[8] T. Husain, *Open Mapping and Closed Graph Theorems in Topological Vector Spaces*, Oxford 1965.
[9] S. Kakutani and V. Klee, *The finite topology of a linear space*, Archiv der Mathematik 14 (1963), pp. 55–58.
[10] J. L. Kelley, *General Topology*, Princeton 1955.
[11] — *Hypercompleteness*, Michigan Math. J. 5 (1958), pp. 235–246.
[12] — and I. Namioka, *Linear Topological Spaces*, Princeton 1963.
[13] Y. Komura, *Some examples on linear topological spaces*, Math. Ann. 153 (1964), pp. 150–162.
[14] G. Köthe, *Topologische Lineare Räume*, Berlin 1960.
[15] — *Vollständige lokalkonvexe Räume abzählbarer Dimension*, Archiv der Mathematik 13 (1962), pp. 377–384.
[16] A. H. Kruse, *Introduction to the Theory of Block Assemblages and Related Topics in Topology*, University of Kansas 1956.
[17] S. Mazur, *On continuous mappings on cartesian products*, Fund. Math. 39 (1952), pp. 229–238.
[18] H. H. Schaefer, *Topological Vector Spaces*, New York 1966.
[19] S. Warner, *The topology of compact convergence on continuous function spaces*, Duke, Math. J. 25 (1958), pp. 265–282.

LOUISIANA STATE UNIVERSITY
BATON ROUGE

Received September 12, 1970 (244)

STUDIA MATHEMATICA T. XLI. (1972)

An extension of Brudno–Mazur–Orlicz theorem

by

A. JAKIMOVSKI and A. LIVNE (Tel Aviv)

Abstract. In this paper the Brudno–Mazur–Orlicz consistency theorem is extended to certain BK-spaces. Let E be a closed subspace of the BK-space D, where $\{e^j\}_{j\geqslant 0}$, $e^j = (0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the j-th place, is a Schauder-basis for E, and $\{e^j\}_{j\geqslant 0}$ together with $e \equiv (1, 1, 1, \ldots)$ is a Schauder-basis for D. With such a BK-space E two dual spaces $(\lambda^* E)$ and (λ^*, D), where associated in [4]. Define a matrix A to be regular with respect to D if its convergence field c_A includes D, $\lim_A e = 1$ and $\lim_A x = 0$ for each $x \in E$. The extension mentioned above is the following: Suppose E is solid and $\{e^j\}_{j\geqslant 0}$ is a Schauder-basis for (λ^*, E). If A and B are regular matrices with respect to D and $c_A \supset c_B \cap \mu^*$, then A and B are consistent on (μ^*, E). For $E = c_0$, and $D = c$ we get the Brudno–Mazur–Orlicz theorem. Examples of spaces E, D satisfying all the assumptions are given.

§ 1. Introduction. In this paper Brudno–Mazur–Orlicz famous consistency theorem stating that *if A and B are regular matrices such that the bounded convergence field of A includes that of B, then A and B are consistent for bounded sequences* (See [2, p. 198], also Mazur and Orlicz [5, Theorem 6] and [6] and Zeller [7, Theorem 6.4]) is extended to pairs of matrices which are "regular" in a certain extended sense. We obtain this extension of Brudno–Mazur–Orlicz theorem by increasing the scope of the principle of "aping sequences". The basic idea of the principle of aping sequences was introduced independently by Agnew [1, Theorem 4.1] and Brudno [2]. A general form of this principle was given by Erdös and Piranian [3, Theorem 1]. This principle states, roughly, that a regular matrix transforms each pair of similar bounded sequences into similar bounded sequences.

Terminology and Notation. We denote by E a closed subspace of co-dimension one of a fixed BK-space $(D, \|\cdot\|_D)$. We assume that $\{e^j\}_{j\geqslant 0}$, $e^j = (0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the j-th place, is a Schauder-basis for E and $\{e^j\}_{j\geqslant 0}$ together with $e = (1, 1, 1, \ldots)$ is a Schauder-basis for D. The space $\lambda^* = (\lambda^*, E)$ is defined (see [4, § 5]) as the set of all infinite sequences y satisfying $\|y\|_{\lambda^*} \equiv \sup_{n\geqslant 0} \sup_{\|x\|_E\leqslant 1} |\sum_{k=0}^{n} x_k y_k| < \infty$. It is known (see [4, § 5]) that $(\lambda^*, \|\cdot\|_{\lambda^*})$ is a BK-space and $e^j \in \lambda^*, j \geqslant 0$. We assume that $\{e^j\}_{j\geqslant 0}$ in a Schauder-basis for λ^*, too. Similarly the space μ^* and

$\|\cdot\|_{\mu^*}$ are defined by means of $\mu^* \equiv (\lambda^*, (\lambda^*, E))$. It is obvious that we have $\sup_{n\geqslant 0} |\sum_{k=0}^{n} x_k y_k| \leqslant \|x\|_E \|y\|_{\lambda^*}$ and $\sup_{n\geqslant 0} |\sum_{k=0}^{n} y_k z_k| \leqslant \|y\|_{\lambda^*} \|z\|_{\mu^*}$ for $x \in E$, $y \in \lambda^*$ and $z \in \mu^*$.

The space E is *solid* (with constant $c \geqslant 1$) if for each $x \in E$ and each real sequence

$$\theta = (\theta_0, \theta_1, \ldots)\{x_k e^{i\theta_k}\}_{k\geqslant 0} \in E \quad \text{and} \quad \|\{x_k e^{i\theta_k}\}_{k\geqslant 0}\|_E \leqslant c\|x\|_E.$$

For an infinite matrix $A = (a_{mn})_{m,n\geqslant 0}$ define

$$c_A \equiv \Big\{x \,|\, (Ax)_m \equiv \sum_{n=0}^{\infty} a_{mn} x_n \text{ convergence for each } m \geqslant 0 \text{ and } \lim_A x \equiv \lim_{m\to\infty} (Ax)_m \text{ exists}\Big\}$$

and $Ax \equiv \{(Ax)_m\}, m \geqslant 0$. The matrix A is *conservative with respect to* D if $D \subset c_A$. The matrix A is *regular with respect to* D if it is conservative with respect to D, $\lim_A e = 1$ and $\lim_A x = 0$ for each $x \in E$. By Theorem (5.1) of [4] we have

THEOREM (1.1). *A matrix $A \equiv (a_{mn})_{m,n\geqslant 0}$ is regular with respect to D if and only if the following three conditions are satisfied:*

(i) $(Ae^j)_m$ *exists for* $j, m \geqslant 0$ *and* $\lim_A e^j = 0$ *for* $j \geqslant 0$

(ii) $(Ae)_m$ *exists for* $m \geqslant 0$ *and* $\lim_A e = 1$

(iii) $\sup_{m\geqslant 0} \|\{a_{mn}\}_{n\geqslant 0}\|_{\lambda^*} \equiv M < \infty$.

For a matrix A regular with respect to D we have

$$(1.1) \qquad \lim_{m\to\infty} \|(0, \ldots, 0, a_{mj}, 0, \ldots)\|_{\lambda^*} = 0 \quad \text{for } j = 0, 1, 2, \ldots$$

§ 2. The principle of aping sequences and Brudno–Mazur–Orlicz theorem. In our discussion of matrices which are regular with respect to D it may be assumed that the matrices are row-finite. This will follow from the following result.

LEMMA (2.1). *Let A be regular with respect to D. Then there exists a pair of strictly increasing sequences $\{m(r)\}, \{n(r)\}$ $(r \geqslant 0)$ of non-negative integers such that*

$$(2.1) \qquad \lim_{r\to\infty} \max_{m(r)\leqslant m < m(r+1)} \|(a_{m0}; \ldots; a_{m,n(r)-1}; 0; \ldots; 0; a_{m,n(r+2)+1}; \ldots)\|_{\lambda^*} = 0.$$

Proof. Sequences $\{m(r)\}, \{n(r)\}$ are chosen inductively so that $m(0) = n(0) = 0$, $n(1)$ is an arbitrary positive integer, $m(r) > m(r-1)$, $n(r) > n(r-1)$,

$$(2.2) \qquad \|(a_{m0}; \ldots; a_{m,n(r)}; 0; \ldots)\|_{\lambda^*} < 2^{-r} \quad \text{for } m \geqslant m(r) \text{ and } r \geqslant 1;$$

and

$$(2.3) \qquad \max_{m(r-2)\leqslant m < m(r-1)} \|(0; \ldots; 0; a_{m,n(r)}; a_{m,n(r)+1}; \ldots)\|_{\lambda^*} < 2^{-r} \quad \text{for } r \geqslant 2.$$

Arbitrary large $m(r)$ satisfying (2.2) exist by (1.1) and arbitrary large $n(r)$ satisfying (2.3) exist, since by Theorem (1.1) $\{a_{mn}\}_{n\geqslant 0} \in \lambda^*$ for $m > 0$ and $\{e^j\}_{j\geqslant 0}$ is a Schauder-basis for λ^*. Now (2.1) follows immediately by (2.2) and (2.3). Q.E.D.

A pair $\{\{m(r)\}, \{n(r)\}\}$ of strictly increasing sequences of non-negative integers satisfying (2.1) is called *a-trim* of the matrix A. By the proof of Lemma (2.1) it follows that for any pair A, B of regular matrices with respect to D there exists a trim for both A and B.

If $\{n(r)\}_{r\geqslant 0}$ is a strictly increasing sequence of non-negative integers, we say that $\xi \equiv \{\xi_k\}_{k\geqslant 0}$ *wanders slowly over* $\{n(r)\}$ provided $\lim_{r\to\infty} \max_{n(r)<k\leqslant n(r+1)} |\xi_k - \xi_{n(r)}| = 0$ (see Erdös and Piranian [3]). If ξ wanders slowly over $\{n(r)\}$ and η wanders slowly over $\{m(r)\}$ and $\xi_{n(r)} - \eta_{m(r)} \to 0$ we say that ξ and η are $\{n(r), m(r)\}$*-similar*. We say that y *apes* x *(boundedly) over* $\{n(r)\}$ provided there exists a (bounded) sequence ξ wandering slowly over $\{n(r)\}$ and having the property that $\{y_k - x_k \xi_k\}_{k\geqslant 0} \in D$.

THEOREM (2.1) (PRINCIPLE OF APING SEQUENCES). *Suppose E is solid. Let A be regular with respect to D, and let $\{\{m(r)\}, \{n(r)\}\}$ be a-trim of A. For $z \in \mu^*$ and $y_k = z_k \xi_k + d_k$, where ξ is bounded and wanders slowly over $\{n(r)\}$ and $d \in D$ denote $s = Az$, $t = Ay$. Then t has a representation $t_n = s_n \eta_n + \delta_n$, where, $\delta \in c$, $\lim_{n\to\infty} \delta_n = \lim_A d_n$, η wanders slowly over $\{m(r)\}$ and ξ and η are $\{n(r), m(r)\}$ — similar.*

For $E = c_0$, $D = c$ and $\|\cdot\|_D$ the sup norm in c we have $\lambda^* = l_1$. In this example $E = c_0$ is solid and all the assumptions made on the spaces E, D, λ^* in Theorem (2.1) are satisfied. Theorem (2.1) for this example is Theorem 1 by Erdös and Piranian [3]. Another example is the following one. Let $\{d_k\}_{k\geqslant 1}$ be a positive sequence satisfying $D_n = \sum_{k=1}^{n} d_k \to +\infty$, $n \to \infty$. For a given $p \geqslant 1$, $x \equiv (x_1, x_2, \ldots)$ is strongly Riesz-summable of order p to a if $\sum_{k=1}^{n} d_k |x_k - a|^p = o(D_n)$, $n \to \infty$. Denote by E the set of all infinite sequences which are strongly Riesz-summable of order p to zero and let D be the set of all infinite sequences strongly Riesz-summable of order p to a finite sum. For the norm $\|x\|_D = \sup_{n\geqslant 1} (D_n^{-1} \sum_{k=1}^{n} d_k |x_k|^p)^{1/p}$ D is a BK-space and E is a closed subspace of co-dimension one in D. Also, E is solid, $\{e^j\}_{j\geqslant 1}$ is a Schauder-basis for E and $\{e^j\}_{j\geqslant 1}$ together with e is a Schauder-basis for D. By the proof of Theorem 4.1 of [4] it follows that $\lambda^* \equiv (\lambda^*, E) = \{y \,|\, \|y\|_p < \infty\}$, where

$$\|y\|_p = \sum_{n=0}^{\infty} 2^{n/p} \Big(\sum_{2^n \leqslant D_k < 2^{n+1}} |y_k d_k^{-1/p}|^q \Big)^{1/q} \quad \text{for } p > 1$$

and

$$\|y\|_1 = \sum_{n=0}^{\infty} 2^{n/p} \max_{\substack{k \\ 2^n \leqslant D_k < 2^{n+1}}} |y_k/d_k|.$$

Also $\|\cdot\|_p$ is equivalent to $\|\cdot\|_{\lambda}^*$. Obviously $\{e^j\}_{j\geqslant 1}$ is a Schauder-basis for λ^*, in this case the space μ^* is the set $\{z \mid \sum_{k=1}^{n} d_k |z_k|^p = O(D_n)\}$.

We need the following result.

Lemma (2.2). *Suppose E is solid with constant $c \geqslant 1$. Then λ^* is solid with the constant c and for each pair $y \in \lambda^*$ and $w \in m$ we have $\{y_k w_k\}_{k\geqslant 0} \in \lambda^*$ and $\|\{y_k w_k\}_{k\geqslant 0}\|_{\lambda^*} \leqslant c\|y\|_{\lambda^*}\|w\|_m$. Also, μ^* is solid with constant c and for each pair $z \in \mu^*$ and $w \in m$ we have $\{z_k w_k\}_{k\geqslant 0} \in \mu^*$ and $\|\{z_k w_k\}_{k\geqslant 0}\|_{\mu^*} \leqslant c\|z\|_{\mu^*}\|w\|_m$.*

Proof. First, for $y \in \lambda^*$ and a real sequence θ we have $\{y_k e^{i\theta_k}\}_{k\geqslant 0} \in \lambda^*$ since

$$\|\{y_k e^{i\theta_k}\}\|_{\lambda^*} = \sup_{n\geqslant 0} \sup_{\|x\|_E \leqslant 1} \Big|\sum_{k=0}^{n} (x_k e^{i\theta_k}) y_k\Big| \leqslant \sup_{n\geqslant 0} \sup_{\|x\|_E \leqslant c} \Big|\sum_{k=0}^{n} x_k y_k\Big| = c\|y\|_{\lambda^*} < \infty.$$

Hence λ^* is solid with constant c. Now for $y \in \lambda^*$ and $w \in m$ we have

$$\begin{aligned}\|\{y_k w_k\}_{k\geqslant 0}\|_{\lambda^*} &\leqslant \|w\|_m \cdot \sup_{n\geqslant 0} \sup_{\|x\|_E \leqslant 1} \sum_{k=0}^{n} |x_k y_k| \\ &= \|w\|_m \cdot \sup_{n\geqslant 0} \sup_{\|x\|_E \leqslant 1} \Big|\sum_{k=0}^{n} (x_k e^{-i\arg(x_k y_k)}) y_k\Big| \\ &\leqslant \|w\|_m \cdot \sup_{n\geqslant 0} \sup_{\|x\|_E \leqslant c} \Big|\sum_{k=0}^{n} x_k y_k\Big| = c\|w\|_m \|y\|_{\lambda^*} < \infty.\end{aligned}$$

This completes the proof of the first part of the lemma. The proof of the second part is similar. Q.E.D.

Proof of theorem (2.1). By Lemma (2.2) $y \in \mu^*$. For $m(r) \leqslant m < m(r+1)$ write

$$t_m = \sum_{n<n(r)} a_{mn} y_n + \xi_{n(r)} \sum_{n(r)\leqslant n \leqslant n(r+2)} a_{mn} z_n + \sum_{n(r)\leqslant n \leqslant n(r+2)} a_{mn} (\xi_n - \xi_{n(r)}) z_n +$$
$$+ \sum_{n(r)\leqslant n \leqslant n(r+2)} a_{mn} d_n + \sum_{n>n(r+2)} a_{mn} y_n \equiv \sum_{k=1}^{5} S_k.$$

We have by (2.1)

$$|S_1 + S_5| \leqslant \|y\|_{\mu^*} \|(a_{m0}; \ldots; a_{m,n(r)-1}; 0; \ldots; 0; a_{m,n(r+2)+1}; \ldots)\|_{\lambda^*} \to 0 \quad (m \to \infty).$$

By (2.1) and Theorem (1.1) we have $S_2 - \xi_{n(r)} s_m \to 0$ $(m \to \infty)$ and $S_4 \to \lim_A d_n$. By Lemma (2.2) and Theorem (1.1).

$$\begin{aligned}|S_3| &\leqslant \big(\max_{n(r)\leqslant n \leqslant n(r+2)} |\xi_n - \xi_{n(r)}|\big) \sum_{n(r)\leqslant n \leqslant n(r+2)} |a_{mn} z_n| \\ &\leqslant \big(\max_{n(r)\leqslant n \leqslant n(r+2)} |\xi_n - \xi_{n(r)}|\big) \cdot \sup_{\theta_n \geqslant 0} \Big|\sum_{n(r)\leqslant n \leqslant n(r+2)} a_{mn} z_n e^{i\theta_n}\Big| \\ &\leqslant \big(\max_{n(r)\leqslant n \leqslant n(r+2)} |\xi_n - \xi_{n(r)}|\big) \|(0; \ldots; 0; a_{m,n(r)}; \ldots; a_{m,n(r+2)-1}; \\ &\qquad\qquad 0; \ldots)\|_{\lambda^*} \|\{z_k e^{i\theta_k}\}\|_{\mu^*} \\ &\leqslant 2c\|z\|_{\mu^*} \|\{a_{mn}\}_{n\geqslant 0}\|_{\lambda^*} \big(\max_{n(r)\leqslant n \leqslant n(r+2)} |\xi_n - \xi_{n(r)}|\big) \\ &\leqslant 2Mc\|z\|_{\mu^*} \big(\max_{n(r)\leqslant n \leqslant n(r+2)} |\xi_n - \xi_{n(r)}|\big) \to 0 \quad (m \to \infty).\end{aligned}$$

This completes the proof. Q.E.D.

As a consequence of Theorem (2.1) we get the following result.

Theorem (2.2) (Consistency theorem). *Suppose E is solid and $e \in \mu^*$. If A and B are regular with respect to D and $c_A \supset c_B \cap \mu^*$, then the matrices A and B are consistent on μ^*.*

Theorem (2.2) for $E = c_0, D = c$ is Brudno–Mazur–Orlicz consistency theorem.

Proof. There is a-trim $\{n(r), m(r)\}$ for both A and B. If A and B are not consistent in μ^*, there exists some $z \in \mu^*$ such that $\lim_A z_n = 1$ and $\lim_B z_n = 0$. Let ξ be a bounded sequence wandering slowly over $\{n(r)\}$ with $\xi_{n(r)} = \sin\sqrt{r}$. By Lemma (2.2) $\{z_k \xi_k\} \in \mu^*$ and by Theorem (2.1) B evaluates the sequence $\{z_k \xi_k\}$ to 0 while A transforms $\{z_k \xi_k\}$ into a sequence which oscillates between 1 and -1, contrary to the hypothesis $c_A \supset c_B \cap \mu^*$. Q.E.D.

References

[1] R. P. Agnew, *Convergence fields of methods of summability*, Ann. of Math. (2) 46 (1945), pp. 93–101.

[2] A. L. Brudno, *Summation of bounded sequences by matrices*, Math. Sbornik (N. S.) 16 (1945), pp. 191–247.

[3] P. Erdös and G. Piranian, *The topologization of a sequence space by Toeplitz matrices*, Michigan Math. J. 5 (1958), pp. 139–148.

[4] A. Jakimovski and A. Livne, *General Kojima–Toeplitz like theorems and consistency theorems*, J. d'Analyse Math. 24 (1971), pp. 323–368.

[5] S. Mazur and W. Orlicz, *Sur les methodes lineaires de sommation*, C. R. Acad. Sci. Paris 196 (1933), pp. 32–34.
[6] — *On linear methods of summability*, Studia Math. 14 (1954), pp. 129–160.
[7] K. Zeller, *Faktorfolgen bei Limitierungsverfahren*, Math. Zeit. 56 (1952), pp. 134–151.

TEL AVIV UNIVERSITY
THE WEIZMANN INSTITUTE OF SCIENCE

Received October 5, 1970 (269)

STUDIA MATHEMATICA T. XLI. (1972)

Approximation von Elementen eines lokalkonvexen Raumes

von

EBERHARD SCHOCK (Bonn)

Zusammenfassung. Der Satz von Bernstein über die Approximationsgeschwindigkeit in Banachräumen wird auf lokalkonvexe Räume übertragen. Dabei zeigt sich, daß Schwartz–Räume durch ein günstiges Approximationsverhalten charakterisiert werden. Weiterhin wird eine Methode beschrieben, die für die Approximation günstigen Teilräume zu ermitteln.

Das (lineare) Approximationsproblem besteht bekanntlich darin, in einem linearen Raum F zu einem Punkt $x \notin F$ ein y zu finden, das bezüglich einer auf $x+F$ definierten Norm p von x minimalen Abstand hat. Dabei wählt man einen lokalkonvexen Raum E mit der Eigenschaft, daß $x+F$ Teilmenge von E ist und daß p die Topologie von E erzeugt.

Abweichend von dieser Auffassung wollen wir hier nicht verlangen, daß die Norm p die Topologie von E erzeugt, sondern lediglich, daß die Norm p auf E stetig ist.

Diese Abschwächung hat Konsequenzen weniger für Fragen der geometrischen Approximationstheorie, also für Aussagen, die invariant sind unter bezüglich p isometrischen Isomorphismen, sondern für Aussagen von mehr topologischem Charakter, die also invariant sind unter (linearen) topologischen Isomorphismen.

In dieser Arbeit sollen vor allem Aussagen untersucht werden, die zusammenhängen mit dem

Satz von Bernstein. *Es sei E ein unendlich dimensionaler Banachraum mit der Einheitskugel U. Dann gibt es für jede positive monotone Nullfolge $\{\alpha_n\}$ und für jede Folge $\{E_n\}$ von Teilräumen von E mit* $\dim E_n = n$ *ein $x \in E$ mit*

$$\varrho(x, U, E_n) := \inf\{p_U(x-y), y \in E_n\} = \alpha_n.$$

(Dabei bezeichnet p_U das Minkowski–Funktional von U, also die Norm von E.)

Banachräume zeigen also ein ungünstiges Approximationsverhalten, man muß daher die Frage nach „besseren" Räumen stellen (dazu vergleiche man auch das von I. M. Gelfand auf der 3. Konferenz über Funktionalanalysis, Moskau 1956, ([5], S. 6) gestellte Problem). Ebenfalls besagt