An extension of Brudno–Mazur–Orlicz theorem

by

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Abstract. In this paper the Brudno–Mazur–Orlicz consistency theorem is extended to certain BK-spaces. Let $E$ be a closed subspace of the BK-space $D$, and $\phi'_{j\in\omega}$ where $1$ is the $j$-th place, is a Schauder-basis for $E$, and $\phi'_{j\in\omega}$ is a Schauder-basis for $D$. With such a BK-space $E$ two dual spaces $(E')$ and $(D')$, where associated in $[4]$. Define a matrix $A$ to be regular with respect to $D$ if its convergence field $c_{D}$ includes $D_{m}$, $\lim_{D_{m}} = 1$ and $\lim_{D_{m}} = 0$ for each $A = B$. The extension mentioned above is the following: Suppose $E$ is solid and $(\phi'_{j\in\omega})$ is a Schauder-basis for $(\phi_{j\in\omega})$. If $A$ and $B$ are regular matrices with respect to $D$ and $A_{D} = B_{D}$, then $A$ and $B$ are consistent on $(\phi_{j\in\omega})$. For $E = D$, and $D = c$ we get the Brudno–Mazur–Orlicz theorem. Examples of spaces $E, D$ satisfying all the assumptions are given.

§ 1. Introduction. In this paper Brudno–Mazur–Orlicz famous consistency theorem stating that if $A$ and $B$ are regular matrices such that the bounded convergence field of $A$ includes that of $B$, then $A$ and $B$ are consistent for bounded sequences (see [2, p. 198], also Mazur and Orlicz [5, Theorem 6] and [6] and Zeller [7, Theorem 6.4]) is extended to pairs of matrices which are “regular” in a certain extended sense. We obtain this extension of Brudno–Mazur–Orlicz theorem by increasing the scope of the principle of “apaining sequences”. The basic idea of the principle of aching sequences was introduced independently by Agnew [1, Theorem 4.1] and Brudno [2]. A general form of this principle was given by Erdős and Piranian [3, Theorem 1]. This principle states, roughly, that a regular matrix transforms each pair of similar bounded sequences into similar bounded sequences.

Terminology and Notation. We denote by $E$ a closed subspace of co-dimension one of a fixed BK-space $(D, \|\cdot\|)$. We assume that $(\phi'_{j\in\omega})_E' = (0, 1, 0, 1, 0, \ldots)$, where $1$ is in the $j$-th place, is a Schauder-basis for $E$ and $(\phi'_{j\in\omega})_D'$ together with $c = (1, 1, 1, \ldots)$ is a Schauder-basis for $D$. The space $E' = (E')$ is defined (see [4, § 3]) as the set of all infinite sequences $y$ satisfying $\|y\|_c = \sup \sum_{i=1}^{n} |y_{i}| < \infty$. It is known (see [4, § 3]) that $(E', \|\cdot\|)$ is a BK-space and $e' = \lambda'$, $j \geq 0$. We assume that $(\phi'_{j\in\omega})_E'$ is a Schauder-basis for $E'$, too. Similarly the space $\mu'$ and
$\|\cdot\|_r$ are defined by means of $\mu^* = (\lambda^*, \Lambda^*, E)$. It is obvious that we have
$$\sup_{n=0}^\infty \sum_{a \in A} |a_n| \leq \|\cdot\|_r$$ and
$$\sup_{n=0}^\infty \sum_{a \in A} |a_n| \leq \|\cdot\|_r$$ for $x \in E$, $y \in \Lambda^*$ and $z \in \mu^*$.

The space $E$ is solid (with constant $c \geq 1$) if for each $x \in E$ and each real sequence
$$\theta = (\theta_1, \theta_2, \ldots) \in (\Lambda^*(\mu^*))_{\infty}^\infty$$ and
$$\|x\|_{\infty} \leq c \|\theta\|_{\infty}.$$

For an infinite matrix $A = (a_{m,n})_{m,n \in \mathbb{N}}$ define
$$c_A = \|\Lambda(\mu^*)_{\infty} \|_{\infty} = \sum_{a \in A} |a_m| \|a_n\|_r$$ convergence for each $m \geq 0$
$$\text{and} \lim_{m \to \infty} c_A = \lim_{m \to \infty} \|\Lambda(\mu^*)_{\infty} \|_{\infty},$$ and $A x = (\Lambda(\mu^*)_{\infty} x)_{m \in \mathbb{N}}$, $m \geq 0$. The matrix $A$ is regular with respect to $D$ if $D < c_A$. The matrix $A$ is regular with respect to $D$ if $D$ is conservative with respect to $D$, $\lim_{m \to \infty} c_A = 1$ and $\lim_{m \to \infty} \|x\|_r = 0$ for each $x \in E$. By Theorem 5.1 of [4] we have

Theorem 5.1. A matrix $A = (a_{m,n})_{m,n \in \mathbb{N}}$ is regular with respect to $D$ if and only if the following three conditions are satisfied:

1. $(A d^m)_m$ exists for $j, m \geq 0$ and $\lim_{m \to \infty} d^m = 0$ for $j \geq 0$
2. $(A e^m)_m$ exists for $m \geq 0$ and $\lim_{m \to \infty} e^m = 1$
3. $\sup_{n=0}^\infty \|\Lambda(\mu^*)_{\infty} \|_{\infty} < M < \infty$.

For a matrix $A$ regular with respect to $D$ we have

\[\lim_{m \to \infty} \|x_m, 0, \ldots, 0, a_n, 0, \ldots\|_r = 0 \quad \text{for} \quad j = 0, 1, 2, \ldots\]

§ 2. The principle of aping sequences and Brudno–Mazur–Oreiz theorem. In our discussion of matrices which are regular with respect to $D$ it may be assumed that the matrices are row-finite. This will follow from the following result.

Lemma 2.1. Let $A$ be regular with respect to $D$. Then there exists a pair of strictly increasing sequences $(m(n)), (n(r)) (r \geq 0)$ of non-negative integers such that

\[\lim_{n \to \infty} \sup_{m \geq n} \|x_m, \ldots, a_n, 0, \ldots, 0, a_n, 0, \ldots\|_r = 0.\]

Proof. Sequences $(m(n)), (n(r))$ are chosen inductively so that $m(0) = m(0) = n(0) = 0$, $m(1)$ is an arbitrary positive integer, $m(r) > m(r-1), n(r) > n(r-1)$,

\[\|x(m, \ldots, a_n, 0, \ldots, 0, a_n, 0, \ldots)\|_r < 2^{-r} \quad \text{for} \quad m \geq m(n) \text{ and } r \geq 1;\]

and

\[\max_{n \geq m(n)} \|x(0, \ldots, 0, a_n, 0, \ldots, a_n, 0, \ldots)\|_r < 2^{-r} \quad \text{for} \quad r \geq 2.\]

Arbitrary large $m(r)$ satisfying (2.2) exist by (1.1) and arbitrary large $n(r)$ satisfying (2.3) exist, since by Theorem 2.1 $(a_{m(n), m(n)+1})^* \Lambda^*$ for $m > 0$ and $(\ell^2_{n(r)})_{n(r)}$ is a Schauder-basis for $\Lambda^*$. Now (2.1) follows immediately by (2.2) and (2.3), O.E.D.

A pair $(m(r)), (n(r))$ of strictly increasing sequences of non-negative integers satisfying (2.1) is called a-tring of the matrix $A$. By the proof of Lemma 2.1 it follows that for any pair $A, B$ of regular matrices with respect to $D$ there exists a trim for both $A$ and $B$.

If $(n(r)), (\xi(n))$ is a strictly increasing sequence of non-negative integers, we say that $\xi = (\xi_n)_{n \in \mathbb{N}}$ wanders slowly over $(n(r))$ provided

\[\lim_{n \to \infty} \|\xi_n - \xi(n)\|_r = 0 \quad \text{see Erdős and Piranian [3]}\]

If $\xi$ wanders slowly over $(n(r))$ and $\eta$ wanders slowly over $(m(r))$ and $\xi(\eta) \to \eta(\xi) \to 0$ we say that $\xi$ and $\eta$ are $(n(r)), (m(r))$-similar. We say that $y : E \to (\ell^2_{n(r)})_{n \in \mathbb{N}}$ is bounded (boundedly) over $(n(r))$ provided there exists a bounded sequence $x$ wandering slowly over $(n(r))$ and having the property that $\|y_n - x_n\|_r = 0$.

Theorem 2.1 (Principle of Aping Sequences). Suppose $E$ is solid. Let $A$ be regular with respect to $D$, and let $(\{m(n))$, $(n(r))$ be a-tring of $A$. For $x \in E$ and $y_n = x_n - \xi_n + \delta_n$, where $\delta_n$ is bounded and wanders slowly over $(n(r))$ and $\xi(n) \to \eta(n) \to 0$ denote $t = \Lambda(\mu^*)_{\infty} + d - A y$. Then $t$ has a representation

\[t_n = e_n - \delta_n - \xi(n), \quad \text{where} \quad \xi(n) \to \eta(n), \quad \text{wander slowly over} \quad (n(r)) \quad \text{and} \quad \eta \text{ and } \xi \text{ are} \quad (n(r)), (m(r)) \text{-similar}.\]

For $E = E_0, D = c$ and $\|\cdot\|_r$ the sup norm in $c$ we have $\Lambda^* = t_1$.

In this example $E = E_0$ is solid and all the assumptions made on the spaces $A, \Lambda^*$ in Theorem 2.1 are satisfied. Theorem 2.1 for this example is by Erdős and Piranian [3]. Another example is the following one. Let $(d_{n+1})_{n \geq 0}$ be a positive sequence satisfying $D_n = \sum_{n=0}^\infty d_n \rightarrow +\infty, n \rightarrow \infty$. For a given $p > 1, z = (z_1, z_2, \ldots)$ is strongly Riesz-summable of order $p$ to a if $\sum_{n=0}^\infty d_n |z_n - a|^p = o(D_n), n \rightarrow \infty$. Denote by $E$ the set of all infinite sequences which are strongly Riesz-summable of order $p$ to zero and let $D$ be the set of all infinite sequences strongly Riesz-summable of order $p$ to a finite sum. For the norm $\|\cdot\|_p$

\[= \sup_{n \in \mathbb{N}} \|D_n^{-1} \sum_{n=0}^\infty |x_n|^{1/p} D_n \|_E \text{ is a $\mathcal{H}$-space and } E \text{ is a closed subspace of co-dimension one in } \mathcal{H}.\]

Also, $E$ is solid, $(\ell^2_{n(r)})_{n(r)}$ is a Schauder-basis for $E$ and $(\ell^2_{n(r)})_{n(r)}$ together is $c$ is a Schauder-basis for $E$. By the proof of Theorem 4.1 of [4] it follows that $\Lambda^* = (\lambda^*, E) = (y \|\cdot\|_r < \infty$, where

\[\|y\|_F = \sum_{n=0}^\infty \left( \sum_{n \in \mathcal{H}, D_n \geq n+1} |y_n d_n^{-1/p} \right)^{1/p} \quad \text{for } p > 1.\]
By (2.1) and Theorem (1.1) we have \( S_{l} - \xi_{0} \alpha_{m} \rightarrow 0 \) (\( m \rightarrow \infty \)) and \( S_{l} \rightarrow \lim \alpha_{n} \). By Lemma (2.2) and Theorem (1.1).

\[
|S_{l}| \leq \left( \max_{n \in \mathbb{N}, 0 \leq n \leq (r+1)} |\xi_{n} - \xi_{0}| \right) \sup_{0 \leq s \leq r} \sum_{n=0}^{\infty} a_{n} \alpha_{n} e^{\theta_{s}} - \alpha_{n} \alpha_{m} |\alpha_{n} - \alpha_{n+\theta_{s}}| \sum_{n=0}^{\infty} a_{n} e^{\theta_{s}} \alpha_{m}.
\]

\[
\leq \left( \max_{n \in \mathbb{N}, 0 \leq n \leq (r+1)} |\xi_{n} - \xi_{0}| \right) \sup_{0 \leq s \leq r} \sum_{n=0}^{\infty} a_{n} \alpha_{n} e^{\theta_{s}} - \alpha_{n} \alpha_{m} |\alpha_{n} - \alpha_{n+\theta_{s}}| \sum_{n=0}^{\infty} a_{n} e^{\theta_{s}} \alpha_{m}.
\]

\[
\leq 2\varepsilon |\varepsilon| \max_{n \in \mathbb{N}, 0 \leq n \leq (r+1)} |\xi_{n} - \xi_{0}| \rightarrow 0 \quad (m \rightarrow \infty).
\]

This completes the proof. Q.E.D.

As a consequence of Theorem (2.1) we get the following result.

**Theorem (2.2) (Consistency Theorem).** Suppose \( E \) is solid and \( \alpha \beta \). If \( A \) and \( B \) are regular with respect to \( D \) and \( c_{A} \supseteq c_{B} \cap \mu \), then the matrices \( A \) and \( B \) are consistent on \( \mu \).

Theorem (2.2) for \( B = c_{A}, D = c \) is Brudno–Maruz-Orlicz consistency theorem.

Proof. There is a-\( \alpha \) \{\( s(r), m(r) \} \) for both \( A \) and \( B \). If \( A \) and \( B \) are not consistent in \( \mu \), there exists some \( c_{A} \beta \) such that \( \lim_{n \to \infty} a_{n} = 0 \) and \( \lim_{n \to \infty} a_{n} = \infty \). Let \( \xi \) be a bounded sequence wandering slowly over \( (n(r)) \) with \( \varepsilon_{n(r)} = \sin \eta \). By Lemma (2.3) \( s_{A}, s_{B} \) and by Theorem (2.1) \( S_{l} \rightarrow 0 \) while \( \xi \) transforms \( s_{A}, s_{B} \) into a sequence which oscillates between 1 and \( -1 \), contrary to the hypothesis \( c_{A} \supseteq c_{B} \cap \mu \). Q.E.D.

**References**


Approximation von Elementen eines lokalkonvexen Raumes

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Zusammenfassung. Der Satz von Bernstein über die Approximationsgeschwindigkeit in Banachräumen wird auf lokalkonvexe Räume übertragen. Dabei zeigt sich, daß Schwarts-Räume durch ein günstiges Approximationsverhalten charakterisiert werden. Weiterhin wird eine Methode beschrieben, die für die Approximation günstigen Teilmengen zu ermitteln.

Das (lineare) Approximationsproblem besteht bekanntlich darin, in einem linearen Raum $F$ zu einem Punkt $x \in F$ ein $y$ zu finden, das bezüglich einer auf $x + F$ definierten Norm $p$ von $x$ minimalen Abstand hat. Dabei wählt man einen lokalkonvexen Raum $E$ mit der Eigenschaft, daß $x + F$ Teilmenge von $E$ ist und daß $p$ die Topologie von $E$ erzeugt.

Abweichend von dieser Auffassung wollen wir hier nicht verlangen, daß die Norm $p$ die Topologie von $E$ erzeugt, sondern lediglich, daß die Norm $p$ auf $E$ stetig ist.

Diese Abschwächung hat Konsequenzen weniger für Fragen der geometrischen Approximationstheorie, also für Aussagen, die invariant sind unter bezüglich $p$ isometrischen Isomorphismen, sondern für Aussagen von mehr topologischem Charakter, die also invariant sind unter (linearen) topologischen Isomorphismen.

In dieser Arbeit sollen vor allem Aussagen untersucht werden, die zusammenhängen mit dem

Satz von Bernstein. Es sei $E$ ein endlich dimensionaler Banachraum mit der Einheitskugel $U$. Dann gibt es für jede positive monotone $N$-Nullfolge $(a_n)$ und für jede Folge $(E_n)$ von Teilräumen von $E$ mit $\dim E_n = n$ ein $x \in E$ mit

$$q(x, U, E_n) = \inf \{p_U(x - y) : y \in E_n \} = a_n.$$

(Dabei bezeichnet $p_U$ das Minkowski-Funktional von $U$, also die Norm von $E$.)

Banachräume zeigen also ein ungünstiges Approximationsverhalten, man muß daher die Frage, nach „besseren“ Räumen stellen (dazu vergleiche man auch das von I. M. Gelfand auf der 3. Konferenz über Funktionalanalysis, Moskau 1956, ([5], S. 6) gestellte Problem). Ebenfalls besagt