$N = \sup\{\overline{\overline{Z_a(j,l)}}\colon (j,l) \in J_a \times L_a,\ a \in \mathscr{A}\} < \infty,$$

hence (a is dropped)

$$|(\boldsymbol{M}\circ\boldsymbol{N})(j,l)| = \Big|\sum_{k\in K} M_{j,k}N_{k,l}\Big| = \Big|\sum_{k\in Z(j,l)} M_{j,k}N_{k,l}\Big| \leqslant \sum_{k\in Z(j,l)} Cq^{|j-k|}Cq^{|k-l|} \leqslant C^2Nq^{|j-l|}.$$

6. Inverses of d.e. matrices.

We begin the analysis with the following

LEMMA 12. *Let $J = J^1 \cup J^2$, $J^1 \cap J^2 = \emptyset$. Then for the $J\times J$-matrix $\boldsymbol{M}$ with*

$$M_{j,k} = \delta_{j,k} \quad \textit{for } (j,k) \in J^1 \times J \tag{31}$$

we have

$$\operatorname{Det} \boldsymbol{M} = \operatorname{Det} \boldsymbol{M}_{(2;2)}. \tag{32}$$

If, moreover, $\operatorname{Det} \boldsymbol{M} \neq 0$, then the inverse $\boldsymbol{N} = \boldsymbol{M}^{-1}$ has the properties

$$N_{j,k} = \delta_{j,k} \quad \textit{for } (j,k) \in J^1 \times J, \tag{33'}$$

$$\boldsymbol{N}_{(2;2)} = (\boldsymbol{M}_{(2;2)})^{-1}, \tag{33''}$$

$$\boldsymbol{N}_{(2;1)} = -\boldsymbol{N}_{(2;2)} \circ \boldsymbol{M}_{(2;1)}. \tag{33'''}$$

Proof. Let us expand the determinant with respect to the j-th row of $\boldsymbol{M}$, $j \in J^1$, and then apply (31):

$$\operatorname{Det} \boldsymbol{M} = \sum_{k \in J} (-1)^{j+k} M_{j,k} \operatorname{Det} \boldsymbol{M}_{(J \setminus \{j\}) \times (J \setminus \{k\})} = \operatorname{Det} \boldsymbol{M}_{(J \setminus \{j\}) \times (J \setminus \{j\})}.$$

Repeating the same procedure $\overline{\overline{J^1}}-1$ times more we get (32).

The proof of (33) is based on the following scheme

$$\boldsymbol{E}_{J\times J} = \left[\begin{array}{c|c} \boldsymbol{E}_{(1;1)} & \boldsymbol{O}_{(1,2)} \\ \hline \boldsymbol{O}_{(2;1)} & \boldsymbol{E}_{(2;2)} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{E}_{(1;1)} & \boldsymbol{O}_{(1;2)} \\ \hline \boldsymbol{M}_{(2;1)} & \boldsymbol{M}_{(2;2)} \end{array}\right] \circ \left[\begin{array}{c|c} \boldsymbol{N}_{(1,1)} & \boldsymbol{N}_{(1;2)} \\ \hline \boldsymbol{N}_{(2;1)} & \boldsymbol{N}_{(2;2)} \end{array}\right] \tag{34}$$

$$= \left[\begin{array}{c|c} \boldsymbol{E}_{(1;1)} \circ \boldsymbol{N}_{(1;1)} + \boldsymbol{O}_{(1;2)} \circ \boldsymbol{N}_{(2;1)} & \boldsymbol{E}_{(1;1)} \circ \boldsymbol{N}_{(1;2)} + \boldsymbol{O}_{(1;2)} \circ \boldsymbol{N}_{(2;2)} \\ \hline \boldsymbol{M}_{(2;1)} \circ \boldsymbol{N}_{(1;1)} + \boldsymbol{M}_{(2;2)} \boldsymbol{N}_{(2;1)} & \boldsymbol{M}_{(2;1)} \circ \boldsymbol{N}_{(1;2)} + \boldsymbol{M}_{(2;2)} \circ \boldsymbol{N}_{(2;2)} \end{array}\right].$$

Comparing the submatrices of the first and the last matrices in (34) we get

$$\boldsymbol{N}_{(1;1)} = \boldsymbol{E}_{(1;1)} \quad \text{and} \quad \boldsymbol{N}_{(1;2)} = \boldsymbol{O}_{(1;2)},$$

i.e. (33′), whence $\boldsymbol{M}_{(2;2)} \circ \boldsymbol{N}_{(2;2)} = \boldsymbol{E}_{(2;2)}$, i.e. (33″), and

$$\boldsymbol{M}_{(2;1)} + \boldsymbol{M}_{(2;2)} \circ \boldsymbol{N}_{(2,1)} = \boldsymbol{O}_{(2;1)}.$$

Applying (33″) we get (33‴).

Let $J = \{j \in \mathscr{Z}, j_1 \leqslant j \leqslant j_2\}$, where $j_1, j_2 \in \mathscr{Z}$. Further let m denote a non-negative integer. We say that the disjoint subsets J^a, J^b and J^c, $J^a \cup J^b \cup J^c = J$, form an *$m$-partition of J of the first kind* whenever

$$\overline{\overline{J}} \geqslant m+1, \tag{35}$$

and $J^b = \{j_0, \ldots, j_0+m\}$ for some $j_0 \in J$, $J^a = \{j \in J; j < j_0\}$, $J^c = \{j \in J: j > j_0+m\}$. The subsets J^a, J^b and J^c form the *m-partition of J of the second kind* if

$$\overline{\overline{J}} \geqslant 2m+2 \tag{36}$$

and $J^a = \{j_1, \ldots, j_1+m\}$, $J^c = \{j_2-m, \ldots, j_2\}$.

Definition 7. Let J^a, J^b and J^c form an m-partition of J of the first (second) kind. A $J\times J$-matrix $\boldsymbol{M}$ is said to be *of m-shape of the first (second) kind* if $M_{j,k} = \delta_{j,k}$ for $(j,k) \in J \times (J^a \cup J^c)$ (for $(j,k) \in J^b \times J$).

It is obvious that Lemma 12 is applicable to matrices of m-shape of the second kind with $J^1 = J^b$ and $J^2 = J^a \cup J^c$. The assumptions of Lemma 12 are satisfied also by transposed matrices of m-shape of the first kind with $J^1 = J^a \cup J^c$ and $J^2 = J^b$. Thus

Lemma 13. *If $\boldsymbol{M}$ is of m-shape of the first (second) kind, then the inverse $\boldsymbol{M}^{-1}$ also is of the m-shape of the first (second) kind, whenever it exists.*

In the remainder of this section $\{\boldsymbol{M}_\alpha: \alpha \in \mathscr{A}\}$ is such a family of matrices that

$$\mathscr{D}(\boldsymbol{M}_\alpha) = J_\alpha \times J_\alpha, \tag{37}$$

where J_α is a finite set of succesive integers for $\alpha \in \mathscr{A}$;

$$\{\boldsymbol{M}_\alpha\} \text{ is d.e.}; \tag{38}$$

$$C = \inf\{|\operatorname{Det} \boldsymbol{M}_\alpha|: \alpha \in \mathscr{A}\} > 0. \tag{39}$$

According to (39) the elements of $\boldsymbol{N}_\alpha \equiv (\boldsymbol{M}_\alpha)^{-1}$ may be estimated as follows

$$|N_{\alpha;j,k}| \leqslant \frac{|\operatorname{Det}(\boldsymbol{M}_\alpha)_{J'\times J''}|}{C}, \tag{40}$$

where $J' = J_\alpha \setminus \{k\}$ and $J'' = J_\alpha \setminus \{j\}$, for $(j,k) \in J_\alpha$, $\alpha \in \mathscr{A}$.

Theorem 2. *Each of the conditions*

(i) $N \equiv \sup\{\overline{\overline{J}}_\alpha: \alpha \in \mathscr{A}\} < \infty$;

(ii) *Both $\boldsymbol{M}_\alpha$ and $\boldsymbol{M}_\alpha^T$ are of m-shape of the second kind*;

(iii) *$\boldsymbol{M}_\alpha$ is of m-shape of the first kind*;

(iv) *$\boldsymbol{M}_\alpha$ is of m-shape of the second kind*;

together with (37)–(39) *implies that*

$$\{\boldsymbol{N}_\alpha\} = \{(\boldsymbol{M}_\alpha)^{-1}\} \text{ is d.e.} \tag{41}$$

Proof. According to (40) and (i) we get

$$|N_{\alpha;j,k}| \leqslant \frac{l!\,C_M^l}{C}, \tag{42}$$

where C_M is the constant from the inequality

$$|M_{\alpha;j,k}| \leqslant C_M q_M^{|j-k|} \tag{43}$$

and $l = \overline{\overline{J}}_\alpha - 1 \leqslant N-1$. Applying Lemma 9 we obtain (41).

For the proof of (ii) let $J_\alpha^a, J_\alpha^b, J_\alpha^c$ form the m-partition of J_α of the second kind. According to Lemma 13 (cf. also the remark following Definition 7) both $\boldsymbol{N}_\alpha$ and $\boldsymbol{N}_\alpha^T$ are of m-shape of the second kind and therefore

$$\boldsymbol{N}_\alpha = \boldsymbol{N}_{\alpha(a,c;a,c)}^{J_\alpha\times J_\alpha} + \boldsymbol{E}_{(b;b)}^{J_\alpha\times J_\alpha}, \qquad \alpha \in \mathscr{A}.$$

According to Lemmas 7, 8 and 10 it is sufficient to prove that $\{\boldsymbol{N}_{\alpha(a,c;a,c)}\}$ is d.e. In view of the equality (cf. Lemma 12)

$$\boldsymbol{N}_{\alpha(a,c;a,c)} = (\boldsymbol{M}_{\alpha(a,c;a,c)})^{-1}$$

we can take $J' = (J_a^a \cup J_a^c)\setminus\{k\}$ and $J'' = (J_a^a \cup J_a^c)\setminus\{j\}$ in inequality (40). Thus also (42) is applicable with $l = \overline{\overline{J_a^a \cup J_a^c}}-1 = 2m+1$. Hence, according to Lemma 9, $\{\boldsymbol{N}_{a(a;a)}\}$ and $\{\boldsymbol{N}_{a(c;c)}\}$ are d.e.

If $j \in J^a$ and $k \in J^c$, then for each $\pi \in \Pi(J', J'')$ (the subindex a is dropped) there exists $j_\pi \in J^a$ such that $k_\pi = \pi(j_\pi) \in J^c$. This enables us to compute the value of

$$\operatorname{Det} \boldsymbol{M}_{J'\times J''} = \sum_{\pi \in \Pi(J', J'')} \operatorname{sgn} \pi M_{j_\pi, k_\pi} \prod_{j \in J'_\pi} M_{j, \pi(j)},$$

where $J'_\pi = J' \setminus \{j_\pi\}$, and hence

$$\begin{aligned} |\operatorname{Det} \boldsymbol{M}_{J'\times J''}| &\leqslant \overline{\overline{\Pi(J', J'')}} \max\{|M_{j_\pi, \pi(j_\pi)}| \cdot C_M^{2m} : \pi \in \Pi(J', J'')\} \\ &\leqslant (2m+1)!\, C_M^{2m+1} q_M^n, \end{aligned}$$

where $n = \min\{|j_\pi - \pi(j_\pi)|: \pi \in \Pi(J', J'')\} \geqslant |j-k| - 2m - 1$. Consequently

$$|N_{a;j,k}| \leqslant \frac{(2m+1)!\, C_M^{2m+1}}{C} q_M^{-(2m+1)} q_M^{|j-k|}$$

for $(j, k) \in J_a^a \times J_a^c$, i.e. $\{\boldsymbol{N}_{a(a;c)}\}$ is d.e. In a quite similar way we obtain that $\{\boldsymbol{N}_{a(c;a)}\}$ is d.e. This completes the proof in the case (ii).

For the proof in the case (iii) (the case (iv) respectively) notice that according to the remark following Definition 7 $\boldsymbol{N} \equiv \boldsymbol{N}_a \equiv \boldsymbol{M}_a^{-1}$ is of the m-shape of the first (of the second) kind for $a \in \mathscr{A}$. Moreover (Lemma 12),

$$\boldsymbol{N} = \boldsymbol{N}_{(;a,c)}^{J\times J} + \boldsymbol{N}_{(;b)}^{J\times J}, \tag{44}$$

where

$$\boldsymbol{N}_{(;a,c)}^{J\times J} = \boldsymbol{E}_{(a,c;a,c)}^{J\times J} \qquad (= \boldsymbol{N}_{(a,c;a,c)}^{J\times J}) \tag{45}$$

and

$$\boldsymbol{N}_{(;b)}^{J\times J} = \boldsymbol{N}_{(b;b)}^{J\times J} - \boldsymbol{M}_{(a,c;b)}^{J\times J} \circ \boldsymbol{N}_{(b;b)}^{J\times J} \qquad (= \boldsymbol{E}_{(b;b)}^{J\times J} - \boldsymbol{N}_{(a,c;a,c)}^{J\times J} \circ \boldsymbol{M}_{(a,c;b)}^{J\times J}). \tag{46}$$

Indeed, it is sufficient to write the matrix $\boldsymbol{N}$ in the form

$$\boldsymbol{N} = \begin{array}{cl} \begin{bmatrix} \boldsymbol{E}_{(a;a)} & \boldsymbol{N}_{(a;b)} & \boldsymbol{O}_{(a;c)} \\ \boldsymbol{O}_{(b;a)} & \boldsymbol{N}_{(b;b)} & \boldsymbol{O}_{(b;c)} \\ \boldsymbol{O}_{(c;a)} & \boldsymbol{N}_{(c;b)} & \boldsymbol{E}_{(c;c)} \end{bmatrix} & \begin{array}{l} \}\, J^a \\ \}\, J^b, \\ \}\, J^c \end{array} \\ \begin{array}{ccc} \underbrace{\qquad}_{J^a} & \underbrace{\qquad}_{J^b} & \underbrace{\qquad}_{J^c} \end{array} & \end{array} \qquad \overline{\overline{J^b}} = m+1,$$

$$\left(\boldsymbol{N} = \begin{array}{cl} \begin{bmatrix} \boldsymbol{N}_{(a;a)} & \boldsymbol{N}_{(a;b)} & \boldsymbol{N}_{(a;c)} \\ \boldsymbol{O}_{(b;a)} & \boldsymbol{E}_{(b;b)} & \boldsymbol{O}_{(b;c)} \\ \boldsymbol{N}_{(c;a)} & \boldsymbol{N}_{(c;b)} & \boldsymbol{N}_{(c;c)} \end{bmatrix} & \begin{array}{l} \}\, J^a \\ \}\, J^b, \\ \}\, J^c \end{array} \\ \begin{array}{ccc} \underbrace{\qquad}_{J^a} & \underbrace{\qquad}_{J^b} & \underbrace{\qquad}_{J^c} \end{array} & \end{array} \qquad \overline{\overline{J^a}} = \overline{\overline{J^c}} = m+1 \right).$$

The family $\{\boldsymbol{M}_{a(b;b)}\}$ $(\{\boldsymbol{M}_{a(a,c;a,c)}^{J_a\times J_a} + \boldsymbol{E}_{(b;b)}^{J_a\times J_a}\})$ satisfies the assumptions of Theorem 2 corresponding to case (i) (to case (ii)). Hence, $\{\boldsymbol{N}_{(b;b)}\}$ is d.e. ($\{\boldsymbol{N}_{(a,c;a,c)}\}$ is d.e.). Applying appropriately Lemmas 7, 8 and 11 to relations (44)–(46) we obtain that $\{\boldsymbol{N}_a\}$ is d.e.

7. Additional partitions of $(-\infty, \infty)$. Let $\{S_n^{[l]}: n = 1, 2, \ldots\}$ denote the l-th sequence of partitions of $(-\infty, \infty)$, $l = 0, 1, \ldots, 5$, where $S_n^{[0]} = \{s_{n;i}^{[0]}: i \in \mathscr{Z}\}$, $n \in \mathscr{N}$, is defined by (10) and $S_n^{[l]} = \{s_{n;i}^{[l]}: i \in \mathscr{Z}\}$, $l = 1, \ldots, 5$, $n \in \mathscr{N}$, are given as follows:

$$s_{n;i}^{[1]} = i \qquad \text{for } i \in \mathscr{Z},\ n \in \mathscr{N}, \tag{47}$$

$$s_{n;i}^{[2]} = \begin{cases} i/2 & \text{for } i \leqslant 0,\ n \in \mathscr{N}, \\ i & \text{for } i > 0,\ n \in \mathscr{N}, \end{cases} \tag{48}$$

$$s_{n;i}^{[3]} = \begin{cases} i & \text{for } i \leqslant n,\ n \in \mathscr{N}, \\ 2i-n & \text{for } i > n,\ n \in \mathscr{N}, \end{cases} \tag{49}$$

$$s_{n;i}^{[4]} = 2 s_{n;i}^{[2]} \qquad \text{for } i \in \mathscr{Z},\ n \in \mathscr{N}, \tag{50}$$

$$s_{n;i}^{[5]} = \begin{cases} s_{1;i}^{[0]} \equiv i & \text{for } i \in \mathscr{Z},\ n = 1, \\ 2^{\mu+1} s_{n;i}^{[0]} & \text{for } i \in \mathscr{Z},\ n \geqslant 2 \end{cases} \tag{51}$$

(for definition of μ refer to Section 3).

Let S denote any partition of $(-\infty, \infty)$. The subspaces $\mathscr{S}^m(S, +)$ and $\mathscr{S}^m(S, -)$ of the space of splines (of order m) $\mathscr{S}^m(S)$, spanned by the B-splines $N_j^{(m)}$ with $j \geqslant 0$ and $j < 0$ respectively are characterized by

Lemma 14. *The spline function f belongs to $\mathscr{S}^m(S, +)$ $(\mathscr{S}^m(S, -))$ if and only if* $\operatorname{supp} f \subset \langle s_{-1}, \infty)$ $(\operatorname{supp} f \subset (-\infty, s_m\rangle)$.

Proof. If $f = \sum_{j\geqslant 0} a_j N_j^{(m)}$ $(f = \sum_{j<0} a_j N_j^{(m)})$, then $\operatorname{supp} f \subset \langle s_{-1}, \infty)$ $(\operatorname{supp} f \subset (-\infty, s_m\rangle)$. According to properties (P.2) and (P.5) each function f has the unique decomposition $f = f_+ + f_-$, where $f_\pm \in \mathscr{S}^m(S, \pm)$. Thus it is enough to prove that $\operatorname{supp} f \subset \langle s_{-1}, s_m\rangle$ implies $f(t) = 0$ for all $t \in (-\infty, \infty)$. But this is established in Lemma 2.

If $S = \{s_i: i \in \mathscr{Z}\}$ and $S' = \{s'_i: i \in \mathscr{Z}\}$ are two partitions of $(-\infty, \infty)$ such that $s_i = s'_i$ for $i = 0, 1, \ldots, n$, $n \geqslant m+1$, then the corresponding spaces $\mathscr{S}_n^m(S)$ and $\mathscr{S}_n^m(S')$ of splines on $\langle s_0, s_n\rangle$ coincide. The corresponding spline partitions of unity $\boldsymbol{N}_n^{(m)} = \{N_{n;j}^{(m)}: j \in J_n^m\}$ and $\boldsymbol{N}_n'^{(m)} = \{N'^{(m)}_{n;j}: j \in J_n^m\}$ differ at most in the first $m+1$ and the last $m+1$ elements. According to Lemma 14

$$N_{n;i}^{(m)} = \sum_{j \in J_n^a} \Phi_{i,j} N'^{(m)}_{n;j} \qquad \text{for } i \in J_n^a,$$

$$N_{n;i}^{(m)} = \sum_{j \in J_n^c} \Phi_{i,j} N'^{(m)}_{n;j} \qquad \text{for } i \in J_n^c,$$

and

$$N_{n;i}^{(m)} = N_{n;i}^{\prime(m)} \quad \text{for } i \in J_n^b,$$

where J_n^a, J_n^b, J_n^c form the m-partition of J_n^m of the second kind. Since each of the spline partitions of unity is a basis in $\mathscr{S}_n^m(S)$, we have

LEMMA 15. *The unique $J_n^m \times J_n^m$-matrix $\boldsymbol{\Phi}_n$ defined by the identity*

$$N_{n;i}^{(m)} = \sum_{j \in J} \Phi_{n;i,j} N_{n;j}^{\prime(m)} \quad \text{for } i \in J$$

may be expressed as follows:

$$\boldsymbol{\Phi}_n = \boldsymbol{\Phi}_{n(a;a)}^{J\times J} + \boldsymbol{E}_{(b;b)}^{J\times J} + \boldsymbol{\Phi}_{n(c,c)}^{J\times J},$$

where J stands for J_n^m.

Notice that $S_{n;i}^{[l]} = i$ for $i = 0, \ldots, n$, $l = 1, 2, 3$. According to the above remarks, for each n all the symbols $\mathscr{S}_n^m(S_n^{[l]})$ with $l = 1, 2, 3$, denote the same space of splines of order m on $\langle 0, n\rangle$ with knots at $0, 1, \ldots, n$. Let the $J_n \times J_n$-matrix $\boldsymbol{\Phi}_n^{[l,k]}$ be defined by the equality

$$N_{n;i}^{[l]} = \sum_{j \in J_n} \Phi_{n;i,j}^{[l,k]} N_{n;j}^{[k]} \quad \text{for } j \in J_n,\ n \in \mathscr{N},\ l, k = 1, 2, 3,$$

where $J_n = J_n^m$ and $\boldsymbol{N}_n^{[l]} = \boldsymbol{N}_n^{(m)[l]}$ is the J_n-system of B-splines restricted to $\langle 0, n\rangle$. Lemma 15 gives

$$\boldsymbol{\Phi}_n^{[l,k]} = \boldsymbol{\Phi}_{n(a,a)}^{[l,k]J\times J} + \boldsymbol{E}_{(b,b)}^{J\times J} + \boldsymbol{\Phi}_{n(c,c)}^{[l,k]J\times J}$$

for $n \geqslant m+1$, $l, k = 1, 2, 3$, with $J = J_n = J_n^m$.

Since $s_{n;i}^{[l]} = s_{m+1;i}^{[l]}$ for $i \in J_n^a \equiv J_{m+1}^a$ and $s_{n;i}^{[l]} = s_{m+1;i-n+m+1}^{[l]}$ for $i \in J_n^c$, $l = 1, 2, 3$, we get by Lemma 1

$$\boldsymbol{\Phi}_{n(a;a)}^{[l,k]} = \boldsymbol{\Phi}_{m+1(a;a)}^{[l,k]}$$

and

$$\Phi_{n;i,j}^{[l,k]} = \Phi_{m+1;i-n+m+1,j-n+m+1}^{[l,k]} \quad \text{for } i, j \in J_n^c,$$

where $l, k = 1, 2, 3$ and $n \geqslant m+1$. Thus we have (cf. Lemma 10)

LEMMA 16. *For each pair (l, k), $l, k = 1, 2, 3$, the family of matrices $\{\boldsymbol{\Phi}_n^{[l,k]}: n \geqslant m+1\}$ is a.d. (almost diagonal) and d.e. (diagonally exponential).*

The Gram matrices $\boldsymbol{G}_n^{(m)[l]} \equiv \boldsymbol{G}_n^{[l]}$, where

(52) $$G_{n;i,j}^{[l]} = (N_{n;i}^{[l]}, N_{n;j}^{[l]}) \quad \text{for } i, j \in J_n$$

are related as follows:

$$\boldsymbol{G}_n^{[l]} = \boldsymbol{\Phi}_n^{[l,k]} \circ \boldsymbol{G}_n^{[k]} \circ (\boldsymbol{\Phi}_n^{[l,k]})^T$$

for $n \geqslant m+1$, $l, k = 1, 2, 3$. On the other hand, $(\boldsymbol{\Phi}_n^{[l,k]})^{-1} = \boldsymbol{\Phi}_n^{[k,l]}$ for $l, k = 1, 2, 3$, $n \geqslant m+1$, hence

$$\boldsymbol{A}_n^{[l]} \equiv (\boldsymbol{G}_n^{[l]})^{-1} = (\boldsymbol{\Phi}_n^{[k,l]})^T \circ \boldsymbol{A}_n^{[k]} \circ \boldsymbol{\Phi}_n^{[k,l]}.$$

According to (50) and Lemma 1, $G_{n;i,j}^{[4]} = 2G_{n;i,j}^{[2]}$ for $i, j \in J_n$, $n \in \mathscr{N}$, i.e.

$$\boldsymbol{A}_n^{[4]} = \tfrac{1}{2}\boldsymbol{A}_n^{[2]} \quad \text{for } n \in \mathscr{N}.$$

Using Lemmas 11 and 16 we obtain

LEMMA 17. *For each l, $l = 2, 3, 4$, $\{\boldsymbol{A}_n^{[l]}: n \geqslant m+1\}$ is d.e. if and only if $\{\boldsymbol{A}_n^{[1]}: n \geqslant m+1\}$ is d.e.*

THEOREM 3. *For each l, $l = 1, 2, 3, 4$, the family $\{\boldsymbol{A}_n^{[l]}: n \in \mathscr{N}\}$ is d.e.*

According to Lemmas 6 and 17 it is sufficient to prove that $\{\boldsymbol{A}_n^{[1]}: n \geqslant m+1\}$ is d.e. and this will be proved in the next three sections.

8. The Gram matrices $\boldsymbol{G}_n^{[1]}$. Let the $J_n \times J_n$-matrices $\boldsymbol{G}_n^{[1]}$ be defined by (52) with $l = 1$, where $\boldsymbol{N}_n^{[1]} = \boldsymbol{N}_n^{(m)[1]}$ is the J_n-system of B-splines (of order m) on $\langle 0, n\rangle$ corresponding to the partition $S_{n;i}^{[1]} = \{s_{n;i}^{[1]} \equiv i\colon i \in \mathscr{Z}\}$. It is obvions that for each $n \in \mathscr{N}$ and $i \in J_n^m$, $N_{n;i}^{(m)[1]} \equiv N_{n;i}^{(m)} = N_i^{(m)}|_{\langle 0,n\rangle}$, where $N_i^{(m)}$ is the ith B-spline corresponding to $S = \{s_i = i\colon i \in \mathscr{Z}\}$, with equidistant knots, $s_i - s_{i-1} = 1$.

Let the numbers $G_l \equiv G_l^{(m)}$ be defined in terms of these B-splines according to (4) and (5) for $l \in \mathscr{Z}$. With use of properties (P.3) and (P.4) we verify that

(53) $$\sum_{l \in \mathscr{Z}} G_l = \sum_{l=-m-1}^{m+1} G_l = 1.$$

Let J_n^a, J_n^b and J_n^c form the m-partition of J_n of the second kind for $n \geqslant m+1$.

LEMMA 18. *The matrices $\boldsymbol{G}_n^{[1]}, n \geqslant m+1$, have the properties*

(T.1) $G_{n;i,j}^{[1]} = G_{j-i}$ *whenever $i \in J_n^b$ or $j \in J_n^b$,*

(T.2) $G_{n;i+n,j+n}^{[1]} + G_{n;i,j}^{[1]} = G_{j-i}$ *for $i, j \in J_n^a$,*

(T.3) $G_{n;i,j}^{[1]} = G_{m+1;i,j}^{[1]}$ *for $i, j \in J_n^a$,*

(T.4) *$\boldsymbol{G}_n^{[1]}$ is symmetric, r.s. (rotatively symmetric) and p.d. (positive definite),*

(T.5) *$\{\boldsymbol{G}_n^{[1]}: n \geqslant m+1\}$ is a.d. and d.e.*

Proof. Equality (T.1) holds whenever $\operatorname{supp} N_i^{(m)} N_j^{(m)} \subset \langle 0, n\rangle$, i.e. for $i \in J_n^b$ or $j \in J_n^b$.

For the proof of (T.2) notice that, according to Lemma 1,

$$G_{n;i,j}^{[1]} + G_{n;i+n,j+n}^{[1]} = \int_{\langle 0,n\rangle} N_i^{(m)} N_j^{(m)} + N_{i+n}^{(m)} N_{j+n}^{(m)}$$
$$= \int_{\langle 0,n\rangle} N_i^{(m)} N_j^{(m)} + \int_{\langle -n,0\rangle} N_i^{(m)} N_j^{(m)} = \int_{\langle -n,n\rangle} N_i^{(m)} N_j^{(m)} = G_{j-i}^{(m)} \equiv G_{j-i}$$

whenever $\operatorname{supp} N_i^{(m)} N_j^{(m)} \subset \langle -n, n\rangle$; but this holds for $(i, j) \in J_n^a \times J_n^a$, $n \geqslant m+1$.

(T.3) follows from the inclusion $(\operatorname{supp} N_i^{(m)} \cap \langle 0, n\rangle) \subset \langle 0, m+1\rangle$ for $i \in J_n^a \equiv J_{m+1}^a$, $n \geqslant m+1$.

The symmetry of $G_n^{[1]}$ is obvious. $G_n^{[1]}$ is p.d. according to (24) and (P.5). For the proof of r.s. notice that the B-splines corresponding to a partition of $(-\infty, \infty)$ with equidistant knots are symmetric in the following sense:

(54) $$N_i^{(m)}(t-s_0) = N_{-i-m}^{(m)}(s_0-t) \quad \text{for } t \in (-\infty, \infty),\ i \in \mathscr{Z}.$$

The proof of (54) is quite similar to that of Lemma 1 and therefore omitted. Applying (54) and Lemma 1 we get

$$G_{n;-m+i,-m+j}^{[1]} = \int_{\langle 0,n\rangle} N_{-m+i}^{(m)} N_{-m+j}^{(m)} = \int_{\langle -n,0\rangle} N_{-i}^{(m)} N_{-j}^{(m)}$$
$$= \int_{\langle 0,n\rangle} N_{n-i}^{(m)} N_{n-j}^{(m)} = G_{n;\,n-i,n-j}$$

for $i, j = 0, 1, \ldots, n+m$.

Notice that

(55) $$G_{n;i,j}^{[1]} = 0 \quad \text{whenever } |i-j| \geqslant m+2$$

and that, according to (T.2), (53) and properties (P.1) and (P.3), we have

(56) $$1 \geqslant G_{n;i,j}^{[1]} > 0 \quad \text{whenever } |i-j| \leqslant m+1.$$

Thus (T.5) is a consequence of Lemma 10.

9. The Toeplitz matrix "inverse" to G. The function $g(z) = g^{(m)}(z)$ defined for $z \in \mathscr{C}$, $z \neq 0$, according to (7), is holomorphic in the annulus

(57) $$\mathscr{D} = \{z \in \mathscr{C}\colon r < |z| < 1/r\},$$

where $r \in (|\gamma_{-1}|, 1)$ and $\gamma_{-1} \equiv \gamma_{-1}^{(m)}$ is the root of g defined by relations (8)–(9). Moreover, there exist constants C_1 and C_2 such that

$$0 < C_1 \leqslant |g(z)| \leqslant C_2 < \infty \quad \text{for } z \in \mathscr{D}.$$

Therefore the function h, defined by

(58) $$h(z) = 1/[g(z)] \quad \text{for } z \in \mathscr{D},$$

is holomorphic and bounded in $\mathscr{D}$. It follows from the Cauchy formula for the coefficients of the Laurent expansion

(59) $$h(z) = \sum_{j \in \mathscr{Z}} H_j z^j \quad \text{for } z \in \mathscr{D}$$

that

(60) $$|H_j| \leqslant \frac{1}{C_1} r^{|j|} \quad \text{for } j \in \mathscr{Z}.$$

Thus the Toeplitz matrix $\boldsymbol{H}$, $\mathscr{D}(\boldsymbol{H}) = \mathscr{Z} \times \mathscr{Z}$, with

$$H_{j,k} = H_{k-j} \quad \text{for } j, k \in \mathscr{Z}$$

forms a one-element d.e. family of matrices. It may be proved that $\boldsymbol{H}$ is symmetric.

The product $1 = (g \cdot h)(z) = g(z)h(z)$ for $z \in \mathscr{D}$ may be represented as the Cauchy product of series (7) and (59), i.e.

$$1 \equiv (g \cdot h)(z) = \sum_{k \in \mathscr{Z}} \Big(\sum_{j \in \mathscr{Z}} G_j H_{k-j}\Big) z^k \quad \text{for } z \in \mathscr{D},$$

whence

$$\sum_{k \in \mathscr{Z}} G_{k-j} H_{l-k} = \sum_{k \in \mathscr{Z}} G_k H_{l-j-k} = \delta_{l-j,0} = \delta_{j,l} \quad \text{for } j, l \in \mathscr{Z}.$$

The symmetry of $\boldsymbol{G}$ and $\boldsymbol{H}$ implies that

$$\sum_{k \in \mathscr{Z}} H_{k-j} G_{l-k} = \delta_{j,l} \quad \text{for } j, l \in \mathscr{Z},$$

i.e.

(61) $$\boldsymbol{H} \circ \boldsymbol{G} = \boldsymbol{G} \circ \boldsymbol{H} = \boldsymbol{E}_{\mathscr{Z} \in \mathscr{Z}}.$$

10. Construction of the matrices $\boldsymbol{A}_n^{[1]}$. Let us define the matrices $\boldsymbol{C}_n$ as follows:

(62) $$C_{n;\,j,k} = \begin{cases} (\gamma_{k-1})^{j+m} & \text{for } (j, k) \in J_n \times J_n^a, \\ H_{j,k} \equiv H_{k-j} & \text{for } (j, k) \in J_n \times J_n^b, \\ (\gamma_{n-m-k+1})^{n-j} & \text{for } (j, k) \in J_n \times J_n^c, \end{cases}$$

for $n \geqslant m+1$. Note that according to (8), (9) and (60) each $\boldsymbol{C}_n$ is r.s. and $\{\boldsymbol{C}_n\colon n \geqslant m+1\}$ is d.e. (cf. also Lemmas 7 and 8).

It will be shown that the matrices

(63) $$\boldsymbol{D}_n = \boldsymbol{G}_n \circ \boldsymbol{C}_n \quad \text{for } n \geqslant m+1$$

have the following properties:

(64) *Each $\boldsymbol{D}_n$ is of m-shape of the second kind;*

(65) *$\{\boldsymbol{D}_n\}$ is d.e.;*

(66) $\lim_{n\to\infty} \operatorname{Det} \boldsymbol{D}_n > 0$.

For the proof of (64) let $j \in J_n^b$. Then, according to Lemma 18, we have for each l, $l \in J_n$,

$$D_{n;j,l} = \sum_{k \in J_n} G_{n;j,k}^{[1]} C_{n;\,k,l} = \sum_{k=j-m-1}^{j+m+1} G_{k-j} C_{n;\,k,l}.$$

Hence, by the definition of γ_i (cf. Lemma 3)

$$D_{n;j,l} = \sum_{k=j-m-1}^{j+m+1} G_{k-j}(\gamma_{l-1})^{k+m} = (\gamma_{l-1})^{m+j} \sum_{k=j-m-1}^{j+m+1} G_{k-j}(\gamma_{l-1})^{k-j}$$

$$= (\gamma_{l-1})^{m+j} g(\gamma_{l-1}) = 0 = \delta_{j,l} \quad \text{for } j \in J_n^b,\ l \in J_n^a.$$

Similarly

$$D_{n;j,l} = \delta_{j,l} = 0 \quad \text{for } j \in J_n^b,\ l \in J_n^c.$$

If $(j, l) \in J_n^b \times J_n^b$, then according to (61)

$$D_{n;j,l} = \sum_{k=j-m-1}^{j+m+1} G_{k-j} H_{l-k} = \sum_{-\infty}^{+\infty} G_{k-j} H_{l-k} = \delta_{j,l}.$$

The d.e. property of $\boldsymbol{D}_n$ follows from Lemmas 11, 18 and the d.e. property of $\boldsymbol{C}_n$.

Inequality (66) is verified as follows: Applying (T.1) and (T.3) of Lemma 18 and (55) we get

$$\boldsymbol{D}_{n(a;a)} = \boldsymbol{G}^{[1]}_{n(a;)} \circ \boldsymbol{C}_{n(;a)} = \boldsymbol{G}^{[1]}_{m+1(a;)} \circ \boldsymbol{C}_{m+1(;a)} = \boldsymbol{D}_{m+1(a;a)}. \tag{67}$$

In view of the r.s. property of $\boldsymbol{D}_n$ for $n \geqslant m+1$ (cf. (62) and Lemmas 5, 18) we have

$$\operatorname{Det} \boldsymbol{D}_{n(a;a)} \operatorname{Det} \boldsymbol{D}_{n(c;c)} = (\operatorname{Det} \boldsymbol{D}_{m+1(a;a)})^2. \tag{68}$$

Let us consider the matrix

$$\boldsymbol{M} = (\boldsymbol{C}_{m+1})^T \circ \boldsymbol{D}_{m+1} = (\boldsymbol{C}_{m+1})^T \circ \boldsymbol{G}_{m+1} \circ \boldsymbol{C}_{m+1}.$$

The matrix $\boldsymbol{C}_{m+1}$ is non-singular as a product of the Vandermonde matrix corresponding to the (different) numbers $\gamma_i^{(m)}$, $i = \pm 1, \ldots, \pm(m+1)$, and the diagonal $J_{m+1} \times J_{m+1}$-matrix $\bar{\boldsymbol{\Gamma}}$ with $\bar{\Gamma}_{i,i} = 1$ for $i \in J_{m+1}^a$ and $\bar{\Gamma}_{i,i} = (\gamma_{-i})^{2m+1} \neq 0$ for $i \in J_{m+1}^c$ ($J_{m+1}^b = \emptyset$). According to (17) $\boldsymbol{M}$ is p.d. and by (20) and (22′) $\boldsymbol{M}_{(a;a)}$ is p.d. too. But

$$\begin{aligned} \boldsymbol{M}_{(a;a)} &= (\boldsymbol{C}^T_{m+1})_{(a;)} \circ \boldsymbol{D}_{m+1(;a)} \\ &= (\boldsymbol{C}^T_{m+1})_{(a;a)} \circ \boldsymbol{D}_{m+1(a;a)} + (\boldsymbol{C}^T_{m+1})_{(a;c)} \circ \boldsymbol{D}_{m+1(c;a)} = \boldsymbol{P} + \boldsymbol{Q}, \end{aligned} \tag{69}$$

where $\boldsymbol{P} = (\boldsymbol{C}^T_{m+1})_{(a;a)} \circ \boldsymbol{D}_{m+1(a;a)}$ and $\boldsymbol{Q} = (\boldsymbol{C}^T_{m+1})_{(a;c)} \circ \boldsymbol{G}_{m+1(c;)} \circ \boldsymbol{C}_{m+1(;a)}$. Applying the properties of $\boldsymbol{G}_{m+1}$ given in Lemma 18 and the definitions of $\boldsymbol{C}_{m+1}$ and γ_i (cf. (62) and Lemma 3 respectively) we may write (the subindex $m+1$ is dropped)

$$\begin{aligned} Q_{i,j} &= \sum_{k=1}^{m+1} C_{k,i}\Big(\sum_{l=-m}^{0} G_{l-k} C_{l,j} + \sum_{l=1}^{m+1} (G_{l-k} - G_{m+1;k-m-1,l-m-1}) C_{l,j}\Big) \\ &= \sum_{k=1}^{m+1} C_{k,i}\Big(\sum_{l=-m}^{m+1} G_{l-k} C_{l,j} - \sum_{l=1}^{m+1} G_{m+1;k-m-1,l-m-1} C_{l,j}\Big) \\ &= \sum_{k=1}^{m+1} C_{k,i}\Big(\sum_{l=-m}^{2m+2} G_{l-k} (\gamma_{j-1})^{m+l} - \sum_{l=1}^{2m+2} G_{m+1;k-m-1,l-m-1} (\gamma_{j-1})^{m+1}\Big) \\ &= -\sum_{k=1}^{m+1} (\gamma_{i-1})^{k+m} \sum_{l=-m}^{m+1} G_{m+1;k-m-1,l} (\gamma_{j-1})^{m+l+m+1} \\ &= -(\gamma_{i-1})^{m+1}\Big(\sum_{k=-m}^{0} \sum_{l=-m}^{m+1} (\gamma_{i-1})^{k+m} G_{m+1;k,l} (\gamma_{j-1})^{m+l}\Big) (\gamma_{j-1})^{m+1} \\ &= -(\boldsymbol{\Gamma} \circ \boldsymbol{P} \circ \boldsymbol{\Gamma})(i,j) \quad \text{for } i, j \in J_{m+1}^a, \end{aligned}$$

where $\mathscr{D}(\boldsymbol{\Gamma}) = J_{m+1}^a \times J_{m+1}^a$ and

$$\Gamma_{i,j} = (\gamma_{i-1})^{m+1} \delta_{i,j} \quad \text{for each } (i, j).$$

According to (69), $\boldsymbol{M} = \boldsymbol{P} - \boldsymbol{\Gamma} \circ \boldsymbol{P} \circ \boldsymbol{\Gamma}$ and therefore the assumptions of Lemma 4 are satisfied (cf. Lemma 3). Hence $\boldsymbol{P}$ is p.d. It follows that $\boldsymbol{D}_{m+1(a;a)}$ is non-singular.

In view of (68), in order to prove (66) it is sufficient to show that $\Delta_n \to 0$ as $n \to \infty$, where $\Delta_n = \operatorname{Det} \boldsymbol{D}_n - \operatorname{Det} \boldsymbol{D}_{n(a;a)} \operatorname{Det} \boldsymbol{D}_{n(c;c)}$. According to Lemma 12

$$\Delta_n = \sum_{\pi \in \Pi''} \operatorname{sgn} \pi \prod_{j \in J_n^a \cup J_n^c} D_{n;j,\pi(j)},$$

where $\Pi'' = \Pi \setminus \Pi'$, $\Pi = \Pi(J_n^a \cup J_n^c, J_n^a \cup J_n^c)$, $\Pi' = \{\pi \in \Pi\colon \pi(J_n^a) = J_n^a\}$. For each $\pi \in \Pi''$ there exists $j_\pi \in J_n^a$ such that $k_\pi \equiv \pi(j_\pi) \in J_n^c$. Writing $J_\pi = (J_n^a \cup J_n^c) \setminus \{j_\pi\}$ for $\pi \in \Pi''$ we obtain

$$\begin{aligned} |\Delta_n| &\leqslant \sum_{\pi \in \Pi''} |D_{n;j_\pi,k_\pi}| \prod_{j \in J_\pi} |D_{n;j,\pi(j)}| \\ &\leqslant \overline{\overline{\Pi''}}\, C_D q_D^{|j_\pi - k_\pi|} C_D^{(2m+1)} \leqslant (2m+2)!\, C_D^{(2m+2)} q_D^l, \end{aligned}$$

where $l = \min\{|j_\pi - k_\pi|\colon \pi \in \Pi''\} = n - m \to \infty$ as $n \to \infty$, hence $\Delta_n \to 0$.

Now we may complete the proof of Theorem 3. The inverse of $\boldsymbol{G}_n^{[1]}$ equals $\boldsymbol{A}_n^{[1]} = \boldsymbol{C}_n \circ \boldsymbol{D}_n^{-1}$. Indeed, $\boldsymbol{G}_n^{[1]} \circ \boldsymbol{C}_n \circ \boldsymbol{D}_n^{-1} = \boldsymbol{D}_n \circ \boldsymbol{D}_n^{-1} = \boldsymbol{E}_{J_n \times J_n}$. The family $\{(\boldsymbol{D}_n)^{-1}\}$ is (according to (64)–(66) and Theorem 2) d.e. and of m-shape of the second kind, i.e. of a.n.c. Thus $\{\boldsymbol{C}_n \circ \boldsymbol{D}_n^{-1}\} \equiv \{\boldsymbol{A}_n^{[1]}\}$ is d.e.

11. The dual bases. Let $\boldsymbol{N}_n$ denote the J_n-system of B-splines of order m corresponding to the partition $S = \{s_i\colon i \in \mathscr{Z}\}$ restricted to $\langle s_0, s_n\rangle$. The J_n-system $\underline{\boldsymbol{N}}_n$, where $\underline{\boldsymbol{N}}_n(i) \equiv \underline{N}_{n;i} \in \mathscr{S}_n^m(S)$, $i \in J_n = J_n^m$, is called a dual basis to $\boldsymbol{N}_n$ whenever

$$(70)\qquad (\underline{N}_{n;i}, N_{n;j}) \equiv \int\limits_{\langle s_0, s_n\rangle} \underline{N}_{n;i} N_{n;j} = \delta_{i;j} \quad \text{for } i, j \in J_n.$$

If $\boldsymbol{G}$ denotes the Gram matrix of $\boldsymbol{N}_n$, then

$$(71)\qquad \underline{N}_{n;i} = \sum_{j \in J_n} A_{i,j} N_{n;j} \quad \text{for } i \in J_n,$$

where $\boldsymbol{A} = \{A_{i,j}\colon (i,j) \in J_n \times J_n\}$ is the inverse of $\boldsymbol{G}$. It follows from (71) that

$$(72)\qquad A_{i,j} = (\underline{N}_i, \underline{N}_j) \quad \text{for } i, j \in J_n.$$

According to (24) both $\boldsymbol{A}$ and $\boldsymbol{G}$ are positive definite, for any $n \in \mathscr{N}$ and any S.

For further considerations we recall the notation introduced in the preceding sections. The integers μ and ν for each n are defined by the equality $2^\mu + \nu = n$, $\mu \geqslant 0$, $1 \leqslant \nu \leqslant 2^\mu$; $\boldsymbol{N}_n^{[l]} \equiv \boldsymbol{N}_n^{(m)[l]}$ denotes the spline partition of unity which is a basis in $\mathscr{S}_n^{[l]} = \mathscr{S}_n^m(S_n^{[l]})$ for $n \in \mathscr{N}$, $l = 0, \ldots, 5$, where the partitions $S_n^{[l]}$ are defined in (10) and (47)–(51).

It is obvious that $N_{n;i}^{[l]} = N_i^{[l]}|_{I_n}$, where $I_n = \langle s_{n;0}^{[l]}, s_{n;n}^{[l]}\rangle$ and $\boldsymbol{N}^{[l]} = \{N_i^{[l]}\colon i \in \mathscr{Z}\}$ is the $\mathscr{Z}$-system of B-splines corresponding to $S_n^{[l]}$. $\boldsymbol{G}_n^{[l]}$ is the Gram matrix of $\boldsymbol{N}_n^{[l]}$. $\boldsymbol{A}_n^{[l]} = (\boldsymbol{G}_n^{[l]})^{-1}$ is the Gram matrix of $\underline{\boldsymbol{N}}_n^{[l]}$, for $n \in \mathscr{N}$, $l = 0, \ldots, 5$, where $\underline{\boldsymbol{N}}_n^{[l]}$ is the dual basis to $\boldsymbol{N}_n^{[l]}$.

LEMMA 19. *The maximal eigenvalues of the (p.d.) matrices* $\boldsymbol{A}_n^{[l]}$ *and* $\boldsymbol{G}_n^{[l]}$ *are uniformly bounded in* n *for* $l = 1, 2, 3, 4$.

Indeed, applying properties (P.1)–(P.4) of the B-splines and inequality (23) we may estimate

$$\lambda_M(\boldsymbol{G}_n^{[l]}) \leqslant \max\Big\{\sum_{i \in J_n} G_{n;i,j}^{[l]}\colon j \in J_n\Big\} \leqslant \max\Big\{\int\limits_{\langle s_0, s_n\rangle} \Big(\sum_i N_i^{[l]}\Big) N_j^{[l]}\colon j \in J_n\Big\}$$
$$\leqslant \max\Big\{\int\limits_{-\infty}^{\infty} N_j^{[l]}\colon j \in \mathscr{Z}\Big\} \leqslant \max\{s_{n;j}^{[l]} - s_{n;j-1}^{[l]}\} \leqslant 2$$

for each $n \in \mathscr{N}$, $l = 1, \ldots, 5$. Similarly,

$$\lambda_M(A_n^{[l]}) \leqslant \sum_{i \in J_n} |A_{n;i,j}^{[l]}| \leqslant C_A \frac{1 + q_A}{1 - q_A}$$

for $n \in \mathscr{N}$, $l = 1, 2, 3, 4$, where C_A and q_A, $0 < q_A < 1$, are the constants from the inequalities (cf. Theorem 3)

$$|A_{n;i,j}^{[l]}| \leqslant C_A q_A^{|i-j|} \quad \text{for } i, j \in J_n,\ n \in \mathscr{N},\ l = 1, 2, 3, 4.$$

12. Estimates for the elements of $A_n^{[5]}$. The aim of this section is to prove

THEOREM 4. *The family of matrices* $\{\boldsymbol{A}_n^{[5]}\colon n \in \mathscr{N}\}$ *is d.e.*

Notice that $S_1^{[5]} = S_1^{[0]}$ and that $S_n^{[5]} = S_n^{[3]}$ whenever $n = 2^{\mu+1}$, i.e. $\nu = 2^\mu$. Thus according to Lemma 6 and Theorem 3 it suffices to consider the matrices with $n \geqslant 2$ for which

$$(73)\qquad 0 < \nu < 2^\mu$$

holds. Till the end of this section we shall say "for each n" instead of "for each n for which (73) holds". The subsets $J_n^a = \{-m, \ldots, 2\nu - m - 1\}$, $J_n^b = \{2\nu - m, \ldots, 2\nu\}$, $J_n^c = \{2\nu + 1, \ldots, n\}$ form an m-partition of J_n of the first kind for each n. According to (73) none of the subsets is empty. Notice that $S_{2\nu}^{[3]} = S_n^{[5]}$ and that $s_{n;i}^{[5]} = s_{n-2\nu;i-2\nu}^{[4]}$ for each $i \in \mathscr{Z}$. Further $J_{2\nu} = J_n^a \cup J_n^b$ and $J_{n-2\nu} = \{i - 2\nu\colon i \in J_n^b \cup J_n^c\}$.

For each n let us define the J_n-system of functions $\boldsymbol{M}_n = \{M_{n;i}\colon i \in J_n\}$ as follows:

$$(74')\qquad M_{n;i}(t) = \begin{cases} \underline{N}_{2\nu;i}^{[3]}(t) & \text{for } t \in \langle 0, 2\nu\rangle, \\ \sum\limits_{j \in J_n^b} W_{i,j} \underline{N}_{n-2\nu;j-2\nu}^{[4]}(t - 2\nu) & \text{for } t \in (2\nu, 2^{\mu+1}\rangle, \end{cases}$$

for $i \in J_n^a \cup J_n^b$ and

$$(74'')\qquad M_{n;i}(t) = \begin{cases} \sum\limits_{j \in J_n^b} W_{i,j} \underline{N}_{2\nu;j}^{[3]}(t) & \text{for } t \in \langle 0, 2\nu\rangle, \\ \underline{N}_{n-2\nu;i-2\nu}^{[4]}(t - 2\nu) & \text{for } t \in (2\nu, 2^{\mu+1}\rangle, \end{cases}$$

for $i \in J_n^c$. Each system $\boldsymbol{M}_n$ has the property

$$(75)\qquad (M_{n;i}, N_{n;k}^{[5]}) = \delta_{i,k} \quad \text{for } i \in J_n,\ k \in J_n^a \cup J_n^c.$$

For the proof of (75) notice that according to Lemma 1 we have

$$(76')\qquad N_{n;k}^{[5]}|_{\langle 0, 2\nu\rangle} = \begin{cases} N_{2\nu;k}^{[3]} & \text{for } k \in J_n^a \cup J_n^b, \\ 0 & \text{for } k \in J_n^c, \end{cases}$$

and

$$(76'')\qquad N_{n;k}^{[5]}|_{\langle 2\nu, 2^{\mu+1}\rangle} = \begin{cases} 0 & \text{for } k \in J_n^a, \\ N_{n-2\nu;k-2\nu}^{[4]} \circ \tau_{2\nu} & \text{for } k \in J_n^b \cup J_n^c, \end{cases}$$

where $\tau_{2\nu}(t) = t - 2\nu$ for $t \in \langle 2\nu, 2^{\mu+1}\rangle$ and $\circ$ denotes the superposition of functions.

Let for $f \in \mathscr{S}^{[3]}_{2\nu}$ and $g \in \mathscr{S}^{[4]}_{n-2\nu}$ $f \dotplus g$ denote the *direct sum* of f and g, i.e.

$$(f \dotplus g)(t) = \begin{cases} f(t) & \text{for } t \in \langle 0, 2\nu\rangle, \\ g(t-2\nu) & \text{for } t \in (2\nu, 2^{\mu+1}\rangle. \end{cases} \tag{77}$$

Now, (76) and the linear independence of B-splines give

LEMMA 20. *The function $f \dotplus g$, where*

$$f = \sum_{i \in J^a_n \cup J^b_n} a_i N^{[3]}_{2\nu;i} \quad \textit{and} \quad g = \sum_{i \in J^b_n \cup J^c_n} b_i N^{[4]}_{n-2\nu;\, i-2\nu},$$

is a spline function with knots at $0 = s^{[5]}_0 < \ldots < s^{[5]}_n = 2^{\mu+1}$ if and only if $a_i = b_i$ for $i \in J^b_n$.

The coefficients $W_{i,j}$ in (74), $i \in J_n$, $j \in J^b_n$, may be chosen in such a way that $M_{n;i} \in \mathscr{S}^{[5]}_n$. Indeed, applying (71) and the notation (77) we may rewrite (74′) and (74″) in the form

$$M_{n;i} = \begin{cases} \displaystyle\sum_{k \in J^a \cup J^b} A^{[3]}_{i,k} N^{[3]}_k + \sum_{k \in J^b \cup J^c} \sum_{j \in I^b} W_{i,j} \bar{A}^{[4]}_{j,k} N^{[4]}_{k-2\nu} & \text{for } i \in J^a_n \cup J^b_n, \\ \displaystyle\sum_{k \in J^a \cup J^b} \sum_{j \in J^b} W_{i,j} A^{[3]}_{j,k} N^{[3]}_k + \sum_{k \in J^b \cup J^c} \bar{A}^{[4]}_{i,k} N^{[4]}_{k-2\nu} & \text{for } i \in J^c_n \end{cases} \tag{78}$$

where $\boldsymbol{A}^{[3]} = \boldsymbol{A}^{[3]}_{2\nu}$, $J^\alpha = J^\alpha_n$ for $\alpha = a, b, c$, and $\bar{\boldsymbol{A}}^{[4]} = \bar{\boldsymbol{A}}^{[4]}_{n-2\nu}$ is the $(J^b_n \cup J^c_n) \times (J^b_n \cup J^c_n)$-matrix with

$$\bar{A}^{[4]}_{i,k} \equiv \bar{A}^{[4]}_{n-2\nu;\, i,k} = A^{[4]}_{n-2\nu;\, i-2\nu, k-2\nu} \quad \text{for each } (i, k). \tag{79}$$

According to Lemma 20 we have

$$A^{[3]}_{i,k} = \sum_{j \in J^b} W_{i,j} \bar{A}^{[4]}_{j,k} \quad \text{for } i \in J^a \cup J^b,\ k \in J^b \tag{80′}$$

and

$$\bar{A}^{[4]}_{i,k} = \sum_{j \in J^b} W_{i,j} A^{[3]}_{j,k} \quad \text{for } i \in J^c,\ k \in J^b. \tag{80″}$$

Let us introduce for each n the matrix $\boldsymbol{W}_n$ with

$$W_{n;i,j} = \begin{cases} 0 & \text{for } i \in J_n,\ j \in J^a_n \cup J^c_n, \\ W_{i,j} & \text{for } i \in J_n,\ j \in J^b_n, \end{cases}$$

the numbers $W_{i,j}$ being the solution of (80). In this notation (80) may be rewritten as follows

$$\boldsymbol{A}^{[3]}_{2\nu(a,b;\,b)} = \boldsymbol{W}_{n(a,b;\,b)} \circ \boldsymbol{U}^{[4]}_n, \tag{81′}$$

$$\bar{\boldsymbol{A}}^{[4]}_{n-2\nu(c;\,b)} = \boldsymbol{W}_{n(c;\,b)} \circ \boldsymbol{U}^{[3]}_n, \tag{81″}$$

where $\boldsymbol{U}^{[4]}_n = \bar{\boldsymbol{A}}^{[4]}_{n-2\nu(b;b)}$ and $\boldsymbol{U}^{[3]}_n = \boldsymbol{A}^{[3]}_{2\nu(b;\,b)}$. According to (20) and Lemma 19 there exists $C < \infty$ such that for each n

$$\tfrac{1}{2} \leqslant \lambda_m(\boldsymbol{U}^{[l]}_n) \leqslant \lambda_M(\boldsymbol{U}^{[l]}_n) \leqslant C \quad \text{for } l = 3, 4. \tag{82}$$

Therefore the assumptions of Theorem 2, condition (i), are satisfied, and hence the inverses $\boldsymbol{V}^{[l]}_n = (\boldsymbol{U}^{[l]}_n)^{-1}$ form a d.e. family of matrices, $l = 3, 4$. According to Lemma 11 the family of matrices $\boldsymbol{W}_n$, where (cf. (81)) $\boldsymbol{W}_{n(a,b;\,b)} = \boldsymbol{A}^{[3]}_{2\nu(a,b;b)} \circ \boldsymbol{V}^{[4]}_n$, $\boldsymbol{W}_{n(c;b)} = \boldsymbol{A}^{[4]}_{n-2\nu(c;\,b)} \circ \boldsymbol{V}^{[4]}_n$, $\boldsymbol{W}_{n(;a,c)} = \boldsymbol{O}_{(;a,c)}$, is d.e and of a.n.r. For the unique spline functions $M_{n;i}$ given by (78) we have

$$M_{n;i} = \sum_{k \in J_n} B_{n;\,i,k} N^{[5]}_{n;k}, \tag{83}$$

where for each n for which (73) holds

$$\begin{aligned} \boldsymbol{B}_{n(a,b;\,a,b)} &= \boldsymbol{A}^{[3]}_{2\nu(a,b;\,a,b)}, \\ \boldsymbol{B}_{n(a,b;\,c)} &= \boldsymbol{W}_{n(a,b;\,b)} \circ \bar{\boldsymbol{A}}^{[4]}_{n-2\nu(b;\,c)}, \\ \boldsymbol{B}_{n(c;\,a)} &= \boldsymbol{W}_{n(c;a)} \circ \boldsymbol{A}^{[3]}_{2\nu(b;a)}, \\ \boldsymbol{B}_{n(c;\,b,c)} &= \bar{\boldsymbol{A}}^{[4]}_{n-2\nu(c;\,b,c)}. \end{aligned} \tag{84}$$

Applying Lemmas 11 and 7 to (84) we infer that the family $\{\boldsymbol{B}_n\}$ is d.e. Substituting (83) into (75) we get that the $J_n \times J_n$-matrix $\boldsymbol{F}_n$ with

$$F_{n;\,i,j} = (M_{n;\,i}, N^{[5]}_{n;\,j}) \quad \text{for } i, j \in J_n$$

has the properties: $\boldsymbol{F}_n = \boldsymbol{B}_n \circ \boldsymbol{G}^{[5]}_n$, $\boldsymbol{F}_{n(a,c;\,a,c)} = \boldsymbol{E}_{(a,c;\,a,c)}$ and $\boldsymbol{F}_{n(b;\,a,c)} = \boldsymbol{O}_{(b;\,a,c)}$, i.e. each $\boldsymbol{F}_n$ is of m-shape of the first kind. Obviously $\{\boldsymbol{F}_n\}$ is d.e. Further, $\mathrm{Det}\boldsymbol{F}_n = \mathrm{Det}\boldsymbol{F}_{n(b;\,b)}$, where according to (76) and (78) (the subindices $n, n-2\nu, 2\nu$ are dropped)

$$\begin{aligned} F_{n;i,j} &= \int\limits_{\langle 0, 2^{\mu+1}\rangle} M_{n;\,i}(t) N^{[5]}_{n;\,j}(t)\,dt \\ &= \sum_{k \in J^a \cup J^b} A^{[3]}_{i,k} G^{[3]}_{k,j} + \sum_{k \in J^b \cup J^c} \sum_{l \in J^b_n} W_{i,l} \bar{A}^{[4]}_{l,k} G^{[4]}_{k-2\nu, j-2\nu} \\ &= \delta_{i,j} + W_{i,j} = (\boldsymbol{E}_{(b;\,b)} + \boldsymbol{W}_{n(b;\,b)})_{(i,j)} \quad \text{for } i, j \in J^b_n. \end{aligned}$$

According to (81′) we have $\boldsymbol{W}_{n(b;\,b)} = \boldsymbol{A}^{[3]}_{2\nu(b;\,b)} \circ \boldsymbol{V}^{[4]}_n$ and hence

$$\boldsymbol{F}_{n(b;\,b)} = (\boldsymbol{U}^{[4]}_n + \boldsymbol{U}^{[3]}_n) \circ \boldsymbol{V}^{[4]}_n.$$

According to (21) and (82) we have

$$\begin{aligned} \mathrm{Det}\boldsymbol{F}_n &= \mathrm{Det}\boldsymbol{F}_{n(b;\,b)} = \mathrm{Det}(\boldsymbol{U}^{[4]}_n + \boldsymbol{U}^{[3]}_n) \cdot \mathrm{Det}\boldsymbol{V}^{[4]}_n \\ &\geqslant \left(\frac{\lambda_m(\boldsymbol{U}^{[4]}_n + \boldsymbol{U}^{[3]}_n)}{\lambda_M(\boldsymbol{U}^{[4]}_n)}\right)^{m+1} \geqslant \left(\frac{1}{C}\right)^{m+1} > 0. \end{aligned}$$

Thus the assumptions of Theorem 2, condition (iii), are satisfied for the matrices $\boldsymbol{F}_n$ and therefore the inverses $\boldsymbol{P}_n = (\boldsymbol{F}_n)^{-1}$ form a d.e. family. Further, according to Lemma 13 each $\boldsymbol{P}_n$ is of m-shape of the first kind, thus the matrix $\boldsymbol{P}_n \circ \boldsymbol{B}_n$ is (i) inverse to $\boldsymbol{G}^{[5]}_n$ and (ii) $\{\boldsymbol{P}_n \circ \boldsymbol{B}_n\} = \{\boldsymbol{A}^{[5]}_n\}$ is d.e.

Thus the proof of Theorem 4 is complete.

13. Proof of Theorem 1. Formulae (10) and (51) which define $S_n^{[0]}$ and $S_n^{[5]}$ respectively allow us to state that (using the detailed notation)

$$G_n^{(m)[5]} = 2^{\mu+1} G_n^{(m)[0]} \quad \text{for } n \geqslant 2$$

and therefore

$$A_n^{(m)[0]} = 2^{\mu+1} A_n^{(m)[5]} \quad \text{for } n \geqslant 2.$$

Now, $2^{\mu} < n \leqslant 2^{\mu+1}$ for $n \geqslant 2$, hence

$$|A_{n;i,j}^{(m)[0]}| \equiv |A_{n;i,j}^{(m)}| \leqslant 2n |A_{n;i,j}^{(m)[5]}| \quad \text{for } i, j \in J_n^m, n \in \mathcal{N}.$$

According to Theorem 4, for each m, $m \geqslant 0$, there exist constants C_m and q_m, $0 < q_m < 1$ such that

$$|A_{n;i,j}^{(m)[0]}| \leqslant n C_m q_m^{|i-j|} \quad \text{for } i, j \in J_n^m, n \in \mathcal{N}.$$

Thus Theorem 1 is proved.

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Applications des ultraproduits à l'étude des espaces et des algèbres de Banach

par

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Sommaire. Un certain nombre de notions précisant les rapports entre les problèmes de caractérisation des classes d'espaces de Banach, et les problèmes de la théorie de la dimension linéaire. Nous posons le problème de caractérisation: une classe C d'espaces de Banach est-elle caractérisée par un ensemble de conditions du type

$$\forall x_1 \forall x_2 \dots \forall x_n ((\|\sum_{i=1}^{n} a_i^j x_i\|)_{i=1,\dots,m} \in F)$$

ou F est un fermé (cône) de (R_+^m) et (a) des matrices réelles données?

La notion d'ultraproduit donne un critère d'étude de telles caractérisations à titre d'exemple nous donnons la caractérisation des espaces isomorphes à des sous-L^p. La méthode permet de bien situer le problème de la dimension linéaire en fonction des propriétés des sous-espaces de dimension finie. Nous l'appliquons à certains espaces d'Orlicz et aux sous algèbres de Banach des algèbres l^p.

Dans cet article, nous donnons des applications de la notion d'ultraproduit dans les espaces de Banach. Les classes d'espaces de Banach stables par ultraproduit, isomorphismes (ou isométries) et sous-espaces se caractérisent par des conditions d'un type simple portant sur la norme. Nous étendons par exemple à une classe $\mathscr{C}$ d'espaces d'Orlicz la propriété: Si B est un espace de Banach, pour qu'il soit isomorphe (avec des bornes données) à un espace C de la classe $\mathscr{C}$, il faut et il suffit que tout sous-espace de dimension finie de B ait cette même propriété. Ce type de conditions, a été trouvé sous des formes particulièrement simples par Grothendieck pour les espaces isomorphes à des espaces de Hilbert [1] et dans [2] et [3] pour les espaces isométriques à des sous-espaces d'espaces L^p.

Le problème est évidemment lié au problème de la dimension linéaire.

Le plan est le suivant:

§ 1. Notion d'ultraproduit dans les espaces de Banach.
§ 2. Compléments sur les Banach réticulés.
§ 3. Rappels et compléments sur les espaces d'Orlicz.
§ 4. Ultraproduits d'Orlicz.
§ 5. Problème de plongements et de finitude.